Cofibre of τ, first take

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Abstract

By following [Isa19], [Ghe18] and [DI10] we explain how the motivic Adams and Adams-Novikov spectral sequences are related to their classical analogues and how the cofibre of the map of motivic spectra τ can be used to compute the classical Adams-Novikov spectral sequence and thus classical stable homotopy groups of spheres.

Contents

1 The motivic Adams spectral sequence

Let F be an algebraically closed field of characteristic 0. Most of the time we will work with $F = \mathbb{C}$. Recall that objects in the stable motivic homotopy category

$$\mathsf{Sp}_F = L_{\mathbb{A}^1,\mathrm{Nis}}(\mathcal{P}(\mathrm{Sm}_F))_*[(S^1)^{-1}, \mathbb{G}_m^{-1}],$$

where S^1 is the usual simplicial circle and $\mathbb{G}_m$ is the punctured affine line $\mathbb{A}^1 \smallsetminus 0$, are bigraded motivic spectra representing generalised motivic cohomology theories. Write $\mathbb{S}_F$ for the motivic sphere spectrum and write $\mathbb{S}_F^{n,w} \coloneqq \mathbb{G}_m^{\wedge w} \wedge \Sigma^{n-w}\mathbb{S}$ for its (n,w)-suspension. The *stable motivic homotopy groups of spheres* are given by

$$\pi_{n,w} \coloneqq \pi_{n,w}(\mathbb{S}_F) \coloneqq \hom_{\mathsf{Sp}_F}(\mathbb{S}_F^{n,w}, \mathbb{S}_F).$$

In motivic weight 0 and over $\mathbb{C}$, they are isomorphic to the classical stable homotopy groups:

$$\pi_{n,0}(\mathbb{S}_\mathbb{C}) \xrightarrow{\sim} \pi_n(\mathbb{S}).$$

Analogous to singular cohomology, there is a cohomology theory called *motivic cohomology* with coefficients in an abelian group, represented by a (motivic) Eilenberg-MacLane spectrum. We will consider motivic cohomology with $\mathbb{F}_2$-coefficients, represented by the mod 2 motivic Eilenberg-MacLane spectrum $M\mathbb{F}_2$. Write $\mathbb{M}_2$ for the bigraded motivic cohomology ring with $\mathbb{F}_2$-coefficients $H^{*,*}(\mathrm{Spec}(F)) = M\mathbb{F}_2^{*,*}(\mathrm{Spec}(F))$. All cohomology groups we write (whether motivic or classical) are to be understood with $\mathbb{F}_2$-coefficients.

Theorem 1.1 (Voevodsky). *The bigraded ring $\mathbb{M}_2$ is isomorphic to the polynomial ring $\mathbb{F}_2[\tau]$ on one generator τ of bidegree $(0,1)$.*

We write $\mathcal{A}_{*,*} = [M\mathbb{F}_2, M\mathbb{F}_2]_{*,*}$ for the ring of stable cohomology operations in mod 2 motivic cohomology. It is called *motivic Steenrod algebra*, and it is generated over $\mathbb{M}_2$ by motivic Steenrod operations Sq^i, as the following theorem shows:

Theorem 1.2 (Voevodsky). *The motivic Steenrod algebra $\mathcal{A}$ is the $\mathbb{M}_2$-algebra generated by elements Sq^{2k} of bidegree $(2k,k)$ and Sq^{2k-1} of bidegree $(2k-1,k-1)$ for all $k \geq 1$, satisfying the following relations for $a < 2b$:*

$$Sq^a Sq^b = \sum_c \binom{b-1-c}{a-2c} \tau^{?} Sq^{a+b-c} Sq^c,$$

where the exponent ? is either 0 or 1, easily determined by degrees.

Over $\mathbb{C}$, we can describe the *dual* motivic Steenrod algebra $\mathcal{A}_{*,*}$ (which we notationally will not distinguish from the motivic Steenrod algebra) even more explicitly by

$$\mathbb{M}_2[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 = \tau\xi_{i+1}),\ |\tau_i| = (2^n-1, 2^{n-1}-1),\ |\xi| = (2^{n+1}-2, 2^n-1).$$

If we invert τ, we will see that we obtain a polynomial algebra that is essentially the same as the classical dual Steenrod algebra. The *motivic Adams spectral sequence* is the trigraded spectral sequence with E_2-page

$$E_2^{s,t,v}(\mathbb{S}_F) = \mathrm{Ext}_{\mathcal{A}}^{s,t,v}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \pi_{t-s,v}((\mathbb{S}_F)_{(2)})$$

with differentials of the form $d_r : E_r^{s,t+s,v} \to E_r^{s-1,t+r,v}$, where s is the homological degree of the Ext-group (the Adams filtration), and (t,v) is the internal bigrading coming from the bigrading on $\mathcal{A}$ and $\mathbb{M}_2$, so t is the topological dimension and v is the motivic weight. For $x \in E_\infty(\mathbb{S})$, we write $\{x\}$ for the set of all elements of $\pi_{*,*}$ that are represented by x.

It is constructed analogously to the classical Adams spectral sequence: starting with the motivic sphere spectrum $\mathbb{S}_F$, one can inductively construct a motivic Adams resolution

$$\begin{array}{ccccccc} & K_2 & & K_1 & & K_0 & \\ & \uparrow & & \uparrow & & \uparrow & \\ \cdots \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & = \mathbb{S}_F \end{array}$$

where each K_i is a motivically finite type wedge of suspensions of $M\mathbb{F}_2$, the maps $X_i \to K_i$ are surjective on mod 2 motivic cohomology, and X_{i+1} is the homotopy fiber of $X_i \to K_i$. Applying $\pi_{*,v}$ gives an exact couple for each v, so a $\mathbb{Z}$-graded family of spectral sequences

indexed by v. The E_2-term is $\mathrm{Ext}_{\mathcal{A}}^{s,t,v}(\mathbb{M}_2, \mathbb{M}_2)$ and abuts to the stable motivic homotopy group $\pi_{t-s,u}((\mathbb{S}_F)_{(2)})$ of the 2-completed motivic sphere spectrum.
The relation of the classical Adams spectral sequence and the motivic Adams spectral sequence we first would like to understand is the following one:

Theorem 1.3 ([DI10]). *After inverting τ, the motivic Adams spectral sequence becomes isomorphic to the classical Adams spectral sequence.*

The functor $\mathsf{Sm}_{\mathbb{C}} \to \mathsf{Sp},\ X \mapsto \Sigma^\infty_+ X(\mathbb{C})$ induces a topological realisation functor

$$\mathrm{Re}_{\mathbb{C}} : \mathsf{Sp}_{\mathbb{C}} \to \mathsf{Sp}$$

uniquely determined up to homotopy by the fact that it preserves homotopy colimits and weak equivalences and that it sends the motivic suspension spectrum of a smooth scheme X to the ordinary suspension spectrum of the topological space of complex-valued points $X(\mathbb{C})$. It is called *Betti realisation* and maps $\mathbb{S}^{n,w}_{\mathbb{C}} \mapsto \mathbb{S}^n$.
It also sends the mod 2 motivic Eilenberg-MacLane spectrum $M\mathbb{F}_2$ to the classical mod 2 Eilenberg MacLane spectrum $H\mathbb{F}_2$, so induces a natural transformation

$$H^{p,q}(X) \to H^p(X(\mathbb{C})),\ \alpha \mapsto \alpha(\mathbb{C}),$$

where we view $H^*(X(\mathbb{C}))$ as bigraded concentrated in weight 0.

Definition 1.4. For X a motivic spectrum, let

$$\theta_X : H^{*,*}(X) \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}] \to H^*(X(\mathbb{C})) \otimes_{\mathbb{F}_2} \mathbb{M}_2[\tau^{-1}]$$

be the $\mathbb{M}_2[\tau^{-1}]$-linear map that takes a class α of weight w in $H^{*,*}(X)$ to $\tau^w \alpha(\mathbb{C})$.

Lemma 1.5. *For X the motivic sphere spectrum $\mathbb{S}_{\mathbb{C}}$ or the mod 2 motivic Eilenberg-MacLane spectrum $M\mathbb{F}_2$, the map θ_X is an isomorphism of bigraded $\mathbb{M}_2[\tau^{-1}]$-modules.*

Proof.

- For $X = \mathbb{S}_{\mathbb{C}}$, the map θ_X is given by $\mathbb{M}_2 \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}] \to \mathbb{F}_2 \otimes_{\mathbb{F}_2} \mathbb{M}[\tau^{-1}]$, which clearly is an isomorphism.
- For $X = M\mathbb{F}_2$, $H^{*,*}(X) = H^{*,*}(M\mathbb{F}_2)$ is the motivic Steenrod algebra $\mathcal{A}$ and $H^*(X(\mathbb{C}) = H^*(H\mathbb{F}_2)$ is the classical Steenrod algebra $\mathcal{A}_{\mathrm{cl}}$. The map θ_X maps Sq^{2k} to $\tau^{-k} Sq^{2k}$ and Sq^{2k-1} to $\tau^{-k} Sq^{2k-1}$. Now $\mathcal{A} \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}]$ is free as an $\mathbb{M}_2[\tau^{-1}]$-module on the admissible monomials, and $\mathcal{A}_{\mathrm{cl}} \otimes_{\mathbb{F}_2} \mathbb{M}[\tau^{-1}]$ is free as an $\mathbb{M}_2[\tau^{-1}]$-module on the admissible monomials, so θ_X is an isomorphism. □

Corollary 1.6. *The map $\mathcal{A}[\tau^{-1}] \cong \mathcal{A} \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}] \to \mathcal{A}_{\mathrm{cl}} \otimes_{\mathbb{F}_2} \mathbb{M}_2[\tau^{-1}]$ that takes Sq^{2k} to $\tau^{-k} Sq^{2k}$ and Sq^{2k+1} to $\tau^{-k} Sq^{2k+1}$ is an isomorphism of bigraded rings.*

Considering a motivic Adams resolution from before, we can apply the topological realization functor to obtain a tower of homotopy fiber sequences of classical spectra.

$$\begin{array}{ccccccccc} & & K_2(\mathbb{C}) & & K_1(\mathbb{C}) & & K_0(\mathbb{C}) & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & X_2(\mathbb{C}) & \longrightarrow & X_1(\mathbb{C}) & \longrightarrow & X_0(\mathbb{C}) & = & \mathbb{S} \end{array}$$

It can be shown that this indeed is a classical Adams resolution, which amounts to showing that the maps $X_i(\mathbb{C}) \to K(\mathbb{C})$ are surjective on mod 2 singular cohomology. Topological realization gives maps $\pi_{p,q}(Z) \to \pi_p(Z(\mathbb{C}))$ for any motivic spectrum Z, so we get a map from the homotopy exact couple of (X, K) to that of $(X(\mathbb{C}), K(\mathbb{C}))$. We obtain a map from the motivic spectral sequence for the motivic sphere spectrum to the classical Adams spectral sequence for the classical sphere spectrum. On E_2-pages:

$$\mathrm{Ext}_{\mathcal{A}}^{s,t,v}(\mathbb{M}_2, \mathbb{M}_2) \longrightarrow \mathrm{Ext}_{\mathcal{A}_{\mathrm{cl}}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2).$$

Note that $\mathrm{Ext}_{\mathcal{A}}^{0,*,*}(\mathbb{M}_2, \mathbb{M}_2) = \hom_{\mathcal{A}}^{*,*}(\mathbb{M}_2, \mathbb{M}_2) = \mathbb{F}_2[\tilde{\tau}]$, where $\tilde{\tau}$ is the dual of τ and has degree $(0, 0, -1)$. By abuse of notation, write τ instead of $\tilde{\tau}$ and write $\tilde{\mathbb{M}}_2$ for $\mathbb{F}_2[\tilde{\tau}]$. With what we have discussed so far, one can show the following:

Proposition 1.7. *There is an isomorphism of rings*

$$\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2) \otimes_{\tilde{\mathbb{M}}_2} \tilde{\mathbb{M}}_2[\tau^{-1}] \cong \mathrm{Ext}_{\mathcal{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau, \tau^{-1}].$$

This gives Theorem 1.3, which in turn implies that the motivic differentials and motivic hidden extension[1] must be compatible with their classical analogues, a key computational tool.

2 Cofibre of τ using Adams

The E_2-page $\mathrm{Ext}_{\mathcal{A}}^{s,t,v}(\mathbb{M}_2, \mathbb{M}_2)$ of the motivic Adams spectral sequence contains a non-trivial element in Adams-filtration 0:

$$\mathbb{M}_2 = \mathbb{F}_2[\tau] \xrightarrow{\cdot\tau} \mathbb{F}_2[\tau] = \mathbb{M}_2,$$

multiplication by τ on $\mathbb{F}_2[\tau]$. This is different from the topological Adams spectral sequence for $\mathbb{S}$, where the only elements in Adams filtration 0 are the identity and the zero map. It can be seen that this element survives to the E_∞-page as it cannot be involved with any differentials for degree reasons, so it detects a map

$$\mathbb{S}_{\mathbb{C}}^{0,-1} \xrightarrow{\tau} (\mathbb{S}_{\mathbb{C}}^{0,0})_{(2)}.$$

To avoid complications about the existence of a non-completed version of this map, we will from now on work 2-completed, that is, in 2-completed spectra obtained by localisation at either the Moore spectrum $\mathbb{S}_{\mathbb{C}}/2$ or the EM-spectrum $M\mathbb{F}_2$, but we will stick to the original notation for 2-completed spectra. The motivic Adams spectral sequence produces a nontrivial map

$$\mathbb{S}_{\mathbb{C}}^{0,-1} \xrightarrow{\tau} \mathbb{S}_{\mathbb{C}}^{0,0}.$$

The Betti relisation functor $\mathrm{Re}_{\mathbb{C}} : \mathrm{Sp}_{\mathbb{C}} \to \mathrm{Sp}$ induces a split surjection $\pi_{n,w}(\mathbb{S}_{\mathbb{C}}) \twoheadrightarrow \pi_n(\mathbb{S})$ with section induced by the constant functor. It sends the map $\mathbb{S}_{\mathbb{C}}^{0,-1} \xrightarrow{\tau} \mathbb{S}_{\mathbb{C}}^{0,0}$ to the identity $\mathbb{S} \xrightarrow{\mathrm{id}} \mathbb{S}_{\mathbb{C}}$, and it sends C_τ to a contractibe spectrum. Hence, C_τ is a purely motivic spectrum

[1]Let α be an element of $\pi_{*,*}$ that is detected by an element a of the motivic Adams spectral sequence. A *hidden extension* by α is a pair of elements b, c of E_∞ such that $ab = 0$ in the E_∞-page; there is an element $\beta \in \{b\}$ such that $\alpha\beta \in \{c\}$; and if there exists an element $\beta' \in \{b'\}$ such that $\alpha\beta' \in \{c\}$, then the Adams filtration of b' is less than or equal to the Adams filtration of b.

living in the kernel of Betti realization, and computationally, the Betti realisation functor can be interpreted as sending the element τ to 1. The above map gives a cofibre sequence

$$\mathbb{S}_{\mathbb{C}}^{0,-1} \xrightarrow{\tau} \mathbb{S}_{\mathbb{C}}^{0,0} \longrightarrow C_\tau \longrightarrow \mathbb{S}_{\mathbb{C}}^{1,-1},$$

where we write C_τ for the cofibre of τ. In other words, the diagram

$$\begin{array}{ccc} \mathbb{S}_{\mathbb{C}}^{0,-1} & \longrightarrow & \mathbb{S}_{\mathbb{C}}^{0,0} \\ \downarrow & & \downarrow \\ * & \longrightarrow & C_\tau \end{array}$$

is a pushout square in $(\mathrm{Sp}_{\mathbb{C}})_{(2)}$. We will now be interested in the motivic Adams spectral sequence for C_τ, which takes the form

$$E_2^{s,t,v}(C_\tau) = \mathrm{Ext}_{\mathcal{A}}^{s,t,v}(H^{*,*}(C_\tau), \mathbb{M}_2) \Rightarrow \pi_{t-s,v}(C_\tau),$$

where the stable homotopy groups of C_τ are to be understood 2-completed. The main tool to compute it is the long exact sequence

$$\cdots \longrightarrow E_2(\mathbb{S}_{\mathbb{C}}) \xrightarrow{\tau} E_2(\mathbb{S}_{\mathbb{C}}) \longrightarrow E_2(C_\tau) \longrightarrow \cdots$$

associated to the cofibre sequence above. We get a short exact sequence

$$0 \longrightarrow \mathrm{coker}(\tau) \longrightarrow E_2(C_\tau) \longrightarrow \ker(\tau) \longrightarrow 0,$$

by which the desired $E_2(C_\tau)$ is almost entirely described. An important method to compute Adams differentials for $E_2(C_\tau)$ is to borrow results about the motivic Adams spectral sequence for $\mathbb{S}_{\mathbb{F}}$, furthermore analyses of brackets and hidden extensions are necessary.

Theorem 2.1 ([Isa19]). *The E_∞-page of the motivic Adams spectral sequence for C_τ is known up to the 63-stem.*

3 The motivic Adams-Novikov spectral sequence

The classical Brown-Peterson spectrum BP which is used to define the classical Adams-Novikov spectral sequence

$$E_2^{s,t}(\mathbb{S}; BP) = \mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*) \Rightarrow \pi_{t-s}(\mathbb{S}_{(2)})$$

has a motivic analogue, the motivic Brown-Peterson spectrum BPL. Analogous to going from the classical Adams spectral sequence to the generalized Adams spectral sequence and the Adams-Novikov spectral sequence, one can obtain the motivic Adams-Novikov spectral sequence

$$E_2^{s,t,v}(\mathbb{S}_F; BPL) = \mathrm{Ext}_{BPL_{*,*}BPL}^{s,t}(BPL_{*,*}, BPL_{*,*}) \Rightarrow \pi_{t-s,v}((\mathbb{S}_{\mathbb{F}})_{(2)})$$

from the motivic Adams spectral sequence by replacing the spectra used. Again, classical and motivic spectral sequence are closely related.

Definition 3.1. Define a trigraded object $\overline{E}_2(\mathbb{S}_F, BPL)$ as follows:

$$\overline{E}_2^{s,t,\frac{s+t}{2}}(\mathbb{S}_F; BPL) \coloneqq E_2^{s,t}(\mathbb{S}, BP),$$
$$\overline{E}_2^{s,t,v}(\mathbb{S}_F; BPL) \coloneqq 0 \quad \text{if } v \neq \frac{s+t}{2}.$$

Theorem 3.2 ([HKO11])**.** *The $E_2(\mathbb{S}_F, BPL)$-page of the motivic Adams-Novikov spectral sequence is isomorphic to the trigraded object*

$$\overline{E}_2(\mathbb{S}_F, BPL) \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[\tau],$$

where τ has degree $(0, 0, -1)$. In other words, in order to produce the motivic E_2-page, start with the classical E_2-page. At degree (s, t), replace each copy of $\mathbb{Z}_2$ or $\mathbb{Z}/2^n$ with a copy of $\mathbb{Z}_2[\tau]$ or $\mathbb{Z}/2^n[\tau]$ respectively, where the generator has weight $\frac{s+t}{2}$.

Analogous to the case of the Adams spectral sequence, we have:

Theorem 3.3. *After inverting τ. the motivic Adams-Novikov spectral sequence is isomorphic to the classical Adams-Novikov spectral sequence tensored over $\mathbb{Z}_2$ with $\mathbb{Z}_2[\tau^{\pm 1}]$.*

4 Cofibre of τ using Adams-Novikov

As in the Adams case, we can say something about the motivic Adams-Novikov spectral sequence for the cofibre of τ.

Lemma 4.1. $E_2(C_\tau; BPL) \cong \overline{E}_2^{s,t,v}(\mathbb{S}_F; BPL)$.

Proof. The cofibre sequence

$$\mathbb{S}_{\mathbb{C}}^{0,-1} \xrightarrow{\tau} \mathbb{S}_{\mathbb{C}}^{0,0} \longrightarrow C_\tau \longrightarrow \mathbb{S}_{\mathbb{C}}^{1,-1}$$

induces a long exact sequence

$$\cdots \longrightarrow E_2(\mathbb{S}_F; BPL) \xrightarrow{\tau} E_2(\mathbb{S}_F; BPL) \longrightarrow E_2(C_\tau; BPL) \longrightarrow \cdots.$$

By Theorem 3.2 the map $E_2(\mathbb{S}_F; BPL) \xrightarrow{\tau} E_2(\mathbb{S}_F; BPL)$ is injective. So $E_2(\mathbb{S}_F; BPL)$ is isomorphic to the cokernel of τ, which again by Theorem 3.2 is isomorphic to $\overline{E}_2^{s,t,v}(\mathbb{S}_F; BPL)$. □

Lemma 4.2. *All differentials in the motivic Adams-Novikov spectral sequence for C_τ vanish.*

Proof. By Lemma 4.1, $E_2(C_\tau; BPL)$ in concentrated in degrees (s, t, v) with $s+t-2v = 0$. But the Adams-Novikov differential d_r increases $s+t-2v$ by $r-1$, so all differentials are zero. □

Lemma 4.3. *There are no hidden τ-extensions in $E_\infty(C_\tau; BPL)$.*

Proof. Let x and y be nonzero elements of $E_\infty(C_\tau; BPL)$ of degrees (s, t, v) and (s, t', v') with $t' > t$. Then $v' \geq v$ since $v = (s+t)/2$ and $v' = (s+t')/2$. The lifts $\{x\}$ live in degree $(t-s, v)$, so $\tau\beta$ for $\beta \in \{x\}$ lives in degree $(t-s, v-1)$ as τ has degree $(0, -1)$. Hence, $\tau\beta$ can't be in the degree of $\{y\}$ which is $(t'-s, v')$. □

Theorem 4.4 ([Isa19]). *There is an isomorphism of bigraded abelian groups*

$$\pi_{s,v}(C_\tau) \xrightarrow{\sim} E_2^{s,2v-s}(\mathbb{S}_F; BP) = \mathrm{Ext}^{s,2v-s}_{BP_*BP}(BP_*, BP_*) \qquad \text{for any } s, v \in \mathbb{Z}.$$

Proof. By Lemma 4.1, $E_2(C_\tau; BPL)$ is isomorphic to $E_2(\mathbb{S}, BP)$ when taking appropriate degrees. By Lemma 4.2, $E_\infty(C_\tau, BPL)$ is also isomorphic to $E_2(\mathbb{S}, BP)$. As in the proof of Lemma 4.3, there are no hidden extensions of any kind for degree reasons. Hence, $\pi_{s,v}(C_\tau)$ is also isomorphic to $E_2(\mathbb{S}, BP)$. □

This is very surprising, as it shows that the homotopy groups of a motivic 2-cell complex, which in principle could be as complicated as $\pi_{*,*}(\mathbb{S}_\mathbb{C})$, are completely algebraic.

Corollary 4.5 ([GI17]). *The group $\pi_{s,v}(C_\tau)$ is zero when $v > s$, $v \leq \frac{1}{2}s$ or $s < 0$, except that $\pi_{0,0}(C_\tau) = \mathbb{Z}_{(2)}$, as depicted in Figure 1.*

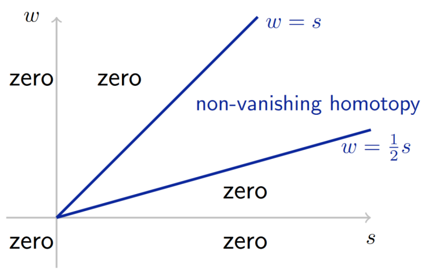

Figure 1: Vanishing regions of the homotopy groups $\pi_{s,w}(C_\tau)$, taken from [Ghe18].

Going the other way, we can also use results about $\pi_{*,*}(C_\tau)$ to compute the classical Adams-Novikov E_2-page. With this technique, the classical Adams-Novikov spectral sequence, including differentials and hidden extensions, could be computed in a larger rang than previously known.

Theorem 4.6 ([Isa19]). *Apart from few uncertainties, the E_∞-page of the classical Adams-Novikov spectral sequence is known through the 59-stem.*

In fact, the isomorphism of Theorem 4.4 can be refined to an isomorphism of rings.

Theorem 4.7 ([Ghe18]). *There exists a unique E_∞-ring structure on C_τ.*

(The module category $\mathrm{Mod}_{C_\tau}(\mathrm{Sp}_\mathbb{C})$ will be considered in more detail in the next talk.)

Theorem 4.8 ([Ghe18]). *The isomorphism*

$$\pi_{*,*}(C_\tau) \cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$$

is an isomorphism of rings which sends Toda brackets in $\pi_{*,*}$ to Massey products in Ext, and vice-versa.

Even more: by [GWX21], the $\mathbb{C}$-motivic Adams spectral sequence $E_*(C_\tau)$ converging to the stable motivic homotopy groups $\pi_{*,*}(C_\tau)$ of the cofibre of τ is (up to reindexing) completely identical to the algebraic Novikov spectral sequence which converges to the E_2-page $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ of the classical Adams-Novikov spectral sequence at $p = 2$ (next talk).

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