Topics in Complex Cobordism

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1 Introduction

Any reduced cohomology theory - that is, a contravariant functor

$$h^* : \mathbf{CW}^{\text{op}} \to \mathbf{AbGp}$$

on the category of CW-pairs to abelian groups satisfying homotopy invariance, exactness, additivity, and with suspension isomorphisms - is representable by an Ω-spectrum as a result of Brown's representability theorem of [Bro62]. This means that there is a sequence $(K_n)_{n\geq 0}$ of CW-complexes, together with weak homotopy equivalences $K_n \to \Omega K_{n+1}$, where ΩK_{n+1} is the loop space of K_{n+1}, such that

$$h^n(X) \cong [X, K_n].$$

Alternatively, we can also take the other point of view: given an Ω-spectrum $E = (E_n)_{n\geq 0}$, we can define the E-cohomology of a pointed CW-complex X by $E^*(X) = [X, E_*]$, which is a reduced cohomology theory. Many interesting cohomology theories are instead defined using CW-spectra, which are pointed CW-complexes $(E_n)_{n\geq 0}$ with basepoint-preserving inclusions $\Sigma E_n \hookrightarrow E_{n+1}$ of the (reduced) suspension $\Sigma E_n := S^1 \wedge E_n$ as a subcomplex of E_{n+1}. Then

$$\widetilde{E}^n(X) := [\Sigma^{-n}X, E] = \varinjlim_{k\to\infty} [\Sigma^k X, E_{n+k}]$$

defines a reduced cohomology theory, which belongs to a generalized cohomology theory $E^* : \mathbf{CW}^{\mathrm{op}} \to \mathbf{AbGp}$. This creates a broad variety of cohomology theories which can be very hard to compute, even for simple spaces. Of particular interest is the cohomology of a point $E^*(pt)$, the *coefficients* of the theory.

This essay will be mainly concerned with *complex cobordism*, a cohomology theory obtained as above by using $E = MU$, the *complex Thom spectrum*. This is constructed as follows: let $\xi \to X$ be a $U(n)$-vector bundle, i.e. a complex vector bundle with Hermitian inner product, over a CW-complex X. Let $\mathbb{D}(\xi)$ and $\mathbb{S}(\xi)$ be the associated unit disc and unit sphere bundles. Then the Thom space $\mathbb{T}(\xi) := \mathbb{D}(\xi)/\mathbb{S}(\xi)$ is a pointed CW-complex. In particular, we can consider the classifying space

$$BU(n) := \mathrm{Gr}_n(H) = \{V \leq H \mid \dim(V) = n\}$$

for $U(n)$, where H an infinite-dimensional complex Hilbert space, together with its tautological bundle

$$\gamma^n \to BU(n).$$

It has the property that any $U(n)$-vector bundle over a paracompact space X is the pullback of γ^n by a map $X \to BU(n)$, unique up to homotopy. Replacing ξ by γ^n gives the Thom space $MU(n) := \mathbb{T}(\gamma^n)$. Now $\gamma^n \oplus \underline{\mathbb{C}} \to BU(n)$ is a rank $(n+1)$-bundle over $BU(n)$, so there is a map $BU(n) \to BU(n+1)$ for which

$$\begin{array}{ccc} \gamma^n \oplus \underline{\mathbb{C}} & \longrightarrow & \gamma^{n+1} \\ \downarrow & \lrcorner & \downarrow \\ BU(n) & \longrightarrow & BU(n+1) \end{array}$$

is a pullback. This induces a map $\mathbb{T}(\gamma^n \oplus \underline{\mathbb{C}}) \to \mathbb{T}(\gamma^{n+1}) = MU(n+1)$, and

$$\mathbb{T}(\gamma^n \oplus \underline{\mathbb{C}}) \cong \mathbb{T}(\gamma^n) \wedge \mathbb{T}(\underline{\mathbb{C}}) \cong \mathbb{T}(\gamma^n) \wedge S^2 = \Sigma^2 MU(n),$$

so we have structure maps $\Sigma^2 MU(n) \to MU(n+1)$. (Note that we can also use double suspensions for the definition of the CW-spectrum and its cohomology theory.) As MU is a *convergent spectrum*, we have

$$\widetilde{MU}^n(X) = [\Sigma^{2k-n}X,\, MU(k)],\ k \text{ large},$$

for the corresponding reduced theory, and working relative to a subspace $A \subseteq X$,

$$\widetilde{MU}^n(X, A) = [\Sigma^{2k-n}(X/A),\, MU(k)],\ k \text{ large}.$$

Products in this theory can also be expressed by maps on MU: the direct sum of complex vector bundles is classified by a multiplication map

$$BU(n) \times BU(m) \to BU(n+m),$$

induced by the block-sum of matrices. By functoriality of Thom-spaces, this descends to a map

$$\mu : MU(n) \wedge MU(m) \to MU(n+m).$$

If two elements $x \in MU^n(X),\ y \in MU^m(X)$ are represented by maps $f : \Sigma^{2k-n}X \to MU(k)$ and $g : \Sigma^{2k-m}X \to MU(k)$, their product will be represented by the composition

$$\Sigma^{4k-n-m}X \to \Sigma^{2k-n}X \wedge \Sigma^{2k-m}X \xrightarrow{f\wedge g} MU(k) \wedge MU(k) \xrightarrow{\mu} MU(2k).$$

Furthermore, using the convention $BU(0) \simeq pt$, we get $MU(0) = S^0$, so the product map on MU induces a map of spectra $\eta : \mathbb{S} \to MU$, where $\mathbb{S} = (S^n)_{n\geq 0}$ is the sphere spectrum. The maps $\mu : MU \wedge MU \to MU$ and $\eta : \mathbb{S} \to MU$ satisfy associativity ($\mu(\mathrm{id} \wedge \mu) \sim \mu(\mu \wedge \mathrm{id})$) and unitality ($\mu(\mathrm{id} \wedge \eta) \sim \mathrm{id} \sim \mu(\eta \wedge \mathrm{id})$) up to homotopy and make MU into a *ring spectrum*.
We start to investigate this cohomology theory in Section 2, and we will see that for a (smooth, closed) manifold X classes in $MU^*(X)$ have a nice geometric interpretation, namely by orientable maps of manifolds $Z \to X$. An important part will be concerned with its coefficient group $MU^*(pt)$, called the *complex cobordism ring*. The structure of this ring as a polynomial algebra on infinitely many generators was first discovered in [Mil60], [Nov60] and [Nov62] by the use of the Adams spectral sequence. In [Qui69] it was first proved to not only be isomorphic to the *Lazard ring*, but also to carry the universal formal group law, which we will elaborate on. Quillen was able to show this making only very little use of abstract homotopy theory.
We will then continue by considering relations of complex cobordism with topological K-theory in Section 3, where we prove a celebrated theorem expressing K-theory in terms of complex cobordism by Conner and Floyd of [CF66]. This comes in the spirit of using the results about $MU^*(pt)$ to make new cohomology theories in the fashion

$$R^*(-) \coloneqq MU^*(-) \otimes_{MU^*(pt)} R$$

for a ring R, and we will see that topological K-theory is in fact just one of them.
Using a CW-spectrum as defined above, we also get a generalized homology theory by setting

$$\widetilde{E}_n(X) \coloneqq \varinjlim_{k\to\infty} \pi_n(E_k \wedge X) = \varinjlim_{k\to\infty} [S^n, MU(k) \wedge X].$$

Applying this to the Thom spectrum MU constructed above, we obtain the theory

$$\widetilde{MU}_n(X) = \pi_n(MU(k) \wedge X),\ k \text{ large},$$

the *complex bordism* of X. Section 4 will be phrased in terms of this homology theory and concerned with the Landweber exact functor theorem of [Lan76], a condition for when the functor $R_*(-) \coloneqq MU_*(-) \otimes_{MU_*(pt)} R$ does indeed define a homology theory. We will explain how this is analogous for cohomology theories and consider examples which opened up new vibrant fields of research.
This essay is based on the courses *Algebraic Topology* and *Characteristic Classes and K-Theory* lectured in the 2022/23 Part III of the Mathematical Tripos. It will freely use their notation, which is standard.

2 Complex cobordism

In this section we will follow [Qui71] to deduce relations of complex cobordism and formal group laws and give a proof for the fact that the complex cobordism ring carries the universal formal group law. We will also see that carrying formal group laws is not unique to complex cobordism, as coefficient rings of all complex oriented cohomology theories do. This will justify to view complex cobordism as in some way universal within these theories.
All manifolds are assumed smooth and embeddable into Euclidean space as a smooth, closed[1] submanifold. We allow different dimensions on different connected components. All maps of manifolds are assumed smooth. We write **Diff** for the category of such manifolds with smooth maps between them. Let $MU^* : \mathbf{Top}^{\mathrm{op}} \to \mathbf{AbGp}$ denote the complex cobordism functor, the generalized cohomology theory with values in the Thom spectrum MU, as defined above.

2.1 Elementary properties of MU^*

We start by giving a geometric interpretation for elements of the cobordism group $MU^*(X)$ under the assumption that X is a manifold. Let $f : Z \to X$ be a map of manifolds. First suppose that $(\dim Z \text{ at } z) - (\dim X \text{ at } f(z))$ is even for all $z \in Z$. If this is constant, we will call it the *dimension* of the map f, written $\dim(f)$. A *complex orientation* of f is an equivalence class of factorizations of the form

$$Z \xhookrightarrow{i} E \xrightarrow{p} X,$$

where $p : E \to X$ is a complex vector bundle and $i : Z \hookrightarrow E$ is an embedding equipped with a complex structure on its normal bundle $\nu_i = TE|_{i(Z)}/TZ$. A complex structure can be given by a bundle map $J : \nu_i \to \nu_i$, where the linear map on fibres $J_x : E_x \to E_x$ satisfies $J_x^2 = -1$. The equivalence relation is given as follows: $(i,p) \sim (i',p')$ if there are inclusions of subbundles $E \hookrightarrow E'',\ E' \hookrightarrow E''$ and i and i' are compatible with the normal complex structure of E'' via an isotopy $i'' : X \times I \hookrightarrow E'' \times I$ over $I = [0,1]$, equipped with a complex structure on its normal bundle that restricts to the corresponding complex structures of E' and E'' on $X \times \{0\}$ and $X \times \{1\}$, respectively. One can show with embedding and isotopy theorems that, making the dimension of E sufficiently large, each complex orientation of f is given by exactly one homotopy class of complex structures on ν_i.
If the difference $(\dim Z \text{ at } z) - (\dim X \text{ at } f(z))$ is odd for all $z \in Z$, a complex orientation is an equivalence class of factorizations as above with E replaced by $E \times \mathbb{R}$. For a general map $f : Z \to X$, we can consider manifolds Z' and Z'' and maps $f' : Z' \to X,\ f'' : Z'' \to X$ with $Z = Z' \sqcup Z''$ so that f' is the even and f'' the odd part of f. Then a complex orientation of f is one for f' and f''.
Two maps of manifolds $f : X \to Z$ and $g : Y \to Z$ are called *transversal* if for all $x \in X,\ y \in Y$ with $f(x) = z = g(y)$ we have $\mathrm{im}(df_x) + \mathrm{im}(dg_y) \cong T_z Z$. If $f : Z \to X$ is complex oriented and $g : Y \to X$ is transversal to f, then the pullback $Y \times_X Z \to Y$ of f along g exists and has an induced complex orientation represented by the pullback of the factorization representing the orientation of f. We call two proper complex oriented maps $f_i : Z_i \to X,\ i = 0,1$, *cobordant* if there is a proper complex oriented map $b : W \to X \times \mathbb{R}$ such that $\varepsilon_i : X \to X \times \mathbb{R},\ \varepsilon_i(x) = (x,i)$ is transversal to b and the pullback of b by ε_i with the induced complex orientation is isomorphic to f_i for $i = 0,1$, as in the commutative diagram below.

[1] By closed we mean compact and without boundary.

$$\begin{array}{ccc} Z_i & \longrightarrow & W \\ {\scriptstyle f_i}\downarrow & \lrcorner & \downarrow{\scriptstyle b} \\ X & \underset{\varepsilon_i}{\longrightarrow} & X \times \mathbb{R} \end{array}$$

The notion of cobordant maps allows us to express the complex cobordism of a manifold X in terms of cobordism classes of maps $Z \to X$. This geometric viewpoint, in the form of the following Proposition, is a generalization of a theorem of Thom from [Tho54] which expresses cobordism groups as homotopy groups.

Proposition 2.1. *For a manifold X,*

$$MU^q(X) \cong \left\{ f : Z \to X \;\middle|\; \begin{array}{c} f \text{ smooth, proper, complex oriented,} \\ \text{of dimension -}q \end{array} \right\} \Big/ \text{cobordism.}$$

Note that if the dimension of X is constant, the assertion $\dim(f) = -q$ precisely means $\dim(Z) = \dim(X) - q$. This description allows us to phrase induced maps on cohomology in terms of cobordism classes: for a map of manifolds $g : Y \to X$ and a proper complex oriented map $f : Z \to X$, we can assume g to be transversal to f by Thom's transversality theorem and homotopy invariance of MU to form the pullback $g^*(f) : Y \times_X Z \to Y$ of f along g. This gives the map

$$MU^*(g) = g^* : MU^q(X) \to MU^q(Y),\ [f] \mapsto [g^*(f)].$$

Furthermore, a proper complex oriented map $g : X \to Y$ of dimension d induces a map

$$g_* : MU^q(X) \to MU^{q-d}(Y),\ [f] \mapsto [g \circ f],$$

as for $[f] \in MU^q(X)$ represented by a proper complex oriented $f : Z \to X$ of dimension $-q$ we have $\dim(Z) = \dim(X) - q = \dim(Y) - (q + \dim(Y) - \dim(X)) = \dim(Y) - (q - d)$, so it follows $\dim(g \circ f) = d - q$ and $[g \circ f] \in MU^{q-d}(Y)$. This induced map is called the *Gysin homomorphism* for complex cobordism.

The structure of $MU^*(X)$ as an abelian group is represented by cobordism classes as follows: if $f_i : Z_i \to X,\ i = 1, 2$, represent two classes, then the sum of these classes is represented by the map $f_1 \sqcup f_2 : Z_1 \sqcup Z_2 \to X$. The inverse of the class of $f : Z \to X$ is the class of f equipped with inverse orientation. If f has even dimension, this is defined as follows: let the orientation of f be represented by a factorization $Z \to \mathbb{C}^n \times X = \underline{\mathbb{C}}^n_X \to X$, then the inverse orientation is represented by the same factorization with the same complex structure on the normal bundle $T_{\underline{\mathbb{C}}^n_X/Z}$, but with a new complex structure on $\mathbb{C}^n$ for which $i \in \mathbb{C}$ acts by $i \cdot (z_1, \ldots, z_n) = (iz_1, \ldots, iz_{n-1}, -iz_n)$.

If x_i is the cobordism class of $f_i : Z_i \to X_i,\ i = 1, 2$, there is an external product $x_1 \boxtimes x_2 \in MU^*(X_1 \times X_2)$ given by the class of the product map $f_1 \times f_2 : Z_1 \times Z_2 \to X_1 \times X_2$. The multiplicative structure of the cobordism ring $MU^*(X)$ is given by $x_1 \cdot x_2 = \Delta^*(x_1 \boxtimes x_2)$, where $x_i \in MU^*(X)$ and $\Delta : X \to X \times X$ is the diagonal.

Example 2.2. The geometric interpretation of complex cobordism (Proposition 2.1) together with the geometric picture of the group operation allows to immediately deduce facts about the complex cobordism of $X = pt$. With $\dim(pt) = 0$ we see that $MU^q(pt) = 0$ for $q > 0$.

To calculate $MU^0(pt)$ we have to consider maps $f : Z \to pt$ with $\dim(Z) = 0$, so Z is discrete.

For the orientation of f only factorizations of the form $Z \to \underline{\mathbb{C}}^n_{pt} \to pt$ are possible, which yield the normal bundle $\nu_i = T\mathbb{C}^n|_{i(Z)} \cong \underline{\mathbb{C}}^n_Z$. This admits the normal and the conjugated complex structure over each point of Z, and furthermore two opposite complex structures over two points of Z cancel out by the cobordism relation. So $MU^0(pt) \cong \mathbb{Z}$, generated by the map $pt \to pt$, where the complex orientation is represented by the factorization through $\underline{\mathbb{C}}^n_{pt}$ with normal complex structure on the normal bundle.

Having discussed the geometric interpretation for complex cobordism, we introduce a first natural transformation for this theory. Let $h : \mathbf{Diff}^{\mathrm{op}} \to \mathbf{Set}$ be a functor. If $g : Y \to X$ is a map of manifolds, write g^* for the map $h(g) : h(X) \to h(Y)$. We assume that for every proper complex oriented map $f : Z \to X$ there is a map $f_* : h(Z) \to h(X)$ such that the following conditions hold:

(1) Suppose that

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{g'} & Z \\ {\scriptstyle f'}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & X \end{array}$$

is a pullback of manifolds with g and f transversal, f proper and complex oriented, f' equipped with the pullback of the complex orientation of f, then $g^* f_* = f'_* g'^*$ as maps $h(Z) \to h(Y)$.

(2) h is homotopy invariant, i.e. if $f_0 \sim f_1 : Y \to X$, then $f_0^* = f_1^*$.

(3) If $f : Z \to X$ and $g : X \to Y$ are proper complex oriented maps and gf is equipped with the composite complex orientation, then $(gf)_* = g_* f_*$.

Note that MU^* satisfies the above properties (1)-(3). The following shows that it is furthermore initial within such functors h, with respect to a fixed element of the evaluation on the one point space $h(pt)$.

Lemma 2.3. *For every $a \in h(pt)$ there exists a unique natural transformation $\theta : MU^* \to h$ commuting with Gysin homomorphisms such that $\theta(1) = a$, where $1 \in MU^0(pt)$ is the cobordism class of the identity map* id_{pt}.

Proof. First assume that θ exists as above. Let $\pi_X : X \to pt$, and let $x \in MU^*(X)$ be a cobordism class represented by a proper complex oriented map $f : Z \to X$ (where x and thus f can have components of different dimensions). Then $x = [f] = [f_*(\mathrm{id}_Z)] = f_*[\mathrm{id}_Z] = f_*\pi_X^*[\mathrm{id}_{pt}] = f_*\pi_X^*(1)$, so applying θ gives $\theta(x) = f_*\pi_Z^*(a) \in h(X)$, showing uniqueness of θ. For the existence we define θ as above, so we have to show the right side does not depend on the choice of the map representing x. For that, let $u : W \to X \times \mathbb{R}$ be proper and complex oriented, transversal to $\varepsilon_i : X \to X \times \mathbb{R}$, $\varepsilon_i(x) = (x, i)$ such that $f_i : Z_i \to X$ is the pullback of u by ε_i with $f = f_0$ and $i = 0, 1$, so that f_0 and f_1 are cobordant.

$$\begin{array}{ccc} & & pt \\ & {\scriptstyle \pi_{Z_i}}\nearrow & \big\uparrow{\scriptstyle \pi_W} \\ Z_i & \xrightarrow{v_i} & W \\ {\scriptstyle f_i}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle u} \\ X & \xrightarrow[\varepsilon_i]{} & X \times \mathbb{R} \end{array}$$

Then $f_{0*}\pi^*_{Z_0}(1) = f_{0*}v_0^*\pi^*_W(1) \overset{(1)}{=} \varepsilon_0^* u_*\pi^*_W(1) \overset{(2)}{=} \varepsilon_1^* u_*\pi^*_W(1) \overset{(1)}{=} f_{1*}v_1^*\pi^*_W(1) = f_{1*}\pi^*_{Z_1}(1)$, so the definition of θ only depends on the cobordism class of x. If $g : Y \to X$, then $g^*(x) = g^*(f)_*\pi^*_{g^*(Z)}(1)$, so $\theta(g^*(x)) = g^*(f)_*\pi^*_{g^*(Z)}(a) \overset{(1)}{=} g^* f_* \pi^*_Z(a) = g^*(\theta(x))$, and hence θ is a natural transformation. If $g : X \to Y$, then $g_*(x) = g_* f_* \pi^*_Z(1) \overset{(3)}{=} (gf)_*\pi^*_Z(1)$, so $\theta(g_*(x)) = (gf)_*\pi^*_Z(a) \overset{(3)}{=} g_* f_* \pi^*_Z(a) = g_*(\theta(x))$, and thus θ commutes with Gysin homomorphisms. □

2.2 Characteristic classes in MU^*

We constructed the complex Thom spectrum MU in such a way, that the n-th piece $MU(n)$ is the Thom space of the universal n-plane bundle over the classifying space $BU(n)$ of the unitary group $U(n)$. In particular, complex cobordism is a cohomology theory with values in a Thom space, so it has a theory of Thom classes for complex vector bundles. We will construct these to consider Euler classes in $MU^*(X)$ and see how we can derive a formal group law over the complex cobordism ring $MU^*(pt)$ out of the behaviour of Euler classes of line bundles under tensor products.

The vector bundles below are assumed to be complex, and we again assume that X is a manifold. Let $\pi : E \to X$ be an n-dimensional vector bundle and let $s : X \to E$ be its zero section, which induces a map to the Thom space $s : X \to \mathbb{T}(E)$. Then $\dim(E) = \dim(X) + 2n$, so for the cobordism class $1 \in MU^0(X)$ of the identity we have $s_*(1) \in MU^{\dim(E)-\dim(X)}(\mathbb{T}(E)) = MU^{2n}(\mathbb{T}(E))$, which is the *Thom class* of E. Furthermore, $s^*s_*(1) \in MU^{2n}(X)$, the *Euler class* of E, denoted $e(E)$.

For the bundle $\pi : E \to X$ there is a projectivisation $\mathbb{P}(\pi) : \mathbb{P}(E) \to X$ constructed as follows: write $E^\# := E \backslash s(X)$ for the complement of the zero section, and define an equivalence relation on $E^\#$ by $v \sim w :\Leftrightarrow \pi(v) = \pi(w)$ and $\exists \lambda \in \mathbb{C}^\times : w = \lambda v$ to obtain $\mathbb{P}(E) := (E^\#)/\sim$. Then $\mathbb{P}(E) \to X$ is given by $[v] \mapsto \pi(v)$. The projectivisation admits the canonical line bundle $L_E := \{(l, v) \in \mathbb{P}(E) \times E \mid v \in l\} \to \mathbb{P}(E),\ (l, v) \mapsto l$.

As for integral cohomology or K-theory, these bundle operations give a projective bundle formula for complex cobordism, for which we refer to 8.1 (p. 50) of [CF66]:

Theorem 2.4. *Let $\mathbb{P}(\pi) : \mathbb{P}(E) \to X$ be the projectivisation map induced by $\pi : E \to X$, let L_E be the canonical line bundle over $\mathbb{P}(E)$, and let $\xi = e(L_E) \in MU^2(\mathbb{P}(E))$. Then*

$$\mathbb{P}(\pi)^* : MU^*(X) \to MU^*(\mathbb{P}(E))$$

makes $MU^*(\mathbb{P}(E))$ into a free $MU^*(X)$-module with basis $1, \xi, \ldots, \xi^{n-1}$, where $n = \dim(E)$.

By $MU^*(X)[\mathbf{t}]$ we denote the graded ring $MU^*(X)[t_1, t_2 \ldots]$ with grading given by $|t_i| = -2i$. As a consequence of Theorem 2.4, $MU^*(\mathbb{P}(E))[\mathbf{t}]$ is a projective algebra which is finitely generated as an $MU^*(X)[\mathbf{t}]$-module. Thus, we can naturally assign an element $c_t(E) \in MU^*(X)[\mathbf{t}]$ to a vector bundle $E \to X$ such that

$$\begin{aligned} c_t(E \oplus E') &= c_t(E) \cdot c_t(E'), \\ c_t(L) &= 1 + \sum_{i>0} t_j e(L)^i, \end{aligned}$$

where $L \to X$ is a line bundle. This is a theorem which holds in more generality for multiplicative cohomology theories on finite CW-pairs, which have a projective bundle formula like

Theorem 2.4. It also holds for integral cohomology, which we will use in Section 2.4. For the proof we again refer to [CF66], Theorem 7.6 (p. 47f) applied to the theory $MU^*(-)[\mathbf{t}]$. Later we will also use Corollary 8.3 (p. 52), which is the analogous result for the theory $MU^*(-)$. So writing $\mathbb{Z}_{\geq 0}^{(\mathbb{N})}$ for the sequences of non-negative integers with finite support, we can write

$$c_t(E) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{(\mathbb{N})}} t^\alpha c_\alpha(E), \tag{2.5}$$

where $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \dots$ and $c_\alpha(E) \in MU^{2|\alpha|}(X)$ with $|\alpha| = \sum_{j \geq 1} j\alpha_j$.

Let $f : Z \to X$ be a complex oriented map of even dimension, and let $Z \xrightarrow{i} E \xrightarrow{p} X$ be a factorization representing the orientation of f as in Section 2.1. Then $f^*(E)$ and ν_i are bundles over Z, and $\nu_f := \nu_i - f^*(E)$ is an element of the K-theory $K^0(Z)$ of Z which depends only on the choice of complex orientation for f. If f has odd dimension, we can represent an orientation by a factorization $Z \xrightarrow{i} E \times \mathbb{R} \to X$ and again form the difference $\nu_f := \nu_i - f^*(E) \in K^0(Z)$. This can be used to define a map

$$s_t : MU^*(X) \to MU^*(X)[\mathbf{t}],\ [f] \mapsto f_*(c_t(\nu_f)),$$

where $[f]$ is the cobordism class of a proper complex oriented map $f : X \to Z$. We can also obtain it using Lemma 2.3, which will additionally show that the map is well-defined. For that, consider the functor

$$h : \mathbf{Diff}^{\mathrm{op}} \to \mathbf{Set},\ \ X \mapsto MU^*(X)[\mathbf{t}],\ \ (f : X \to Y) \mapsto (f^* : MU^*(Y)[\mathbf{t}] \to MU^*(X)[\mathbf{t}]),$$

with new Gysin homomorphisms defined by

$$(f : X \to Y)\ \mapsto\ (f_! : MU^*(X)[\mathbf{t}] \to MU^*(Y)[\mathbf{t}],\ x \mapsto f_*(c_t(\nu_f) \cdot x)).$$

Note that h satisfies the properties (1)-(3) of Section 2.1. So Lemma 2.3 gives a natural transformation $s_t : MU^* \to h$ which commutes with Gysin homomorphisms. That is, the components $s_t : MU^*(-) \to MU^*(-)[\mathbf{t}]$ satisfy $s_t(f_*(x)) = f_!(s_t(x)) = f_*(c_t(\nu_f) \cdot s_t(x))$ for any proper complex oriented map $f : X \to Y$. The map s_t is called *Landweber-Novikov operation*, and we will use it in the considerations of the following sections.

Having defined these operations we now come to the main part of this section: the relation of complex cobordism (and later, more generally, complex oriented cohomology theories) to formal group laws.

Definition 2.6. Let R be a commutative ring and let $F \in R[\![x, y]\!]$ be a power series in two variables with coefficients in R. Then F is a *(commutative, one-dimensional) formal group law*, abbreviated by *fgl*, if it satisfies

$$\begin{aligned} F(0,x) = F(x,0) &= x && \text{(neutral element)},\\ F(x, F(y,z)) &= F(F(x,y), z) && \text{(associativity)},\\ F(x,y) &= F(y,x). && \text{(symmetry)}. \end{aligned}$$

The following central proposition about Euler classes of line bundles shows that the behaviour of Euler classes under tensor products can be used to derive a formal group law, which has coefficients in the complex cobordism of a point.

Proposition 2.7. *There is a unique power series $F(x,y) = \sum_{i,j\geq 0} c_{ij} x^i y^j \in MU^*(pt)[\![x,y]\!]$ with $c_{ij} \in MU^{2-2i-2j}(pt)$ such that*

$$e(L_1 \otimes L_2) = F(e(L_1), e(L_2))$$

for any two line bundles $L_1, L_2 \to X$, where X is a manifold. Moreover, F is a formal group law.

Proof. Using the tautological bundle $\gamma_{\mathbb{C}}^{1,n+1} \to \mathbb{CP}^n$, the Gysin sequence for complex cobordism, and induction on n we have $MU^*(\mathbb{CP}^n) \cong MU^*(pt)[z]/(z^{n+1})$ as for ordinary cohomology, where $z = e(\gamma_{\mathbb{C}}^{1,n+1})$ is the Euler class of the line bundle $\gamma_{\mathbb{C}}^{1,n+1}$. Using Theorem 2.4 and tensoring over $MU^*(pt)$, we obtain

$$MU^*(\mathbb{CP}^n \times \mathbb{CP}^n) \cong MU^*(pt)[z_1, z_2]/(z_1^{n+1}, z_2^{n+1}),$$

where $z_i = e(\pi_i^*(\gamma_{\mathbb{C}}^{1,n+1}))$. So we can express $e(\pi_1^*(\gamma_{\mathbb{C}}^{1,n+1}) \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,n+1})) \in MU^*(\mathbb{CP}^n \times \mathbb{CP}^n)$ as $\sum_{0\leq i,j\leq n} c_{ij}^n z_1^i z_2^j$ with $c_{ij}^n \in MU^{2-2i-2j}(pt)$.
Now if $m > n$, the Euler class $e(\pi_1^*(\gamma_{\mathbb{C}}^{1,m+1}) \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,m+1}))$ of $\pi_1^*(\gamma_{\mathbb{C}}^{1,m+1}) \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,m+1}) \to \mathbb{CP}^m \times \mathbb{CP}^m$ is mapped to $e(\pi_1^*(\gamma_{\mathbb{C}}^{1,n+1}) \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,n+1}))$ by the quotient map

$$MU^*(pt)[z_1, z_2]/(z_1^{m+1}, z_2^{m+1}) \to MU^*(pt)[z_1, z_2]/(z_1^{n+1}, z_2^{n+1}),$$

so it follows $c_{ij}^n = c_{ij}^m$ for $i, j \leq n$. This shows that with letting $n \to \infty$ we have a well-defined power series $F(x,y) = \sum_{i,j\geq 0} c_{ij} x^i y^j$ with coefficients in $MU^*(pt)$, where

$$MU^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong MU^*(pt)[z_1, z_2], \; z_i = e(\pi_i^*(\gamma_{\mathbb{C}}^{1,\infty})).$$

Then the claim $e(L_1 \otimes L_2) = F(e(L_1), e(L_2))$ holds for the pullbacks of the tautological bundles $L_i = \pi_i^*(\gamma_{\mathbb{C}}^{1,n+1}) \to \mathbb{CP}^n \times \mathbb{CP}^n$ for all n by construction of F. Now any line bundle $L \to X$ is the pullback of the tautological bundle on $\mathbb{CP}^n$ along some map $f : X \to \mathbb{CP}^n$ for some n, and for two line bundles $L_1, L_2 \to X$ there are maps $f_i : X \to \mathbb{CP}^n$ for some $n \gg 0$ which pull back $\gamma_{\mathbb{C}}^{1,n+1}$ to L_i. Then $L_1 \otimes L_2 = (f_1 \times f_2)^*(\pi_1^*(\gamma_{\mathbb{C}}^{1,n+1}) \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,n+1}))$ holds, which gives

$$\begin{aligned} e(L_1 \otimes L_2) &= (f_1 \times f_2)^* e(\pi_1^*(\gamma_{\mathbb{C}}^{1,n+1}) \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,n+1})) \\ &= (f_1 \times f_2)^* F(e(\pi_1^*(\gamma_{\mathbb{C}}^{1,n+1})), e(\pi_2^*(\gamma_{\mathbb{C}}^{1,n+1}))) \\ &= F(e(L_1), e(L_2)). \end{aligned}$$

So the claimed equation above also holds in the general case. We finally verify that $F(x,y)$ is a formal group law. The neutral element equation follows from

$$\begin{aligned} F(0, e(\pi_2^*(\gamma_{\mathbb{C}}^{1,n+1}))) &= F(e(\pi_1^*(\underline{\mathbb{C}}_{\mathbb{CP}^n}^1)), e(\pi_2^*(\gamma_{\mathbb{C}}^{1,n+1}))) = e(\pi_1^*(\underline{\mathbb{C}}_{\mathbb{CP}^n}^1) \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,n+1})) \\ &= e(\underline{\mathbb{C}}_{\mathbb{CP}^n \times \mathbb{CP}^n}^1 \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,n+1})) = e(\pi_2^*(\gamma_{\mathbb{C}}^{1,n+1})), \quad \text{for all } n. \end{aligned}$$

For associativity, we compute

$$\begin{aligned} F(e(\gamma_{\mathbb{C}}^{1,n+1}), F(e(\gamma_{\mathbb{C}}^{1,n+1}), e(\gamma_{\mathbb{C}}^{1,n+1}))) &= e(\pi_1^*(\gamma_{\mathbb{C}}^{1,n+1}) \otimes (\pi_2^*(\gamma_{\mathbb{C}}^{1,n+1}) \otimes \pi_3^*(\gamma_{\mathbb{C}}^{1,n+1}))) \\ = e((\pi_1^*(\gamma_{\mathbb{C}}^{1,n+1}) \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,n+1})) \otimes \pi_3^*(\gamma_{\mathbb{C}}^{1,n+1})) &= F(F(e(\gamma_{\mathbb{C}}^{1,n+1}), e(\gamma_{\mathbb{C}}^{1,n+1})), e(\gamma_{\mathbb{C}}^{1,n+1})), \end{aligned}$$

for all n. Furthermore, $e(\pi_1^*(\gamma_{\mathbb{C}}^{1,n+1}) \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,n+1})) = e(\pi_2^*(\gamma_{\mathbb{C}}^{1,n+1}) \otimes \pi_1^*(\gamma_{\mathbb{C}}^{1,n+1}))$ holds for all n, which implies $c_{ij} = c_{ji}$, giving commutativity. □

Remark 2.8. In more generality, we could consider not only complex cobordism, but complex oriented cohomology theories. These are multiplicative[2] cohomology theories

$$h^* : \mathbf{CW}_\mathrm{f}^{\mathrm{op}} \to \mathbf{Rng}_\bullet$$

with $\mathbf{CW}_\mathrm{f}$ the category of finite CW-pairs, which have a theory of Euler classes: for every complex line bundle $L \to X$ over a finite CW-complex X there is an Euler class $e^h(L) \in h^2(X)$ such that

(i) e^h is natural for bundle maps, and

(ii) $h^*(\mathbb{CP}^n) \cong h^*(pt)[z]/z^{n+1}$, where $z = e^h(\gamma_{\mathbb{C}}^{1,n+1})$, for all $n \geq 0$.

Such a complex orientation[3] is equivalent to having an isomorphism $h^*(\mathbb{CP}^\infty) \cong h^*(pt)[z]$, and then z will correspond to the Euler class $e^h(\gamma_{\mathbb{C}}^{1,\infty})$. If h^* is complex oriented, then Proposition 2.7 holds for h, i.e. there is a formal group law $F_h(x,y)$ over $h^*(pt)$ satisfying

$$e^h(L_1 \otimes L_2) = F_h(e^h(L_1), e^h(L_2)).$$

This is proved exactly as Proposition 2.7, essentially relying on the fact that $\mathbb{CP}^\infty$ is classifying for line bundles and the properties $L \otimes \underline{\mathbb{C}}^1 \cong L,\ L_1 \otimes L_2 \cong L_2 \otimes L_1,\ (L_1 \otimes L_2) \otimes L_3 \cong L_1 \otimes (L_2 \otimes L_3)$ of the tensor product of line bundles. This formal group law F_h is referred to as the group law of the theory h.

Example 2.9. (i) For integral cohomology $h^* = H^*(-;\mathbb{Z})$ we have $H^*(pt;\mathbb{Z}) = \mathbb{Z}$, and the formula $e^H(L_1 \otimes L_2) = e^H(L_1) + e^H(L_2)$ holds for line bundles $L_1, L_2 \to X$. So the associated formal group law is the additive group law $F_H(x,y) = x + y$ over $\mathbb{Z}$.

(ii) For complex K-theory $h^* = K^*(-)$ we have $K^*(pt) = \mathbb{Z}[\beta, \beta^{-1}],\ |\beta| = -2$. In this case the Euler class behaves differently: for a line bundle $L \to X$ we have $e^K(L) = \Lambda_{-1}(\overline{L}) = \Lambda^0(\overline{L}) - \Lambda^1(\overline{L}) = 1 - \overline{L}$, which gives

$$\begin{aligned} e^K(L_1 \otimes L_2) &= 1 - \overline{L_1 \otimes L_2} = 1 - \overline{L_1} \otimes \overline{L_2} \\ &= 1 - \overline{L_1} + 1 - \overline{L_2} - (1 - \overline{L_1} - \overline{L_2} + \overline{L_1} \otimes \overline{L_2}) \\ &= e^K(L_1) + e^K(L_2) - e^K(L_1)e^K(L_2) \end{aligned}$$

for two line bundles $L_1, L_2 \to X$. The associated group law is $F_K(x,y) = x + y - xy$ in the $\mathbb{Z}/2$-graded case, $x + y - \beta xy$ in the $\mathbb{Z}$-graded case (note that sign conventions vary for the definition of the K-theory Euler class and the associated group law, so that the sign of the xy-term might be switched).

[2]That is, each graded cohomology group is compatibly equipped with the structure of a graded ring.

[3]In fact, there are many equivalent ways to express the notion of complex orientability for multiplicative cohomology theories. Another is the following: The theory h is *complex orientable* if the restriction map $h^2(\mathbb{CP}^\infty) \to h^2(\mathbb{CP}^1)$ is surjective, and in this case a *complex orientation* is the choice of an element of $h^2(\mathbb{CP}^\infty)$ which restricts to the canonical generator of the reduced theory $\widetilde{h}^1(\mathbb{CP}^1)$.

2.3 A structure theorem for $MU^*(X)$

This section is concerned with the main theorem of [Qui71], Theorem 5.1, p. 47, which explains the structure of the graded ring $MU^*(X)$ for spaces with homotopy type of a finite CW-complex. It will be the basis for further results on the cobordism ring $MU^*(pt)$ in Section 2.4. In that respect this section will be the central one, as it considers the technical details underpinning the remarkable results about $MU^*(pt)$ to come.
Choosing a basepoint for the space X, the elements of $MU^*(X)$ which vanish under the restriction map $MU^*(X) \to MU^*(pt)$ induced by the inclusion $pt \hookrightarrow X$ of the basepoint form an ideal denoted $\widetilde{MU}^*(X) \trianglelefteq MU^*(X)$, the reduced complex cobordism of (X, pt). Let $C = \langle c_{ij} \mid i, j \geq 0 \rangle \leq MU^{\text{ev}}(pt)$ be the subring generated by the coefficients of the formal group law of Proposition 2.7, which is contained in even degrees.

Theorem 2.10. *If X has homotopy type of a finite CW complex, then*

$$MU^*(X) = C \cdot \bigoplus_{q \geq 0} MU^q(X),$$
$$\widetilde{MU}^*(X) = C \cdot \bigoplus_{q > 0} MU^q(X).$$

We can derive an immediate consequence of this theorem: in Example 2.2 we saw that $MU^q(pt) = 0$ for $q > 0$ and $MU^0(pt) \cong \mathbb{Z}$ using the geometric interpretation of complex cobordism, so $MU^*(pt) = C \cdot \bigoplus_{q \geq 0} MU^q(pt) = C \cdot \mathbb{Z}$. As C lies in even degrees this gives:

Corollary 2.11. *$MU^{\text{even}}(pt) \cong C$ and $MU^{\text{odd}}(pt) = 0$.*

We will prove Theorem 2.10 at the end of this section and first consider some technical results the proof requires. Let G be a compact Lie group, and let

$$h : \mathbf{GTop}^{\text{op}} \to \mathbf{Rng}_{\bullet}$$

be a multiplicative equivariant[4] cohomology theory for G-spaces which has Thom classes and so Euler classes for complex G-vector bundles. That is, if X is a G-space and $\pi : E \to X$ is a vector bundle with an action of G on E by bundle maps which makes π equivariant (and when we say *G-vector bundle* below, we will always mean such a bundle), then there is an Euler class $e^h(E) \in h^*(X)$ in the theory h for E.

Example 2.12. Let $Q \to B$ be a principal G-bundle over a manifold B, i.e. a locally trivial fibre bundle with a G-action on the total space such that G acts freely and transitively on each fibre. We write $Q \times_G X$ for the quotient space $(Q \times X)/G$, with the canonical G-action on the product given by $g(q, x) = (g^{-1}q, gx)$. An important example which will be used later for such an equivariant theory as above is the functor

$$MU^*(Q \times_G -) : \mathbf{GTop}^{\text{op}} \to \mathbf{Rng}_{\bullet}, \; X \mapsto MU^*(Q \times_G X),$$
$$(f : Y \to X) \mapsto \; ((1_Q \times_G f)^* : MU^*(Q \times_G X) \to MU^*(Q \times_G Y)),$$

where Gysin homomorphisms are defined similarly by

$$(1 \times_G f)_* : MU^*(Q \times_G X) \to MU^*(Q \times_G Y)$$

for proper G-maps $f : X \to Y$ with equivariant complex orientation (as defined in Section 2.1). If $E \to X$ is a G-vector bundle, then its Euler class in the theory $h = MU^*(Q \times_G -)$ is given by the complex cobordism Euler class: $e^h(E) := e(Q \times_G E) \in MU^*(Q \times_G X)$.

[4]That is, the functor is defined for G-equivariant maps between G-spaces.

Lemma 2.13. *Let $f : Z \to X$ be a proper complex oriented G-map of even dimension, represented by a factorization $Z \xrightarrow{i} E \xrightarrow{p} X$, so that we have a commutative diagram*

$$\begin{array}{ccc} Z^G & \xrightarrow{r_Z} & Z \\ {\scriptstyle r_X}\downarrow & & \downarrow{\scriptstyle i} \\ X^G & \xrightarrow[i^G]{} & X \end{array}$$

with horizontal maps inclusions of the fixpoint submanifolds. Using the eigenbundle decomposition for G-equivariant vector bundles of $r_X^*(E)$ and $r_Z^*(\nu_i)$, let $\mu(E)$ and $\mu(i)$ be the sums of the eigenbundles corresponding to the nontrivial irreducible representations of G. If $z \in h(Z)$, then

$$e^h(\mu(E)) \cdot r_X^* f_*(z) = f_*^G(e^h(\mu(i)) \cdot r_Z^*(z)).$$

Proof sketch. First consider the situation

$$\begin{array}{ccc} W & \xrightarrow{j'} & Z \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow[j]{} & X \end{array}$$

where Y and Z are submanifolds of X and for $W := Y \cap Z$ we have $TW_x = TY_x \cap TZ_x$ for all $x \in W$. If $F := TX/(TY + TZ)|_W$ is the excess bundle of the intersection $Y \cap Z$, then replacing X by a tubular neighbourhood and a computation show that $j^* i_*(z) = i'_*(e(F) \cdot (j')^*(z))$ for $z \in MU^*(Z)$. The same argument applies in the equivariant case. In particular if $h = MU^*(Q \times_G -)$, then applying $Q \times_G -$ to the square in the statement and using the derived formula gives the desired formula in this specific equivariant case. For the details, we refer to [Qui71, Section 3, p. 37ff]. □

Using such equivariant theories as introduced above, we can complement the Landweber-Novikov operation with the definition of a second cohomology operation, the Steenrod operation: if G acts on the set $\{1, 2, \ldots, k\}$, we define the *external Steenrod operation*

$$P_{\text{ext}} : MU^{-2q}(X) \to h^{-2qk}(X^k),\ f_*(1) \mapsto (f^k)_*(1),$$

where $f : Z \to X$ is a proper complex oriented map of even dimension $2q$ and $f^k : Z^k \to X^k$ is the k-fold product of f, which is a G-map by permutation of the factors. Also, f^k has a natural equivariant complex orientation given by the factorization $Z^k \xrightarrow{i^k} E^k \xrightarrow{p^k} X^k$, since the dimension of f is even. By composing with the map induced by the diagonal $\Delta : X \to X^k$, we obtain

$$P_{\text{int}} : MU^{-2q}(X) \xrightarrow{P_{\text{ext}}} h^{-2qk}(X^k) \xrightarrow{\Delta^*} h^{-2qk}(X),\ f_*(1) \mapsto \Delta^*(f^k)_*(1),$$

the *(internal) Steenrod k-th power operation.* We will see the relation of the Steenrod operation and the Landweber-Novikov operation below.

Under assumptions on G we will furthermore define two representations of G which then induce vector bundles we can consider the Euler classes of: if G acts transitively on $\{1, \dots, k\}$, let $W := \{(z_1, \dots, z_k) \in \mathbb{C}^k \mid \sum_{i=1}^k z_i = 0\} \leq \mathbb{C}^k$, a subspace of dimension $k-1$. Then

$$\rho : G \to \mathrm{GL}(W),\ g \mapsto ((z_1, \dots, z_k) \mapsto (z_{g.1}, \dots, z_{g.k}))$$

defines a representation of G on W by permuting the coordinates.
For a G-space X, a representation $G \to \mathrm{GL}(V)$ gives a G-vector bundle $X \times V \xrightarrow{\mathrm{pr}_1} X$ with fibre V over X, which has a diagonal G-action given by $g(x, v) = (x \cdot g, v \cdot g)$. So the representation ρ from above gives a vector bundle also written ρ over X, and we write $e^h(\rho) \in h^{2(k-1)}(X)$ for its Euler class in the theory h.

Lemma 2.14. *In the above situation, suppose $f : Z \to X$ is a proper complex oriented map of dimension $2q$ and $m \in \mathbb{Z}$ with $m > \dim(Z)$, so that $\underline{\mathbb{C}}_Z^m + \nu_f \in K^0(Z)$ is represented by a vector bundle (where ν_f was defined in Section 2.2). Then*

$$e^h(\rho)^m P_{\mathrm{int}}(f_*(1)) = f_* e^h(\rho \otimes (\underline{\mathbb{C}}_Z^m + \nu_f)) \in h^{2m(k-1)-2qk}(X).$$

Proof. As we assumed $m > \dim(Z)$, the complex orientation of f can be represented by a factorization $Z \xrightarrow{i} E = \underline{\mathbb{C}}_X^m \xrightarrow{p} X$ through a trivial bundle of dimension m, together with a complex structure on the normal bundle ν_i. Then with $\nu_f = \nu_i - f^*(E) = \nu_i + \underline{\mathbb{C}}_Z^m$ it follows $\nu_i = \underline{\mathbb{C}}_Z^m + \nu_f$.
Now we apply Lemma 2.13 to the map $f^k = p^k \circ i^k : Z^k \to X^k$ with $z = 1$. This gives

$$e^h(\mu(E^k)) \cdot r_X^* f_*^k(1) = (f^k)_*^G e^h(\mu(i^k)) \cdot r_Z^*(1).$$

Then $\mu(E^k) = \mu((\underline{\mathbb{C}}_X^m)^k) = \rho \otimes \underline{\mathbb{C}}_X^m$, $\mu(i) = \rho \otimes \nu_i = \rho \otimes (\underline{\mathbb{C}}_Z^m - \nu_f)$, and $(X^k)^G = X$, so the fixpoint-submanifold $r_{X^k} : (X^k)^G \to X^k$ for the G-action is $\Delta : X \to X^k$. So the above equation simplifies to

$$e^h(\rho)^m \cdot \Delta^*(f^k)_*(1) = f_* e^h(\rho \otimes (\underline{\mathbb{C}}_Z^m + \nu_f)),$$

and the claim follows by definition of P_{int}. □

Now take $G = \mathbb{Z}/k$, the cyclic group of order k, acting cyclically on $\{1, \dots, k\}$ by setting $1_{\mathbb{Z}/k}.n$ as $n+1$ for $n < k$ and as 1 for $n = k$. As a smooth 0-manifold with discrete topology, we can view $\mathbb{Z}/k$ as a compact Lie group. The case $G = \mathbb{Z}/p$ will be useful for the proof of Theorem 2.10 after performing a localization at $(p) \trianglelefteq \mathbb{Z}$, p a prime number. In this specific case a second representation of G on $\mathbb{C}$ is given by

$$\eta : G = \mathbb{Z}/k \to \mathrm{GL}(\mathbb{C}),\ 1_{\mathbb{Z}/k} \mapsto (x \mapsto x \cdot \exp(2\pi i/k)).$$

For a representation $G \to \mathrm{GL}(V)$, we can form the G-vector bundle on a point $V \to pt$, which has an h-Euler class $e^h(\rho) \in h^*(pt)$. Note that the G-vector bundle $X \times V \to X$ over a G-space X we introduced above is pulled back from the bundle $V \to pt$, and so its Euler class which was also written $e^h(\rho) \in h^*(X)$ is pulled back from the Euler class $e^h(\rho) \in h^*(pt)$ by naturality of Euler classes.
So for the two representations $\rho : G \to \mathrm{GL}(W)$ and $\eta : G \to \mathrm{GL}(\mathbb{C})$ we defined we get vector bundles over a point written $\rho : W \to pt$ and $\eta : \mathbb{C} \to pt$. Using the theory $h = MU^*(Q \times_G -)$

from Example 2.12, we obtain Euler classes $w = e^h(\rho) \in h^{2(k-1)}(pt) = MU^{2(k-1)}(Q \times_G pt) = MU^{2(k-1)}(B)$ and $v = e(\eta) \in h^2(pt) = MU^2(B)$. We now come to the algebraic relation between the Steenrod k-th power operation and the Landweber-Novikov operation of Section 2.2, using the Euler classes we just defined. This relation will be central for the proof of Theorem 2.10.

Proposition 2.15. *Let $Q \to B$ be a principal $\mathbb{Z}/k$-bundle over a manifold B and let*

$$P : MU^{-2q}(X) \xrightarrow{P_{\text{int}}} MU^{-2qk}(Q \times_{\mathbb{Z}/k} X) \to MU^{-2qk}(B \times X)$$

be the Steenrod k-th power operation for the functor $MU^*(Q \times_{\mathbb{Z}/k} -)$. For the Landweber-Novikov operation

$$s_t : MU^*(X) \to MU^*(X)[\boldsymbol{t}]$$

defined in Section 2.2 and a sequence $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_{\geq 0}^{(\mathbb{N})}$ of non-negative integers with finite support we write $s_\alpha(x)$ for the coefficient of t^α in $s_t(x)$ like in (2.5). Then the Steenrod operation P is related to the Landweber-Novikov operation s_t by the formula

$$w^{n+q} P(x) = \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^{(\mathbb{N})} \\ l(\alpha) \leq n}} w^{n-l(\alpha)} a(v)^\alpha s_\alpha(x) \quad \text{with } l(\alpha) = \sum_{j \geq 1} \alpha_j, \tag{2.16}$$

which holds for every $x \in MU^{-2q}(X)$. Here $n \in \mathbb{Z}$ is sufficiently large with respect to $\dim(X)$ and the integer q. We write $a(v)^\alpha = a_1(v)^{\alpha_1} a_2(v)^{\alpha_2} \ldots$ for power series $a_j(z) \in C[\![z]\!]$ with coefficients in the subring $C \leq MU^{\text{ev}}(pt)$, which is generated by the coefficients of the formal group law F of Proposition 2.7. Furthermore, we have the formula

$$w = e(\rho) = (k-1)!\, v^{k-1} + \sum_{j \geq k} b_j v^j, \tag{2.17}$$

relating the Euler classes v and w, where $b_j \in C$.

Proof. Let $F(x, y) \in C[\![x, y]\!]$ be the formal group law of Proposition 2.7. For $i \in \mathbb{Z}$, we define $[i]_F(z) \in C[\![z]\!]$ inductively by $[1]_F(z) := z$ and $[i]_F(z) := F(z, [i-1]_F(z))$, so that $[i]_F(z) = iz + \text{terms of higher order}$.

Consider the situation of Lemma 2.14 with $h = MU^*(Q \times_G -)$. Let L be a line bundle over Z on which $G = \mathbb{Z}/k$ acts trivially, and consider the bundle $\rho : Z \times V \to Z$ for the representation ρ we introduced above, where Z is a trivial G-space. We can form their tensor product

$$\rho \otimes L = (Z \times V) \otimes L \to Z$$

which we consider as a G-bundle, where G acts trivially on Z. For its h-Euler class we have $e^h(\rho \otimes L) \in MU^*(Q \times_G Z)$, and with $Q \times_G Z = (Q \times Z)/G = Q/G \times Z = B \times Z$ it follows $e^h(L \otimes L) \in MU^*(B \times Z)$. We compute

$$\begin{aligned} e(\rho \otimes L) &= \prod_{i=1}^{k-1} e(\eta^i \otimes L) \overset{2.7}{=} \prod_{i=1}^{k-1} F(e(\eta^i), e(L)) \\ &= \prod_{i=1}^{k-1} F([i]_F(v), e(L)) = w + \sum_{j \geq 1} a_j(v) e(L)^j, \end{aligned}$$

where $a_j(z) \in C[\![z]\!]$ and $w = e(\rho) = (k-1)!\, v^{k-1} + \sum_{j \geq k} b_j(v) v^j$ with $b_j \in C$. For a sum of line bundles $E = L_1 \oplus \ldots \oplus L_r$ using the notation (2.5) we have

$$\begin{aligned} e(\rho \otimes E) &= \prod_{i=1}^{r} e(\rho \otimes L_i) = \prod_{i=1}^{r} w + \sum_{j \geq 1} a_j(v) e(L_i)^j \\ &= \sum_{l(\alpha) \leq r} w^{r-l(\alpha)} a(v)^{\alpha} c_{\alpha}(E), \end{aligned}$$

where $l(\alpha) = \sum_j \alpha_j$. By the splitting principle this holds for all r-dimensional vector bundles. Then by definition of the Landweber-Novikov operation $s_\alpha(x) = s_\alpha(f_*(1)) = f_*(c_\alpha(\nu_f))$ for $\alpha \in \mathbb{Z}_{\geq 0}^{(\mathbb{N})}$, and we get

$$f_* e(\rho \otimes \nu_f) = \sum_{l(\alpha) \leq n} w^{n-l(\alpha)} a(v)^{\alpha} f_* c_\alpha(\nu_f) = \sum_{l(\alpha) \leq n} w^{n-l(\alpha)} a(v)^{\alpha} s_\alpha(x).$$

Then substituting the above for the right side of Lemma 2.14 gives the desired equation. □

We need another technical lemma to prove Theorem 2.10. For this, fix a positive integer $k \in \mathbb{Z}_{>0}$, a manifold Y, and let

$$\Theta(z) := [k]_F(z)/z = k + d_1 z + d_2 z^2 + \ldots \in C[\![z]\!].$$

The group $\mathbb{Z}/k$ acts on $S^{2n-1} \subseteq \mathbb{C}^n$ by $1_{\mathbb{Z}/k} \mapsto (x \mapsto x \cdot \exp(2\pi i/k))$. This gives a principal $\mathbb{Z}/k$-bundle $S^{2n-1} \to S^{2n-1}/(\mathbb{Z}/k)$, so with $\eta : \mathbb{Z}/k \to \mathrm{GL}(\mathbb{C})$ as above we have an induced line bundle $S^{2n-1} \times_{\mathbb{Z}/k} \mathbb{C} \to S^{2n-1}/(\mathbb{Z}/k)$. Then forming the product with Y gives a line bundle over $S^{2n-1}/(\mathbb{Z}/k) \times Y$ and thus an Euler class $v_n \in MU^2(S^{2n-1}/(\mathbb{Z}/k) \times Y)$. Furthermore, denote by $j_n : S^{2n-1}/(\mathbb{Z}/k) \to S^{2n+1}/(\mathbb{Z}/k)$ the map induced by the inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$. Then we have the following:

Lemma 2.18. *Let $x \in MU^q(S^{2n-1}/(\mathbb{Z}/k) \times Y)$ such that $x \cdot v_{n+1} = 0$. Then there is an element $y \in MU^q(pt \times Y)$ such that $y \cdot \Theta(v_n) = j_n^*(x) \in MU^q(S^{2n-1}/(\mathbb{Z}/k) \times Y)$.*

Proof. The proof is a careful consideration of the Gysin sequence for the theory $h^*(-) = MU^*(- \times Y)$ for a fixed manifold Y. We will refer to section 4 of [Qui71] for one detail in the proof.
If $E \to X$ is a complex vector bundle of dimension n with sphere bundle $\pi : \mathbb{S}(E) \to X$, then the Gysin sequence for the above theory reads

$$h^{q-2n}(X) \xrightarrow{\cdot e^h(E)} h^q(X) \xrightarrow{\pi^*} h^q(\mathbb{S}(E)) \xrightarrow{\pi_*} h^{q-2n-1}(X).$$

For the representation $\eta^n : \mathbb{Z}/k \to \mathrm{GL}(\mathbb{C}^n),\ 1_{\mathbb{Z}/k} \mapsto (x \mapsto x \cdot \exp(2\pi i/k))$ and the $\mathbb{Z}/k$-bundle $S^1 \to S^1/(\mathbb{Z}/k)$ we can form the induced bundle $\mathbb{C}^n \times_{\mathbb{Z}/k} S^1 \to S^1/(\mathbb{Z}/k)$. Its sphere bundle is given by the map $p_n : S^{2n-1} \times_{\mathbb{Z}/k} S^1 \to S^1/(\mathbb{Z}/k)$, induced by the projection onto the second factor. This gives a diagram of Gysin sequences:

$$\begin{array}{ccccccc} \xrightarrow{\cdot v_1^{n+1}} h^q(S^1/(\mathbb{Z}/k)) & \xrightarrow{p_{n+1}^*} & h^q(S^{2n+1} \times_{(\mathbb{Z}/k)} S^1) & \xrightarrow{(p_{n+1})_*} & h^{q-2n-1}(S^1/(\mathbb{Z}/k)) & \longrightarrow \\ \downarrow \mathrm{id} & & \downarrow (j_n')^* & & \downarrow \cdot v_1 & \\ \xrightarrow{\cdot v_1^n} h^q(S^1/(\mathbb{Z}/k)) & \xrightarrow{p_n^*} & h^q(S^{2n-1} \times_{(\mathbb{Z}/k)} S^1) & \xrightarrow{(p_n)_*} & h^{q-2n+1}(S^1/(\mathbb{Z}/k)) & \longrightarrow \end{array} \tag{2.19}$$

where $j'_n : S^{2n-1} \times_{(\mathbb{Z}/k)} S^1 \to S^{2n+1} \times_{(\mathbb{Z}/k)} S^1$ is induced by the inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$. The left square with only Euler classes clearly commutes. The middle square commutes as all maps are induced by maps of spaces which commute. Commutativity of the right square is nontrivial, and it is discussed in Lemma 4.6, p. 45 of [Qui71].
Using Proposition 2.1, $MU^2(S^1/(\mathbb{Z}/k)) = 0$, which implies that $v_1 \in h^2(S^1/(\mathbb{Z}/k))$ is zero as it is induced from a line bundle over $S^1/(\mathbb{Z}/k)$. Let $\pi_{n+1} : S^{2n+1} \times_{\mathbb{Z}/k} S^1 \to S^{2n+1}/(\mathbb{Z}/k)$ be induced by the projection onto the first factor, which is the sphere bundle of the line bundle $S^{2n+1} \times_{\mathbb{Z}/k} \mathbb{C} \to S^{2n+1}/(\mathbb{Z}/k)$ induced by η, so we get a Gysin sequence

$$h^{q+1}(S^{2n+1} \times_{(\mathbb{Z}/k)} S^1) \xrightarrow{(\pi_{n+1})_*} h^q(S^{2n+1}/(\mathbb{Z}/k)) \xrightarrow{\cdot v_{n+1}} h^{q+2}(S^{2n+1}/(\mathbb{Z}/k)).$$

Now assume $x \in MU^q(S^{2n-1}/(\mathbb{Z}/k) \times Y)$ such that $x \cdot v_{n+1} = 0$. Then $x = (\pi_{n+1})_*(z)$ for some $z \in h^{q+1}(S^{2n+1} \times_{(\mathbb{Z}/k)} S^1)$, so $j_n^*(x) = j_n^*(\pi_{n+1})_*(z) = (\pi_n)_*(j'_n)^*(z)$. Now consider the diagram (2.19): as $v_1 = 0$, $(p_n)_*(j'_n)^*(z) = v_1 \cdot (p_{n+1})_*(z) = 0$, so $(j'_n)^*(z) \in \ker((p_n)_*) = \mathrm{im}(p_n^*)$, so there is some $z' \in h^{q+1}(S^1/(\mathbb{Z}/k))$ with $(j'_n)^*(z) = p_n^*(z')$.
Let $i : pt \hookrightarrow S^1/(\mathbb{Z}/k)$ be the inclusion induced by identifying $\mathbb{Z}/k$ with the k-th roots of unity. Then we can write $z' = y' \cdot 1 + y \cdot i_*(1)$, where $y' \in h^{q+1}(pt)$ and $y \in h^q(pt)$.
Now $\pi_n^*(1) = 1 = p_n^*(1)$, so by the Gysin sequence above $(\pi_n)_* p_n^*(1) = (\pi_n)_* \pi_n^*(1) = 0$. Also, $i_*(1)$ is the cobordism class of the map $i : pt = (\mathbb{Z}/k)/(\mathbb{Z}/k) \hookrightarrow S^1/(\mathbb{Z}/k)$, and so $p_n^* i_*(1)$ is the cobordism class of the pullback $p_n^*(i)$, which is

$$S^{2n-1} \cong S^{2n-1} \times_{(\mathbb{Z}/k)} \mathbb{Z}/k \hookrightarrow S^{2n-1} \times_{(\mathbb{Z}/k)} S^1.$$

Then $(\pi_n)_* p_n^* i_*(1)$ is the cobordism class of the composition with the projection π_n, which is

$$S^{2n-1} \to S^{2n-1} \times_{(\mathbb{Z}/k)} S^1 \to S^{2n-1}/(\mathbb{Z}/k),$$

a principal $\mathbb{Z}/k$-bundle. By the next Proposition 2.20, this cobordism class is given by $\Theta(v_n)$, as introduced before this lemma. ☐

Proposition 2.20. *Let $f : Q \to B$ be a principal $\mathbb{Z}/k$-bundle, and let $L := Q \times_{\mathbb{Z}/k} \mathbb{C} \to B$ be the line bundle associated to the representation $\eta : \mathbb{Z}/k \to GL(\mathbb{C})$. Then $f_* = \Theta(e(L)) \in MU^0(B)$.*

Proof. There is a canonical embedding $j : Q \to L,\ q \mapsto [q, 0]$, where we write the elements of L as equivalence classes $[q, z]$ for $q \in Q,\ z \in \mathbb{C}$, with equivalence relation given by $(q \cdot 1_{\mathbb{Z}/k}, z) \sim (q, \zeta \cdot z)$ with $\zeta = \exp(2\pi i/k)$. Let $i : B \to L, b \mapsto 0_b$ be the zero section of the line bundle L, which we write as $p : L \to B$. Then the line bundle $p^*(L) = L \times_B L \xrightarrow{\pi_1} L$ has a canonical section $s : L \to L \times_B L,\ l \mapsto (l, l)$, which is transversal to the zero set $\{(l, 0_{p(l)}) \mid l \in L\} \subseteq L \times_B L$ and vanishes on $i(B)$, as $s(0_b) = (0_b, 0_b)$. So using s for a trivialization outside $i(B)$, the bundle $p^*(L)$ extends to a line bundle $M \to L_+ = L \sqcup pt$ over the one-point compactification of L, where we suppose that B is compact and that $e(M) = i_*(1)$ with

$$i_* : MU^q(B) \xrightarrow{\sim} MU^{q+2}(L_+, pt)$$

is the Thom isomorphism. Then the bundle $p^*(L)^{\otimes k}$ which is trivialized outside $i(B)$ by the section $s^{\otimes k}$ extends to the bundle $M^{\otimes k}$.
There is a section $t : L \to p^*(L)^{\otimes k},\ [q, z] \mapsto ([q, z], [q^{\otimes k}, z^k] - [q^{\otimes k}, 0])$ which extends to a section of $M^{\otimes k}$, which is smooth away from pt and transversal to the zero set of $M^{\otimes k}$, it vanishes on $j(Q) \subseteq L$. Then $j_*(1) = e(M^{\otimes k}) = [k]_F(i_*(1)) = i_*(1) \cdot \Theta(i_*(1)) = i_*\Theta(i^* i_*(1)) = i_*\Theta(e(L))$. Finally, from $i \circ f = j \Rightarrow i_* f_* = j_*$ and the fact that i_* is an isomorphism, we get $f_*(1) = \Theta(e(L))$. ☐

Eventually, we will need the following result of homotopy theory:

Lemma 2.21. *If X has the homotopy type of a finite CW-complex, then $MU^q(X)$ is a finitely generated abelian group.*

Proof sketch. Considering spaces with the homotopy type of a finite CW-complex, it suffices to prove the statement for the latter. An induction over cells reduces to showing that each group $MU^q(pt)$ is a finitely generated abelian group. For any CW-spectrum E we have $E_*(pt) = \pi_*(E) = [\mathbb{S}, E] = [\Sigma^\infty pt_+, E] = E^*(pt)$ where $\mathbb{S} = (S^n)_{n\geq 0}$ is the sphere spectrum. So we are considering the homotopy group $\pi_{-q}(MU) = MU^q(pt)$.
By the mod $\mathcal{C}$ Hurewicz theorem for $\mathcal{C}$ the Serre-class of finitely generated abelian groups, a space or a spectrum has all homotopy groups finitely generated if and only if it has all integral homology groups finitely generated. Now $H_*(MU;\mathbb{Z}) \cong H_*(BU;\mathbb{Z})$ by the Thom isomorphism. The latter of these groups is dual to the cohomology $H^*(BU;\mathbb{Z}) = \mathbb{Z}[c_1, c_2, \dots]$ which is finitely generated in each degree. □

We now have everything required to prove Theorem 2.10.

Proof of Theorem 2.10. First assume that

$$\widetilde{MU}^{\mathrm{ev}}(X) = C \cdot \bigoplus_{q>0} MU^{2q}(X). \tag{2.22}$$

From the axioms of a generalized cohomology theory follow the suspension isomorphisms

$$\begin{aligned}
\widetilde{MU}^j(X) &\cong \widetilde{MU}^{j+1}(\Sigma X) = \widetilde{MU}^{j+1}(S^1 \wedge X),\\
MU^j(X) &\cong \widetilde{MU}^{j+1}(SX) = \widetilde{MU}^{j+1}(S^1 \times X/\{*\} \times X), \text{ and}\\
MU^j(X) &\cong \widetilde{MU}^{j+2}(S^2X) = \widetilde{MU}^{j+2}(S^2 \times X/\{*\} \times X)
\end{aligned}$$

if X is connected and has a basepoint. Then with (2.22) it follows

$$\begin{aligned}
MU^*(X) &= MU^{\mathrm{odd}}(X) \oplus MU^{\mathrm{ev}}(X) = \bigoplus_{q\in\mathbb{Z}} MU^{2q-1}(X) \oplus \bigoplus_{q\in\mathbb{Z}} MU^{2q}(X)\\
&\cong \bigoplus_{q\in\mathbb{Z}} \widetilde{MU}^{2q}(SX) \oplus \bigoplus_{q\in\mathbb{Z}} \widetilde{MU}^{2q+2}(S^2X) \cong C \cdot \bigoplus_{q>0} MU^{2q}(SX) \oplus C \cdot \bigoplus_{q>0} MU^{2q}(S^2X)\\
&\cong C \cdot \bigoplus_{q\geq 0} MU^{2q+1}(X) \oplus C \cdot \bigoplus_{q\geq 0} MU^{2q}(X) = C \cdot \bigoplus_{q\geq 0} MU^q(X),\\
\widetilde{MU}^*(X) &= \widetilde{MU}^{\mathrm{odd}}(X) \oplus \widetilde{MU}^{\mathrm{ev}}(X) = \bigoplus_{q\in\mathbb{Z}} \widetilde{MU}^{2q-1}(X) \oplus \widetilde{MU}^{\mathrm{ev}}(X)\\
&\cong \bigoplus_{q\in\mathbb{Z}} \widetilde{MU}^{2q}(\Sigma X) \oplus \widetilde{MU}^{\mathrm{ev}}(X) \cong C \cdot \bigoplus_{q>0} MU^{2q}(\Sigma X) \oplus C \cdot \bigoplus_{q>0} MU^{2q}(X)\\
&\cong C \cdot \bigoplus_{q\geq 0} MU^{2q+1}(X) \oplus C \cdot \bigoplus_{q>0} MU^{2q}(X) = C \cdot \bigoplus_{q>0} MU^q(X).
\end{aligned}$$

So it suffices to prove (2.22). For this, let $R \coloneqq C \cdot \bigoplus_{q>0} MU^{2q}(X)$, the right-hand side of (2.22). Then R and $\widetilde{MU}^{\mathrm{ev}}(X)$ are submodules of the $\mathbb{Z}$-module $MU^*(X)$, so it suffices to show that $R_{(p)} = \widetilde{MU}^{\mathrm{ev}}(X)_{(p)}$ for any prime $p \in \mathbb{Z}$. We have $R^{2j}_{(p)} = \widetilde{MU}^{2j}(X)_{(p)}$ for $j > 0$, so

$$R^{-2j}_{(p)} = \widetilde{MU}^{-2j}(X)_{(p)} \tag{2.23}$$

holds for $j < 0$. Pursue by induction on $q \geq 0$ and assume (2.23) holds for all $j < q$. Then in what follows we show it also holds for $j = q$. Let $x \in \widetilde{MU}^{-2q}(X)$. Using the principal $\mathbb{Z}/p$-bundle $S^{2m+1} \to S^{2m+1}/(\mathbb{Z}/p)$ in (2.16) of Proposition 2.15, we have

$$w^{n+q}P(x) = \sum_{l(\alpha)\leq n} w^{n-l(\alpha)}a(v)^{\alpha}s_{\alpha}(x) \tag{2.24}$$

for some large $n \gg 0$. This equation holds in $MU^{2n-2q}(S^{2m+1}/(\mathbb{Z}/p) \times X)$ for all m, where $a_j(z) \in C[\![z]\!]$. Again v is the Euler class of the line bundle induced by $\eta : \mathbb{Z}/p \to \mathrm{GL}(\mathbb{C})$, which is the v_{m+1} of Lemma 2.18, and

$$w = (p-1)!\, v^{p-1} + \sum_{j\geq p} b_j v^j$$

with $b_j \in C$ by (2.17) of Proposition 2.15. As p is prime, $(p-1)!$ is not divisible by p, so it is a unit in $\mathbb{Z}_{(p)}$, and from the formula for w above we get $v^{p-1} = w \cdot \theta(v)$ for some unit $\theta(z) \in C_{(p)}[\![z]\!]^{\times}$. If $\alpha \neq 0$ and $[f] = x \in \widetilde{MU}^{-2q}(X)$, then $c_{\alpha}(\nu_f)$ has positive grading, and f_* lowers degrees by $2q$, so $s_{\alpha}(x) = f_*(c_{\alpha}(\nu_f))$ has grading $> -2q$. Hence, $s_{\alpha}(x) \in R$ by the induction hypothesis (which was that (2.23) holds for $j < q$, and we can also assume it holds for these gradings without localizing). So from (2.24) we obtain

$$\begin{aligned} w^{n+q}P(x) &= w^n \cdot \underbrace{s_0(x)}_{=x} + \sum_{0<l(\alpha)\leq n} w^{n-l(\alpha)}a(v)^{\alpha}s_{\alpha}(x), \text{ and so} \\ w^n(w^qP(x) - x) &= \sum_{0<l(\alpha)\leq n} w^{n-l(\alpha)}a(v)^{\alpha} \underbrace{s_{\alpha}(x)}_{\in R}, \text{ which gives} \\ v^m(w^qP(x) - x) &= \psi(v) \quad \in MU^*(S^{2m+1}/(\mathbb{Z}/p) \times X)_{(p)} \end{aligned} \tag{2.25}$$

for some $m \geq 1$ and $\psi(z) \in R_{(p)}[\![z]\!]$ using the relation $v^{p-1} = w \cdot \theta(v)$. Assume that m is minimal satisfying such a relation. We will show that $m = 1$ by reducing the equation two times to arrive at an equation of the same form which contradicts the minimality of m in the case $m > 1$.
Let $i^* : MU^*(S^{2m+1}/(\mathbb{Z}/p) \times X) \to MU^*(pt \times X) \cong MU^*(X)$ be induced by the inclusion $i : pt \times X \hookrightarrow S^{2m+1}/(\mathbb{Z}/p) \times X$ of any point. Then i^* sends $v \in MU^*(S^{2m+1}/(\mathbb{Z}/p))$ (or more precisely, its pullback along the first projection) to zero, so applying i^* to (2.25) yields $\psi(0) = 0$, so ψ has constant term 0, and we can write it as $\psi(z) = z\varphi(z)$ for some $\varphi \in R_{(p)}[\![z]\!]$, so that

$$v\underbrace{(v^{m-1}(w^qP(x) - x) - \varphi(v))}_{\in MU^{2(m-1)-2q}(S^{2m+1}/(\mathbb{Z}/p))_{(p)}} = 0. \tag{2.26}$$

Then by Lemma 2.18 there is an element $y \in MU^{2(m-1)-2q}(X)_{(p)}$ with

$$v^{m-1}(w^qP(x) - x) = \varphi(v) + y \cdot \Theta(v) \in MU^*(S^{2m-1}/(\mathbb{Z}/p) \times X)_{(p)}. \tag{2.27}$$

Now, let $j^* : MU^*(S^{2m+1}/(\mathbb{Z}/p) \times X) \to MU^*(S^{2m+1}/(\mathbb{Z}/p \times pt)$ be the homomorphism induced by the inclusion $j : S^{2m+1}/(\mathbb{Z}/p) \times pt \hookrightarrow S^{2m+1}/(\mathbb{Z}/p) \times X$ of the base point of X. As x was chosen in $\widetilde{MU}^{-2q}(X)$, applying j^* to (2.27) gives $y' \cdot \Theta(v) = 0$, where y' is the image of y under $MU^*(X)_{(p)} \to MU^*(pt)_{(p)}$. Now $y - y' \in \widetilde{MU}^*(X)_{(p)}$, so we can assume $y \in \widetilde{MU}^*(X)_{(p)}$ by subtracting y', and then the equation (2.27) still holds.

If $m > 1$, then $2(m-1)-2q > -2q$, so as we assumed y is a reduced class we get $y \in R_{(p)}$ by the induction hypothesis as before. As $\varphi(z) \in R_{(p)}[\![z]\!]$ and $\Theta(z) \in C[\![z]\!]$, the right side of (2.27) is in $R_{(p)}[\![v]\!]$. But this would contradict the minimality of m in (2.25). So $m = 1$, and we arrive at the equation

$$w^q P(x) - x = \varphi(v) + y \cdot \Theta(v) \in MU^*(S^{2m-1}/(\mathbb{Z}/p) \times X)_{(p)}.$$

Recall that $\Theta(z) = p+d_1z+d_2z^2+\ldots \in C[\![z]\!]$. So $i^* : MU^*(S^{2m-1}/(\mathbb{Z}/p)\times X) \to MU^*(pt\times X) \cong MU^*(X)$ maps $v \mapsto 0$, so $\Theta(v) \mapsto p$, $\varphi(v) \mapsto \varphi(0)$ and $w^q \mapsto 0$ if $q > 0$. The term $P(x)$ has X-component x^p, so i^* maps $P(x) \mapsto x^p$. Together we get

$$-x = \varphi(0) + py \quad \text{if } q > 0, \tag{2.28}$$

$$x^p - x = \varphi(0) + py \quad \text{if } q = 0, \tag{2.29}$$

which hold in $\widetilde{MU}^{-2q}(X)_{(p)}$. For $q > 0$, we have $\widetilde{MU}^{-2q}(X)_{(p)} = R^{-2q}_{(p)} + p \cdot \widetilde{MU}^{-2q}(X)_{(p)}$, where $(\subseteq)$ holds by (2.28) and $(\supseteq)$ always holds. $MU^{-2q}(X)$ is a finitely generated abelian group by Lemma 2.21, so $\widetilde{MU}^{-2q}(X)$ is finitely generated as well, and so $\widetilde{MU}^{-2q}(X)_{(p)}$ is a finitely generated $\mathbb{Z}_{(p)}$-module. Then $MU^{-2q}(X)_{(p)} = R^{-2q}_{(p)}$ follows with an application of Nakayama's lemma.
For $q = 0$, the mapping $x \mapsto x^p - x$ takes $\widetilde{MU}^0(X)_{(p)}$ into $R^0_{(p)} + p\widetilde{MU}^0(X)_{(p)}$ by (2.29), so it is zero on $\widetilde{MU}^0(X)_{(p)}/(R^0_{(p)} + p\widetilde{MU}^0(X)_{(p)})$. Also, the ideal $\widetilde{MU}^0(X)_{(p)} \trianglelefteq MU^0(X)_{(p)}$ is nilpotent. So for an element x of $\widetilde{MU}^0(X)_{(p)}/(R^0_{(p)} + p\widetilde{MU}^0(X)_{(p)})$ the relations $x^p = x$ and $x^N = 0$ for some $N \gg 0$ hold. Then $x = x^p = x^{p^N} = x^N \cdot x^{p^N-N} = 0$, so $\widetilde{MU}^0(X)_{(p)} = R^0_{(p)} + p\widetilde{MU}^0(X)_{(p)}$ and finally Nakayama's lemma as above gives $MU^0(X)_{(p)} = R^0_{(p)}$, which concludes the proof. □

2.4 The complex cobordism ring $MU^*(pt)$

The aim of this section is to understand the complex cobordism ring $MU^*(pt)$ and show that it is isomorphic to the *Lazard ring*, which we will define below. Together with this we will also show that the formal group law over $MU^*(pt)$ we introduced in Proposition 2.7 is the universal formal group law. The result for the structure of $MU^*(X)$ from Section 2.3 will be central to deduce that $MU^*(pt)$ is generated by the coefficients of this formal group law.
Let $\mathbf{Rng}_\bullet$ be the category of graded rings and consider the functor

$$G : \mathbf{Rng}_\bullet \to \mathbf{Set}, \quad R_\bullet = \bigoplus_{n\in\mathbb{Z}} R_n \mapsto \{F(x,y) = \sum_{i,j\geq 0} a_{ij}x^iy^j \in R[\![x,y]\!] \mid F \text{ fgl}, a_{ij} \in R_{i+j-1}\}.$$

It is representable by some graded ring $\mathcal{L}$ called *Lazard ring*, which has a universal formal group law $F_{\text{uni}}(x,y) \in \mathcal{L}[\![x,y]\!]$ over it. That is, $F_{\text{uni}}(x,y) = \sum_{i,j\geq 0} l_{ij}x^iy^j$, where $\mathcal{L}$ is generated by the coefficients l_{ij} with relations imposed by the formal group law, which are

(1) $l_{10} = l_{01} = 1$,

(2) $l_{ij} = l_{ji}$ for all i, j and $l_{i0} = 0$ for $i \neq 1$,

(3) the relations imposed by the associativity law.

So $\mathcal{L}$ has the presentation $\mathcal{L} \cong \mathbb{Z}[l_{ij} \mid i,j \geq 0]/(\text{relations (1), (2), (3)})$.
The Lazard ring has the following universal property: for any ring R with formal group law $G(x,y) \in R[\![x,y]\!]$, there is a unique ring homomorphism $\delta : \mathcal{L} \to R$ mapping F_{uni} to G. Indeed, as for $\sum_{i,j} \delta(l_{ij})x^i y^j = \delta(F_{\text{uni}}) = G = \sum_{i,j} a_{ij}x^i y^j$ we have to define δ on the generators of $\mathcal{L}$ by $\delta(l_{ij}) = a_{ij}$, and by the formal group law properties δ is a homomorphism. Conversely, for a homomorphism $\delta : \mathcal{L} \to R$, the unique formal group law on R corresponding to δ is $\sum_{i,j} \delta(l_{ij})x^i y^j \in R[\![x,y]\!]$. This gives a bijection

$$\operatorname{Hom}_{\text{Rng}}(\mathcal{L}, R) \xleftrightarrow{\sim} \{\text{Formal group laws on } R\}. \tag{2.30}$$

In fact, $\mathcal{L}$ has a simpler structure as the presentation above might suggest, by the following theorem due to Lazard from [Laz55]:

Theorem 2.31. *$\mathcal{L} \cong \mathbb{Z}[x_1, x_2, x_3, \ldots]$, where x_i has grading $-2i$.*

The aim for the remainder of this section will be to show that $MU^*(pt)$ is isomorphic to $\mathcal{L}$. The choice of grading in Theorem 2.31 may seem odd, but it will turn out to be the right one from the topological perspective of complex cobordism.
As ordinary integral cohomology $H^* : \mathbf{Top}^{\text{op}} \to \mathbf{AbGp}$ satisfies the axioms (1)-(3) of Section 2.1, Lemma 2.3 for every manifold X gives a homomorphism $\varepsilon : MU^*(X) \to H^*(X)$ commuting with Gysin homomorphisms, so in particular it preserves Thom classes, and so also Euler classes. It induces $\varepsilon : MU^*(X)[\mathbf{t}] \to H^*(X)[\mathbf{t}]$, where again $[\mathbf{t}]$ stands for infinitely many indeterminates $[t_1, t_2, t_3 \ldots]$. Set

$$\beta := MU^*(X) \xrightarrow{s_t} MU^*(X)[\mathbf{t}] \xrightarrow{\varepsilon} H^*(X)[\mathbf{t}],$$

where s_t is the Landweber-Novikov operation defined in Section 2.2. The map β is called *Boardman map*, and it satisfies

$$\begin{aligned}\beta(f_*(z)) &= \varepsilon(s_t(f_*(z))) = \varepsilon(f_!(s_t(z))) \\ &= \varepsilon(f_*(c_t^{MU}(\nu_f) \cdot s_t(z))) = f_*(\varepsilon(c_t^{MU}(\nu_f) \cdot s_t(z))) \\ &= f_*(c_t^H(\nu_f) \cdot \varepsilon(s_t(z))) = f_*(c_t^H(\nu_f) \cdot \beta(z))\end{aligned}$$

for every proper complex oriented map $f : Z \to X$, where we write c_t^H and c_t^{MU} for the elements associated to a vector bundle over X in the polynomial rings of H^* and MU^* respectively as in Section 2.2. We will also adopt this notation for Euler classes and write e^H and e^{MU} for integral and complex cobordism Euler classes. Then as noted above, ε sends e^{MU} to e^H. With the formula $c_t^H(L) = 1 + \sum_{j>0} t_j e^H(L)^j$ we obtain

$$\begin{aligned}\beta(e^{MU}(L)) &= \beta(\mathrm{id}_*(e^{MU}(L))) = \mathrm{id}_*(c_t^H(\nu_{\mathrm{id}}) \cdot \beta(e^{MU}(L))) \\ &= c_t^H(L) \cdot \beta(e^{MU}(L)) = (1 + \sum_{j>0} t_j e^H(L)^j) \cdot e^H(L) \\ &= \sum_{j\geq 0} t_j e^H(L)^{j+1} \text{ with } t_0 = 1,\end{aligned}$$

for every line bundle $L \to X$. Considering two line bundles $L_1, L_2 \to X$ and plugging in $L_1 \otimes L_2$ above we have

$$\begin{aligned}\beta(e^{MU}(L_1 \otimes L_2)) &= \sum_{j\geq 0} t_j\, e^H(L_1 \otimes L_2)^{j+1} = \sum_{j\geq 0} t_j\, (e^H(L_1) + e^H(L_2))^{j+1}, \text{ so} \\ \beta F_{MU}(e^{MU}(L_1), e^{MU}(L_2)) &= \sum_{j\geq 0} t_j\, F_H(e^H(L_1), e^H(L_2))^{j+1},\end{aligned}$$

where $F_{MU}(x,y) = \sum_{i,j\geq 0} c_{ij} x^i y^j \in MU^*(pt)[\![x,y]\!]$ is the complex cobordism group law over $MU^*(pt)$ from Proposition 2.7, and $F_H(x,y) = x + y \in \mathbb{Z}[\![x,y]\!]$ is the additive group law corresponding to integral cohomology by Example 2.9. Using a universality argument as in the proof of Proposition 2.7 we obtain the formula

$$\beta F_{MU}(\theta_t(x), \theta_t(y)) = \theta_t(x+y), \text{ with } \theta_t(z) = \sum_{j\geq 0} t_j z^{j+1},\ t_0 = 1.$$

So there are ring homomorphisms

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\delta} & MU^*(pt) & \xrightarrow{\beta} & H^*(pt)[\mathbf{t}] = \mathbb{Z}[\mathbf{t}], \\ F_{\text{uni}} & \longmapsto & F_{MU} & \longmapsto & \theta_t^*(x+y), \end{array}$$

where δ exists by the universal property of the Lazard ring $\mathcal{L}$ with $\delta(\mathcal{L}_q) \subseteq MU^{-2q}(pt)$. For an invertible power series $\theta(z)$ and a formal group law $G(x,y)$, the conjugation is given by $\theta^*(G(x,y)) = \theta(G(\theta^{-1}(x), \theta^{-1}(y)))$. As $t_0 = 1$, the series $\theta_t(z)$ from above is invertible, so $\theta_t^*(x+y) = \theta_t(\theta_t^{-1}(x) + \theta_t^{-1}(y)) = \beta F(\theta_t(\theta_t^{-1}(x)), \theta_t(\theta_t^{-1}(y))) = \beta F(x,y)$.

Theorem 2.32. *The map δ is an isomorphism and β is injective. Consequently, by Theorem 2.31,*

$$MU^*(pt) \cong \mathbb{Z}[x_1, x_2, x_3, \ldots], \text{ with } |x_i| = -2i.$$

Proof. The map δ sends $F_{\text{uni}} \mapsto F_{MU}$, so $l_{ij} \mapsto c_{ij}$. By Corollary 2.11, $MU^*(pt)$ is generated by the c_{ij}, so δ is surjective. We will show that $\beta \circ \delta : \mathcal{L} \to \mathbb{Z}[\mathbf{t}]$ induces an isomorphism of rationalizations $\mathbb{Q} \otimes \mathcal{L} \to \mathbb{Q} \otimes \mathbb{Z}[\mathbf{t}] \cong \mathbb{Q}[\mathbf{t}]$. A homomorphism $u : \mathbb{Z}[\mathbf{t}] \to R$ can be identified with the (invertible) power series $\theta_u(z) \coloneqq \sum_{j\geq 0} u(t_j) z^{j+1},\ t_0 = 1$, as u is uniquely determined by the images of the $t_j, j \geq 1$. Using the bijection (2.30), the composite $u \circ \beta \circ \delta : \mathcal{L} \to R$ corresponds to the formal group law $u\beta\delta(F_{\text{uni}}) = u(\theta_t^*(x+y)) = \theta_u^*(x+y)$ on R.
By [Frö68, Proposition 1, p. 96] any formal group law over a $\mathbb{Q}$-algebra R is of the form $\theta_a^*(x+y)$ for a unique power series $\theta_a = \sum_j a_j z^j$, which is called the *logarithm* of the group law. So for any $\mathbb{Q}$-algebra R the assignment $(u : \mathbb{Z}[\mathbf{t}] \to R) \mapsto \theta_u^*$ is bijective, and again with (2.30), $\beta \circ \delta$ induces a bijection

$$\mathrm{Hom}_{\mathbf{Rng}}(\mathbb{Z}[\mathbf{t}], R) \xleftrightarrow{\sim} \mathrm{Hom}_{\mathbf{Rng}}(\mathcal{L}, R).$$

So $\mathrm{id}_{\mathbb{Q}} \otimes (\beta \circ \delta) : \mathbb{Q} \otimes \mathcal{L} \to \mathbb{Q} \otimes \mathbb{Z}[\mathbf{t}] \cong \mathbb{Q}[\mathbf{t}]$ is an isomorphism. By Theorem 2.31, the Lazard ring $\mathcal{L}$ is torsion free, so $\beta \circ \delta$ is injective. Therefore, δ is also injective, hence an isomorphism, and so $\beta = (\beta \circ \delta) \circ \delta^{-1}$ must also be injective. □

Example 2.33. The isomorphism $\mathcal{L} \cong MU^*(pt)$ of Theorem 2.32 which maps F_{uni} to F_{MU} allows us to build new cohomology theories from formal group laws over rings: given a ring R and a formal group law $G \in R[\![x,y]\!]$, there is a unique ring homomorphism $\varphi : MU^*(pt) \to R$ sending the universal formal group law F over $MU^*(pt)$ to G. In particular, φ makes R into an $MU^*(pt)$-module. Then we can consider the functor

$$R^*(-) \coloneqq MU^*(-) \otimes_{MU^*(pt)} R.$$

It clearly is a homotopy invariant functor and satisfies excision, and we can compute its coefficient ring:

$$R^*(pt) = MU^*(pt) \otimes_{MU^*(pt)} R \cong R,$$

where the isomorphism is given by $a \otimes r \mapsto \varphi(a) \cdot r$. Also, $R^*(pt)$ carries G as a formal group law over it, in the sense of Remark 2.8.
Indeed, we have $R^*(\mathbb{CP}^n \times \mathbb{CP}^n) \cong R^*(pt)[z_1, z_2]/(z_1^{n+1}, z_2^{n+1}) \cong R[z_1, z_2]/(z_1^{n+1}, z_2^{n+1})$. Writing the complex cobordism Euler class $e^{MU}(\pi_1^*(\gamma_{\mathbb{C}}^{1,n+1}) \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,n+1})) \in MU^*(\mathbb{CP}^n \times \mathbb{CP}^n)$ as $\sum_{i,j \le n} c_{ij}^n z_1^i z_2^j \in MU(pt)[z_1, z_2]$, the R-Euler class $e^R(\pi_1^*(\gamma_{\mathbb{C}}^{1,n+1}) \otimes \pi_2^*(\gamma_{\mathbb{C}}^{1,n+1})) \in R^*(\mathbb{CP}^n \times \mathbb{CP}^n)$ is $\sum_{i,j \le n}(c_{ij}^n \otimes 1_R) z_1^i z_2^j$, which corresponds to $\sum_{i,j \le n} \varphi(c_{ij}) z_1^i z_2^j$ under the above isomorphism. As the $\varphi(c_{ij})$ are the coefficients of the formal group law G, we get back G as a group law over $R^*(pt)$ as claimed.
But in general, the functor $R^*(-)$ is not a generalized cohomology theory, as one can not expect that tensoring with an arbitrary ring preserves exact sequences. The assumption that R is flat over $MU^*(pt)$ would make $R^*(pt)$ into a cohomology theory, but this assumption turns out to be too strong in practise. With the *Landweber exact functor theorem* Section 4 will give a more refined condition on R.

3 Relations of complex cobordism to K-theory

We already saw in Example 2.33 how to use the results about $MU^*(pt)$ and in particular Theorem 2.32 to construct new complex oriented cohomology theories by forming the tensor product $MU^*(-) \otimes_{MU^*(pt)} R$. This section is concerned with a special application of this procedure. That is, we show that using the cohomology ring $K^*(pt) = \mathbb{Z}$ of ($\mathbb{Z}/2$-graded) K-theory instead of R, we indeed obtain the (complex) K-theory functor. This then expresses K-theory purely in terms of complex cobordism.
At first sight it might seem surprising that complex cobordism, represented by maps from suspensions of a space to the complex Thom spectrum, is so closely related to the set of vector bundles the space admits. But there are relations of K-theory to the unitary group $U(n)$ which might hint in this direction: once chosen a Hermitian inner product for an n-dimensional complex vector bundle $E \to X$, we can choose the transition functions $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{C})$, for (U_α) a trivializing open cover of X, to take values in $U(n)$, which is equivalent data as choosing a Hermitian inner product for the bundle E.
Furthermore, K-theory can not only be constructed in terms of the vector bundles a space admits, but also as a representable functor in the sense of Section 1: letting $U(\infty) := \bigcup_{n \ge 1} U(n)$, we can define an Ω-spectrum $(E_n)_{n \ge 0}$ by

$$E_n := \begin{cases} U(\infty) & \text{for } n \text{ odd,} \\ \Omega U(\infty) & \text{for } n \text{ even,} \end{cases}$$

where $\Omega^2 U(\infty) \cong U(\infty)$ by the Bott map, and this spectrum represents complex K-theory (cf. [Hil71, p. 15]). In this section we follow [CF66], Chapter I & II, and we will again indicate which specific results we are quoting.

3.1 Basic constructions and cohomology operations

As usual, topological K-theory is viewed as a (two-periodic) functor $K^* : \mathbf{CW}_{\mathrm{f}}^{\mathrm{op}} \to \mathbf{Rng}_\bullet$ on the category of finite CW-pairs (or a more general category as compact Hausdorff spaces). We will introduce a different interpretation of $K^0(X, A)$ for finite CW-pairs (X, A) without the usual procedure of taking $K^0(X, A)$ to be $K^0(X/A)$. For this, consider triples (ξ_0, ξ_1, φ), where $\xi_0, \xi_1 \to X$ are vector bundles over X and $\varphi : \xi_1|_A \xrightarrow{\sim} \xi_2|_A$ is a bundle isomorphism. Two triples (ξ_0, ξ_1, φ) and (η_0, η_1, θ) are isomorphic, written as

$$(\xi_0, \xi_1, \varphi) \cong (\eta_0, \eta_1, \theta),$$

if $\xi_0 \cong \eta_0,\ \xi_1 \cong \eta_1$ as vector bundles, and restricted to A these isomorphisms commute with φ and θ. Furthermore, write

$$(\xi_0, \xi_1, \varphi) \sim (\eta_0, \eta_1, \theta),$$

if there exist vector bundles $\rho, \nu \to X$ such that $(\xi_0 \oplus \rho, \xi_1 \oplus \rho, \varphi) \cong (\eta_0 \oplus \nu, \eta_1 \oplus \nu, \theta)$. Then $K^0(X, A)$ is defined as the set of equivalence classes of such triples, which are written $d(\xi_0, \xi_1, \varphi)$. Component wise application of the Whitney sum and the tensor product gives a ring structure on $K^0(X, A)$. Note that in the case $A = \varnothing$, classes can be written as $d(\xi_0, \xi_1)$, and the mapping $d(\xi_0, \xi_1) \mapsto \xi_0 - \xi_1$ defines an isomorphism to the usual K-group $K^0(X)$. If x_0 is a basepoint of a finite CW-complex X, then the kernel of $K^0(X) \to K^0(\{x_0\})$, denoted $\widetilde{K}^0(X)$, defines the reduced K-group as usual. Then the homomorphism

$$K^0(X, x_0) \to \widetilde{K}^0(X),\ d(\xi_0, \xi_1, \varphi) \mapsto \xi_0 - \xi_1$$

is an isomorphism, and for a finite CW-pair (X, A) we have natural isomorphisms $K^0(X, A) \cong K^0(X/A, x_0) \cong \widetilde{K}^0(X/A)$.

A first goal will be to define two cobordism operations. The first one is a multiplicative natural transformation $MU^* \to K^*$ from complex cobordism into K-theory, the second one is a natural transformation $MU^* \to H^*$ into integral cohomology. A key ingredient to this will be to represent an element of complex cobordism by a map using

$$\widetilde{MU}^n(X) = [\Sigma^{2k-n} X,\ MU(k)],\ k \text{ large.} \tag{3.1}$$

A further ingredient for constructing the morphism into K-theory is to be able to associate to principal $U(n)$-bundles an element of the complex cobordism of a certain Thom space, which we shall construct now.

Let $\xi \to X$ be a principal $U(n)$-bundle over a finite CW-complex X, and let V be a complex vector space of dimension n. Then $U(n)$ acts on the exterior algebra

$$\Lambda V = T(V)/\langle v \otimes v \mid v \in V\rangle = \bigoplus_{i \geq 0} V^{\otimes i}/\langle v \otimes v \mid v \in V\rangle,$$

and (as for the $\mathbb{C}$-linear representations of Section 2) with

$$\Lambda(\xi) \coloneqq (\xi \times \Lambda V)/U(n) \to X$$

we obtain a complex vector bundle on X. Replacing ΛV by the subspaces $\Lambda^{\mathrm{odd}} V$ and $\Lambda^{\mathrm{ev}} V$ of odd respectively even grading yields bundles $\Lambda^{\mathrm{ev}}(\xi) \to X,\ \Lambda^{\mathrm{odd}}(\xi) \to X$. Letting D^{2n}, $S^{2n-1} \subseteq V$ be the unit disk and the unit sphere in V, we have bundles

$$\mathbb{D}(\xi) \coloneqq (\xi \times D^{2n})/U(n) \to X \quad \text{and} \quad \mathbb{S}(\xi) \coloneqq (\xi \times S^{2n-1})/U(n) \to X.$$

Then $\xi' \coloneqq \pi^*(\xi) \to \mathbb{D}(\xi)$ is a principal $SU(n)$-bundle induced by $\pi : \mathbb{D}(\xi) \to X$, so we obtain the complex vector bundle $\Lambda(\xi') \to \mathbb{D}(\xi)$, and similarly $\Lambda^{\mathrm{ev}}(\xi'),\ \Lambda^{\mathrm{odd}}(\xi')$.

For $v \in V$ there is a linear map $F_v : \Lambda V \to \Lambda V,\ x \mapsto v \wedge x$, and it has an adjoint given by

$F_v^* : \Lambda V \to \Lambda V$, $\langle F_v^*(x), y\rangle = \langle x, F_v(y)\rangle = \langle x, v \wedge x\rangle$. Then if $v \neq 0$, $\varphi_v := F_v + F_v^* : \Lambda V \to \Lambda V$ defines a $\mathbb{C}$-linear isomorphism $\Lambda^{\mathrm{odd}} V \cong \Lambda^{\mathrm{ev}} V$, which gives a map

$$\varphi : \Lambda(\xi') \to \Lambda(\xi'),\ [e, v, y] \mapsto [e, v, \varphi_v(y)],$$

where $[e, v, y] \in \Lambda(\xi')$ is the orbit of a point in $\xi \times D^{2n} \times \Lambda V$ under the action of $U(n)$. So we have a triple $(\Lambda^{\mathrm{ev}}(\xi'),\ \Lambda^{\mathrm{odd}}(\xi'), \varphi)$ as defined above, and we set

$$\mathcal{J}(\xi) := d(\Lambda^{\mathrm{ev}}(\xi'),\ \Lambda^{\mathrm{odd}}(\xi'), \varphi)\ \in K^0(\mathbb{D}(\xi), \mathbb{S}(\xi)) \cong \widetilde{K}^0(\mathbb{T}(\xi)),$$

where $\mathbb{T}(\xi) = \mathbb{D}(\xi)/\mathbb{S}(\xi)$ is the Thom space of the bundle ξ.
Furthermore, we have that $i_x^*(\mathcal{J}(\xi))$ is a generator of $\widetilde{K}^0(D_x^n, S_x^n) \cong \widetilde{K}^0(S^{2n})$ for every $x \in X$, where $i_x : (D_x^n, S_x^n) \hookrightarrow (\mathbb{D}(\xi), \mathbb{S}(\xi))$ is the inclusion of the fibre over x, and it is a generator of the free $K^*(pt)$-module $K^*(S^{2n})$. Using a Thom isomorphism argument (cf. [CF66, p. 24f]), this implies

$$K^0(X) \cong K^0(\mathbb{D}(\xi), \mathbb{S}(\xi)) \cong \widetilde{K}^0(\mathbb{T}(\xi)).$$

In particular, we can assign to a principal $U(n)$-bundle $\xi \to X$ the element $\mathcal{J}(\xi) \in \widetilde{K}^0(\mathbb{T}(\xi))$, which then corresponds to an element in $K^0(X)$ under the above isomorphism.
We can use this construction in a specific situation to obtain the first cohomology operation: for an infinite dimensional Hilbert space H and $k \in \mathbb{Z}_{\geq 0}$, let

$$EU(k) := \{e_1, \ldots, e_k \in H \mid (e_i, e_j) = \delta_{ij}\}$$

be the space of orthonormal k-frames with natural action of $U(n)$. Let $\zeta^k : EU(k) \to BU(k)$ be the universal principal $U(k)$-bundle, where $BU(k)$ is the classifying space of $U(k)$ as introduced in Section 1, so that with

$$E(k) := EU(k) \times_{U(k)} \mathbb{C}^k = (EU(k) \times \mathbb{C}^k)/U(k) \to BU(k)$$

we get the tautological bundle $\gamma^k \to BU(k)$ we constructed to define $MU(k)$. Then $MU(k) = \mathbb{T}(\zeta^k) = \mathbb{D}(\zeta^k)/\mathbb{S}(\zeta^k)$ using the definitions from above.

Definition 3.2. For a CW-complex X define a map

$$\mu_X : \widetilde{MU}^*(X) \to \widetilde{K}^*(X)$$

as follows: let $x \in \widetilde{MU}^n(X)$, and represent x by a map $f : S^{2k-n} \wedge X \to MU(k)$ for some large $k \in \mathbb{Z}_{\geq 0}$ using (3.1). Then let $\mu_X(x)$ be the image of $\mathcal{J}(\zeta^k) \in \widetilde{K}^0(\mathbb{T}(\zeta^k)) = \widetilde{K}^0(MU(k))$ under the composition

$$\widetilde{K}^0(MU(k)) \xrightarrow{f^*} \widetilde{K}^0(S^{2k-n} \wedge X) \cong \widetilde{K}^{n-2k}(X) \cong \widetilde{K}^n(X).$$

Lemma 3.3. *The map $\mu_X : \widetilde{MU}^*(X) \to \widetilde{K}^*(X)$ from Definition 3.2 defines a multiplicative natural transformation $\mu : \widetilde{MU}^* \to \widetilde{K}^*$.*

Proof. By homotopy invariance of the functor K^*, the map μ_X does not depend on the choice of the map f representing $x \in \widetilde{MU}^n(X)$.
Let $g : Y \to X$ be a map. Without restriction, we can choose the same k for $\widetilde{MU}^n(Y)$ and $\widetilde{MU}^n(X)$. Then there is an induced map $\bar{g} : S^{2k-n} \wedge Y \to S^{2k-n} \wedge X$. If $x \in \widetilde{MU}^n(X)$ is

represented by $f : S^{2k-n} \wedge X \to MU(k)$, it follows $\mu_Y(g^*(x)) = \mu_Y([f \circ \bar{g}]) = (f \circ \bar{g})^*(\mathcal{J}(\zeta)) = \bar{g}^* \circ f^*(\mathcal{J}(\zeta^k)) = g^*(\mu_X(x))$, where the isomorphism $\widetilde{K}^0(S^{2k-n} \wedge X) \cong \widetilde{K}^n(X)$ is used. This shows naturality.
The fact that μ_X is multiplicative follows by representing a product $x \cdot y$ for $x \in \widetilde{MU}^n(X)$ and $y \in \widetilde{MU}^m(X)$, represented by maps $f : S^{2k-n} \wedge X \to MU(k)$ and $g : S^{2k-m} \wedge X \to MU(k)$ respectively, by the composition

$$\Sigma^{2k-n-m} X \to \Sigma^{2k-n} X \wedge \Sigma^{2k-m} X \xrightarrow{f \wedge g} MU(k) \wedge MU(k) \to MU(2k)$$

and applying it to $\mathcal{J}(\zeta^{2k})$. □

The principal $U(k)$-bundle $\zeta^k : EU(k) \to BU(k)$ shows that $BU(k) = EU(k)/U(k)$. Now the group $U(k-1)$ acts on $EU(k)$ via the inclusion $U(k-1) \hookrightarrow U(k)$, and this gives a fibration

$$BU(k-1) \simeq EU(k)/U(k-1) \to EU(k)/U(k) \simeq BU(k)$$

with fibre $U(k)/U(k-1) \simeq S^{2k-1}$, which corresponds to the sphere bundle of the tautological bundle $\gamma^k : E(k) \to BU(k)$. Since $E(k) = EU(k) \times_{U(k)} \mathbb{C}^k$, the unit disk bundle of γ^k is given by $BU(k)$. So we get $MU(k) \simeq BU(k)/BU(k-1)$, which we can use to analogously define a cohomology operation into integral cohomology.

Definition 3.4. For a pair of spaces (X, A), define a map

$$\eta_{(X,A)} : \widetilde{MU}^*(X, A) \to \widetilde{H}^*(X, A; \mathbb{Z})$$

as follows: let $x \in \widetilde{MU}^n(X, A)$, and represent x by a map $f : S^{2k-n} \wedge (X/A) \to MU(k)$ for some large $k \in \mathbb{Z}_{\geq 0}$ using (3.1). Let $\gamma^k \to BU(k)$ be the tautological bundle. Then using $MU(k) \cong BU(k)/BU(k-1)$ we can identify $\widetilde{H}^*(MU(k))$ with the ideal in $H^*(BU(k))$ generated by the Euler class $c_k \in H^{2k}(BU(k))$ of γ^k. Let $c'_k \in H^{2k}(MU(k))$ be the element corresponding to c_k. Then define $\eta_{(X,A)}(x)$ as the image of c'_k under the composition

$$\widetilde{H}^{2k}(MU(n)) \xrightarrow{f^*} H^{2k}(S^{2k-n} \wedge (X/A)) \cong \widetilde{H}^n(X/A) \cong H^n(X, A).$$

Again, this defines a natural transformation of cohomology theories.

3.2 A cobordism interpretation for K-theory

In this section we show that K-theory may be rephrased in terms of complex cobordism, as stated in the following theorem, [CF66, 10.1 Theorem, p. 60]:

Theorem 3.5. *For every finite CW-pair (X, A), there is an isomorphism of $\mathbb{Z}/2$-graded rings*

$$K^*(X, A) \cong MU^*(X, A) \otimes_{MU^*(pt)} \mathbb{Z}.$$

Here $\mathbb{Z}$ is naturally an $MU^*(pt)$-module: for even gradings there is the homomorphism of abelian groups $\mu_{pt} : MU^{2n}(pt) \to K^{2n}(pt) \cong \mathbb{Z}$ of Definition 3.2, and $MU^{2n+1}(pt) = 0$ for all n by Corollary 2.11, so we have a ring homomorphism

$$\mu_{pt} : MU^*(pt) \to \mathbb{Z},$$

and for $x \in MU^*(pt)$ we can view $\mu_{pt}(x)$ as an integer. Write $x \cdot a$ for $\mu_{pt}(x) \cdot a,\ a \in \mathbb{Z}$.

Remark 3.6. Another formulation of Theorem 3.5 is

$$K^*(X,A) \cong MU^*(X,A) \otimes_{MU^*(pt)} K^*(pt),$$

where the isomorphism is one of $\mathbb{Z}$-graded rings. But using Bott periodicity and the fact that $K^{2n+1}(pt) = 0 = MU^{2n+1}(pt)$, this really is the same statement as in the formulation above. Furthermore, it can be shown that the homomorphism $\mu_{pt} : MU^*(pt) \to \mathbb{Z}$ is in fact the one the universal property of the Lazard ring $MU^*(pt)$ induces, meaning that the statement of Theorem 3.5 can entirely be viewed as in the fashion of Example 2.33.

We introduce some more maps and notation for the proof of Theorem 3.5. For a finite CW-pair (X,A), let

$$\Gamma^*(X,A) \coloneqq MU^*(X,A) \otimes_{MU^*(pt)} \mathbb{Z}.$$

We regard $\Gamma^*(X,A)$ as a $\mathbb{Z}/2$-graded ring by setting $\Gamma^0(X,A) \coloneqq MU^{\mathrm{ev}}(X,A) \otimes_{MU^*(pt)} \mathbb{Z}$, $\Gamma^1(X,A) \coloneqq MU^{\mathrm{odd}}(X,A) \otimes_{MU^*(pt)} \mathbb{Z}$. Then the map

$$\beta_{(X,A)} : MU^*(X,A) \to \Gamma^*(X,A),\ c \mapsto c \otimes 1$$

defines an epimorphism of $MU^*(pt)$-modules. Furthermore, the map

$$MU^*(X,A) \times \mathbb{Z} \to K^*(X,A),\ (c,n) \mapsto n \cdot \mu_{(X,A)}(c)$$

is bilinear and by the universal property of the tensor product induces the map

$$\hat{\mu}_{(X,A)} : MU^*(X,A) \otimes_{MU^*(pt)} \mathbb{Z} \to K^*(X,A),\ c \otimes n \mapsto n \cdot \mu_{(X,A)}(c),$$

which is a homomorphism of $\mathbb{Z}/2$-graded $MU^*(pt)$-modules, using the fact that $\mu_{(X,A)}$ is multiplicative. So the following diagram commutes:

$$\begin{array}{ccc} MU^*(X,A) & \xrightarrow{\beta} & \Gamma^*(X,A) \\ & \searrow^{\mu} & \downarrow{\hat{\mu}} \\ & & K^*(X,A) \end{array}$$

In Section 2.2 we used Theorem 7.5 of [CF66] to assign to a vector bundle $E \to X$ an element $c_t(E)$ in the multiplicative cohomology theory $MU^*(X)[\mathbf{t}]$. There is an analogous result for principal $U(n)$-bundles which also holds in general for multiplicative cohomology theories on finite CW-pairs, see [CF66, Theorem 7.6]. We will apply it for K-theory and state it in the version of Corollary 8.3 in [CF66].

Lemma 3.7. *There exists a unique function assigning to each principal $U(n)$-bundle $\xi \to X$, where X is a finite CW-complex, an element*

$$\underline{c}(\xi) = 1 + \underline{c}_1(\xi) + \ldots + \underline{c}_m(\xi) \in MU^*(X),$$

where $\underline{c}_k(\xi) \in MU^{2k}(X)$, such that

(1) the assignment $\xi \mapsto \underline{c}(\xi)$ commutes with pullbacks,

(2) *if* ξ, η *are principal* $U(n)$, $U(m)$*-bundles over* X *respectively, then* $\underline{c}(\xi \oplus \eta) = \underline{c}(\xi) \cdot \underline{c}(\eta)$, *and*

(3) *for the Hopf principal* $U(1)$*-bundle* $\xi_n : S^{2n+1} \to S^{2n+1}/S^1 = \mathbb{CP}^n$ *over* $\mathbb{CP}^n$*, we have* $\underline{c}(\xi_n) = 1 + e(i_n^*(\gamma_{\mathbb{C}}^{1,\infty}))$*, where* $e(i_n^*(\gamma_{\mathbb{C}}^{1,\infty}))$ *is the Euler class of the pullback of the tautological bundle* $\gamma_{\mathbb{C}}^{1,\infty}$ *over* $\mathbb{CP}^\infty$ *under the inclusion* $i_n : \mathbb{CP}^n \hookrightarrow \mathbb{CP}^\infty$.

Then property (2) in particular implies that $\underline{c}_1(\xi \oplus \eta) = \underline{c}_1(\xi) + \underline{c}_1(\eta)$ for principal $U(n)$- and $U(m)$-bundles $\xi, \eta \to X$. Using the fact that a principle $U(n)$-bundle is equivalent data to a complex vector bundle with fixed Hermitian inner product[5], there is a homomorphism

$$\underline{c}_1 : K^0(X) \to MU^2(X)$$

of abelian groups. Write $\Phi : \widetilde{K}^0(X) \xrightarrow{\sim} K^2(X)$ for the period isomorphism. By [CF66, p. 58] we have $\widetilde{K}^0(X) \xrightarrow{\underline{c}_1} \widetilde{MU}^2(X) \xrightarrow{\mu_X} \widetilde{K}^2(X) \xrightarrow{\Phi^{-1}} \widetilde{K}^0(X) = -\mathrm{id}_{\widetilde{K}^0(X)}$. Then we can define

$$\chi_X := \widetilde{K}^0(X) \cong K^2(S^2 \wedge X) \xrightarrow{\Phi^{-1}} K^0(S^2 \wedge X) \xrightarrow{\underline{c}_1} MU^2(S^2 \wedge X) \cong \widetilde{MU}^0(X),$$

giving rise to a map $\chi_{(X,A)} : K^0(X,A) \to MU(X,A)$ for pairs, for which again the composition with $\mu_{(X,A)}$ satisfies $K^0(X,A) \xrightarrow{\chi_{(X,A)}} \widetilde{MU}^2(X,A) \xrightarrow{\mu_{(X,A)}} K^0(X,A) = -\mathrm{id}_{K^0(X,A)}$. Setting

$$\hat{\chi}_{(X,A)} := K^*(X,A) \xrightarrow{\chi_{(X,A)}} MU^*(X,A) \xrightarrow{\beta} \Gamma^*(X,A),$$

we have $\hat{\mu}\hat{\chi} = \hat{\mu}\beta\chi = \mu\chi = -\mathrm{id}_{K^*(X,A)}$. In particular, $\hat{\mu}_{(X,A)}$ is always surjective. We can now use the defined operations to prove $K^*(X,A) \cong MU^*(X,A) \otimes_{MU^*(pt)} \mathbb{Z}$, where it remains to show the injectivity of $\hat{\mu}_{(X,A)}$.

Proof of Theorem 3.5. (1) First consider the case in which $H^*(X,A;\mathbb{Z})$ is a free abelian group which is trivial in odd gradings. In this case it follows from a spectral sequence argument (see e.g. [CF64, Theorem 18.1, p. 49]) that there is a homogeneous basis (α_j) for the $MU^*(pt)$-module $MU^*(X,A)$, such that $(\eta_{(X,A)}(\alpha_j))$ is a basis for $H^*(X,A)$, where we use the cohomology transformation $\eta_{(X,A)} : MU^*(X,A) \to H^*(X,A)$, as introduced in Definition 3.4. It then follows that

$$MU^*(X,A) \cong H^*(X,A) \otimes_{\mathbb{Z}} MU^*(pt)$$

as $MU^*(pt)$-modules via the mapping $\alpha_j \mapsto \eta_{(X,A)}(\alpha_j) \otimes 1$.
For $\mu_{(X,A)} : MU^*(X,A) \to K^*(X;A)$ from Definition 3.2 the composition $ch \circ \mu_{(X,A)}$ with the Chern character $ch : K^*(X;A) \to H^*(X,A;\mathbb{Q})$ by [CF66, Theorem 6.4, p. 35] maps α_j to an element of $H^*(X,A;\mathbb{Q})$ which in the highest degree has the element $\pm\eta_{(X,A)}(\alpha_j)$. Using the Atiyah-Hirzebruch spectral sequence, the $\mu_{(X,A)}(\alpha_j)$ generate $K^*(X,A)$ as a $\mathbb{Z}/2$-graded free $K^*(pt)$-module.
Now consider the kernel of $\mu_{(X,A)}$: from $\hat{\mu}\beta = \mu$ it follows $\ker(\beta_{(X,A)}) \subseteq \ker(\mu_{(X,A)})$. Conversely, assume that an $MU^*(pt)$-linear combination of the α_j is sent to zero by $\mu_{(X,A)}$. Then, as the $\mu_{(X,A)}(\alpha_j)$ generate $K^*(X,A)$ as a free $K^*(pt)$-module, the coefficients must be in the kernel of the homomorphism making $K^*(pt) = \mathbb{Z}$ an $MU^*(pt)$-module, which is the map

[5] Taking isomorphism classes, the map assigning to a principle $U(n)$-bundle $\xi \to X$ the n-dimensional complex vector bundle with Hermitian inner product $\xi \times_{U(n)} \mathbb{C}^n \to X$ is bijective, because the $U(n)$-representation on $\mathbb{C}^n$ is faithful.

$\mu_{pt} : MU^*(pt) \to \mathbb{Z}$. Then $\beta_{(X,A)} : MU^*(X,A) \to \Gamma^*(X,A)$ sends the combination to zero as well, as it is an $MU^*(pt)$-module homomorphism. This shows $\ker(\mu_{(X,A)}) \subseteq \ker(\beta_{(X,A)})$, so the kernels are equal. As $\beta_{(X,A)}$ is surjective, $\hat{\mu}_{(X,A)}$ must be injective, so we have an isomorphism $\hat{\mu}_{(X,A)} : \Gamma^*(X,A) \cong K^*(X,A)$.

(2) Now consider the case in which $a \in \ker(\hat{\mu}_{(X,A)}) \subseteq \Gamma^*(X,A)$ such that there exists a map of pairs $f : (X,A) \to (Y,B)$ for which $H^*(Y,B)$ is free abelian with generators in even grading and $a = f^*(b)$ for some $b \in \Gamma^*(Y,B)$. We will show that this implies $a = 0$. The diagram in this situation looks like this:

$$\begin{array}{ccc} \Gamma^*(Y,B) & \xrightarrow{f^*,\ b \mapsto a} & \Gamma^*(X,A) \\ \hat{\chi}_{(Y,B)} \uparrow\downarrow \hat{\mu}_{(Y,B)} & & \hat{\chi}_{(X,A)} \uparrow\downarrow \hat{\mu}_{(X,A)},\ a \mapsto 0 \\ K^*(Y,B) & \xrightarrow[f^*]{} & K^*(X,A) \end{array}$$

As $\underline{c}$ from Lemma 3.7 commutes with pullbacks, χ is natural, and β is natural as well, so $\hat{\chi}$ is natural. Similarly, naturality of $\hat{\mu}$ follows from naturality of μ. So the diagram commutes. By the assumptions we made on $H^*(Y,B)$ we know from part (1) that $\hat{\mu}_{(Y,B)}$ is an isomorphism, and with $\hat{\mu}\hat{\chi} = -\mathrm{id}$ it follows that $\hat{\chi}_{(Y,B)}$ is surjective. So there is a $b' \in K^*(Y,B)$ with $\hat{\chi}_{(Y,B)}(b') = b$. Then $a = f^*(b) = f^*(\hat{\chi}_{(Y,B)}(b')) = \hat{\chi}_{(X,A)}(f^*(b'))$, so $0 = \hat{\mu}_{(X,A)}(a) = \hat{\mu}_{(X,A)}(\hat{\chi}_{(X,A)}(f^*(b'))) = -f^*(b')$, and $-a = \hat{\chi}_{(X,A)}(-f^*(b')) = \hat{\chi}_{(X,A)}(0) = 0$, so $a = 0$.

(3) For the general case one separately shows that $\hat{\mu}_{(X,A)} : \Gamma^j(X,A) \to K^j(X,A)$ is an isomorphism for $j = 0$ and $j = 1$. It suffices to consider the even-graded case $j = 0$, as the other one is analogous. Furthermore, as β is surjective, it will be enough to consider a general element in complex cobordism. So let $a \in MU^{\text{ev}}(X,A)$, which we can write as $a = a_{2k} + a_{2k+2} + \ldots + a_{2k+2l}$ with $a_{2k+2i} \in MU^{2k+2i}(X,A),\ 0 \le i \le l$. By [CF66, Corollary 6.5, p. 37] the composition

$$MU_{2m}(pt) \cong MU^{-2m}(pt) \xrightarrow{\mu_{pt}} K^0(pt) \cong \mathbb{Z}$$

maps the bordism class $[M^{2m}]$ of a closed weakly compact manifold to $(-1)^m \cdot Td[M^{2m}]$, where $Td[M^{2m}]$ is the Todd genus[6] of the manifold M^{2m}, and $MU_*(pt)$ is the complex bordism ring, the homology ring of the homology theory with values in the Thom spectrum MU. In particular, there exists an element $b_{-2} \in MU^{-2}(pt)$ with $\mu_{pt}(b_{-2}) = 1$. With this element it follows $\beta_{(X,A)}(a) = \beta_{(X,A)}(\underbrace{a_{2k} + a_{2k+2} \cdot b_{-2} + \ldots + a_{2k+2l} \cdot (b_{-2})^l}_{\in MU^{2k}(X,A)})$, so there is an element $a' \in MU^{2k}(X,A)$ with $\beta_{(X,A)}(a) = \beta_{(X,A)}(a')$. Using (3.1) represent a' by a map

$$f : S^{2n} \wedge (X/A) \to MU(k+n)$$

for n large enough. Performing $2n$ suspensions of X/A and using suspension isomorphisms we obtain an element $S^{2n}(a') \in \widetilde{MU}^{2k+2n}(S^{2n} \wedge (X/A))$, which is in the image of the map

$$f^* : \widetilde{MU}^*(MU(k+n)) \to \widetilde{MU}^*(S^{2n} \wedge (X/A))$$

[6]The *Todd genus* of an almost complex manifold M^n is defined as follows: with $T(x) := \frac{x}{1-\exp(-x)}$ we define $T_n(c_1,\ldots,c_n)$ by $1 + T_1(c_1)x + T_2(c_1,c_2)x^2 + \ldots + T_n(c_1,\ldots,c_n)x^n + \ldots = T(c_1 \cdot x) \cdot \ldots \cdot T(c_n \cdot x)$, then $Td[M^n] := T_n(c_1,\ldots,c_n)[M^n] \in \mathbb{Q}$, where the $c_1,\ldots,c_n$ are the Chern classes of the holomorphic tangent bundle of M^n, and $[M^n] \in H_{2n}(M^n;\mathbb{Z})$ is its fundamental homology class. Cf. [Hir95, p. 93].

on reduced cobordism. Hence, $S^{2n}(\beta_{(X,A)}(a)) = S^{2n}(\beta_{(X,A)}(a'))$ is in the image of the map

$$f^* : \widetilde{\Gamma}^*(MU(k+n)) \to \widetilde{\Gamma}^*(S^{2n} \wedge (X/A))$$

on tensored cobordism. Then if $\hat{\mu}_{(X,A)} : \Gamma^*(X,A) = \widetilde{\Gamma}^*(X/A) \to K^*(X,A)$ maps $\beta_{(X,A)}(a)$ to zero, then $\hat{\mu}_{S^n \wedge (X/A)} : \widetilde{\Gamma}^*(S^n \wedge (X/A)) \to \widetilde{K}^*(S^n \wedge (X/A))$ maps $S^{2n}(\beta_{(X,A)}(a))$ to zero. But now we are in the situation of part (2), as $S^{2n}(\beta_{(X,A)}(a))$ is in the image of f^*, and the integral cohomology $H^*(MU(k+n))$ is free abelian and vanishes in odd grading. So $S^{2n}(\beta_{(X,A)}(a)) = 0$, and therefore $\beta_{(X,A)}(a) = 0 \in \Gamma^*(X,A)$. Thus, $\hat{\mu}_{(X,A)}$ is injective on even gradings, and we have an isomorphism

$$\hat{\mu}_{(X,A)} : \Gamma^0(X,A) \xrightarrow{\sim} K^0(X,A).$$

Repeating the argument for odd gradings shows that $\hat{\mu}_{(X,A)} : \Gamma^1(X,A) \to K^1(X,A)$ is an isomorphism as well, so it follows $\Gamma^*(X,A) \cong K^*(X,A)$ as $\mathbb{Z}/2$-graded rings, which concludes the proof. □

Remark 3.8. An important thing to notice regarding Theorem 3.5 is that its proof heavily relied on the existence of the cohomology operation $\mu : MU^* \to K^*$, which closely related both theories. Lemma 2.3 under assumptions ensures the existence of transformations $MU^* \to h$. As pointed out in Example 2.33, one can not expect that $MU^*(-) \otimes_{MU^*(pt)} R$ defines a cohomology theory for R an arbitrary $MU^*(pt)$-module. One could expect that the situation is different when you start with a multiplicative cohomology theory $h^* : \mathbf{CW}_{\mathrm{f}}^{\mathrm{op}} \to \mathbf{AbGp}_{\bullet}$ and form the functor $MU^*(-) \otimes_{MU^*(pt)} h^*(pt)$ using a given homomorphism $MU^*(pt) \to h^*(pt)$. Although MU^* and h^* might be related by a cohomology operation in the first place, $h^*(X) \cong MU^*(X) \otimes_{MU^*(pt)} h^*(pt)$ might not hold, and furthermore $MU^*(-) \otimes_{MU^*(pt)} h^*(pt)$ need not be a cohomology theory. With integral cohomology we will see an example for this in the next section.

4 The Landweber exact functor theorem

This section is concerned with the question when the contravariant functor

$$R^*(-) \coloneqq MU^*(-) \otimes_{MU^*(pt)} R : \mathbf{CW}_{\mathrm{f}}^{\mathrm{op}} \to \mathbf{AbGp}_{\bullet}$$

defines a cohomology theory. Instead of cohomology, we will be considering the homology theory complex bordism as defined in Section 1 via $\widetilde{MU}_n(X) = \pi_n(MU(k) \wedge X)$, k large. Recall that $E_*(pt) = \pi_*(E) = [\mathbb{S}, E] = [\Sigma^\infty pt_+, E] = E^*(pt)$ holds for any CW-spectrum E, where $\mathbb{S} = (S^n)_{n \geq 0}$ is the sphere spectrum. So we see that the complex bordism ring $MU_*(pt)$ is isomorphic to the cobordism ring $MU^*(pt)$, and we write it as $MU_*(pt) \cong \mathbb{Z}[x_1, x_2, \ldots]$, where each generator x_i can be chosen in $MU_{2i}(pt)$ (so the grading has opposite signs compared to the complex cobordism ring). If the covariant functor

$$R_*(-) \coloneqq MU_*(-) \otimes_{MU_*(pt)} R : \mathbf{CW} \to \mathbf{AbGp}_{\bullet}$$

is a homology theory for an $MU_*(pt)$-module R, then the analogous contravariant theory from above will also be a cohomology theory on finite CW-complexes (which can be shown by precomposing with Spanier-Whitehead duality).

We already noticed in Example 2.33 that the assumption that R is a flat $MU_*(pt)$-module (i.e.,

$\mathrm{Tor}_1^{MU_*(pt)}(N,R) = 0$ for all $MU_*(pt)$-modules N) makes $R_*(-)$ into a homology theory, as it always satisfies homotopy invariance, excision and additivity. So flatness of R ensures $R_*(-)$ has long exact sequences. As this flatness assumption is too strong in practise, one natural approach is to relax it to the weaker condition $\mathrm{Tor}_1^{MU_*(pt)}(MU_*(X),R) = 0$ for finite CW-complexes X. This approach leads to the homological condition *Landweber flatness* which we will elaborate on by following [Lan76], where it was first introduced in the form of the *exact functor theorem.*

4.1 Landweber flatness

Let **MU** denote the category of comodules over the self homology[7] $MU_*(MU)$ which are finitely presented as $MU_*(pt)$-modules[8]. Note this is an additive category, and in particular it makes sense to speak about exactness in this category. If X is a finite CW-complex, then $MU_*(X)$ is a finitely presented $MU_*(pt)$-module, and $MU_*(X)$ has a natural comodule structure

$$\psi : MU_*(X) \to MU_*(MU) \otimes_{MU_*(pt)} MU_*(X)$$

as defined in [Ada69, Lecture 3]. So $MU_*(X)$ itself is an object of **MU**. Now fix generators $x_i \in MU_{2i}(pt)$ of the bordism ring $MU_*(pt) \cong \mathbb{Z}[x_1, x_2, \ldots]$ such that for each prime p, all Chern numbers of x_{p^n-1} are divisible by p (e.g., we could take $x_{p-1} = [\mathbb{CP}^{p-1}]$, using the geometric interpretation of complex bordism in terms of manifolds). Then we can define the prime ideals

$$I(p,n) := (p, x_{p-1}, \ldots, x_{p^{n-1}-1}) \trianglelefteq MU_*(pt)$$

for all primes p and $n \geq 0$ (the quotient $MU_*(pt)/I(p,n)$ is isomorphic to a polynomial ring over $\mathbb{Z}/p$ which is an integral domain, so the ideals are indeed prime). We now define the notion of Landweber flatness for $MU_*(pt)$-modules.

Definition 4.1. An $MU_*(pt)$-module R is *Landweber flat* if for each prime p and each integer $n > 0$, multiplication by p on R and by x_{p^n-1} on $R/I(p,n)R$ is injective (i.e., these elements are not zero divisors in the respective rings).

The following theorem asserts that this condition is sufficient to preserve exactness when tensoring with R on the category **MU**.

Theorem 4.2 (Landweber exact functor theorem)**.** *Let R be an $MU_*(pt)$-module. Then the functor*

$$\textbf{\textit{MU}} \to \textbf{\textit{AbGp}}_\bullet, \; M \mapsto M \otimes_{MU_*(pt)} R$$

is exact if and only if R is Landweber flat.

From the fact that the functor $R^*(-) := MU^*(-) \otimes_{MU^*(pt)} R$ only possibly misses exactness and that $MU_*(X)$ itself is an object of **MU** for finite CW-complexes X, we can deduce the following:

[7] The maps $MU \wedge MU \to MU$ and $\mathbb{S} \to MU$ introduced in Section 1 can be used to define a diagonal map $\Delta : MU_*(MU) \to MU_*(MU) \otimes_{MU_*(pt)} MU_*(MU)$ and a counit map $\varepsilon : MU_*(MU) \to MU_*(pt)$ satisfying $(\mathrm{id} \otimes \Delta) \otimes \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$ and $(\mathrm{id} \otimes \varepsilon) \otimes \Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \Delta$. That is, they make $MU_*(MU)$ into an $MU_*(pt)$-coalgebra. For the definition of the maps see [Ada69, Lecture 3].

[8] So objects of **MU** are finitely presented $MU_*(pt)$-modules R with a linear map $\rho : R \to R \otimes MU_*(MU)$ satisfying $(\mathrm{id} \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}) \circ \rho$ and $(\mathrm{id} \otimes \varepsilon) \circ \rho = \mathrm{id}$ for Δ and ε the maps of the $MU_*(pt)$-coalgebra $MU_*(MU)$.

Corollary 4.3. *If R is Landweber flat, then the functors*

$$R_*(-) = MU_*(-) \otimes_{MU_*(pt)} R : \boldsymbol{CW}_{\mathrm{f}} \to \boldsymbol{AbGp}_{\bullet} \text{ and}$$
$$R^*(-) = MU^*(-) \otimes_{MU^*(pt)} R : \boldsymbol{CW}_{\mathrm{f}}^{\mathrm{op}} \to \boldsymbol{AbGp}_{\bullet}$$

define generalized homology respectively cohomology theories.

Remark 4.4. For the homology theory $R_*(-)$ we can also replace $\mathbf{CW}_{\mathrm{f}}$ by the category of CW-complexes **CW**, as tensor products commute with direct limits. But in the cohomology case, this is not true in general. It can in fact be shown that Landweber flatness is equivalent to $R^*(-)$ being a cohomology on the finite CW-complexes.

Remark 4.5. Corollary 4.3 can be used to show isomorphisms of (co-)homology theories: if E is a module spectrum over the ring spectrum MU via a map of spectra $MU \to E$, then this induces homomorphisms $MU_*(X) \to E_*(X)$ for all CW-complexes X, and we have a naturality square

$$\begin{array}{ccc} MU_*(pt) & \longrightarrow & E_*(pt) \\ \downarrow & & \downarrow \\ MU_*(X) & \longrightarrow & E_*(X) \end{array}$$

which by the universal property of the tensor product induces a comparison map

$$MU_*(X) \otimes_{MU_*(pt)} E_*(pt) \to E_*(X).$$

Now if $E_*(pt)$ is Landweber flat, then varying X the left-hand side by Corollary 4.3 defines a homology theory, and using Zeeman's comparison theorem for homology theories it can be shown that the comparison map from above defines an isomorphism of homology theories. This can analogously be done for cohomology, where the result again holds for finite CW-complexes. Furthermore, it can be shown that the complex Thom spectrum MU is initial within the CW-spectra E defining complex oriented cohomology theories. So if for such E the cohomology ring $E^*(pt)$ happens to be Landweber flat, then we know there is an isomorphism of cohomology theories

$$MU^*(-) \otimes_{MU^*(pt)} E^*(pt) \xrightarrow{\sim} E^*(-)$$

which holds on the category of finite CW-complexes. We will see below that this generalizes Theorem 3.5, the cobordism interpretation of K-theory.

The proof of Theorem 4.2 uses the following result which we will quote.

Theorem 4.6 (Filtration theorem for MU)**.** *We have the following:*

(1) Every finitely generated prime ideal $\mathfrak{p} \trianglelefteq MU_*(pt)$ which additionally is a subcomodule of $MU_*(pt)$ is either the zero ideal or of the form $I(p,n)$ for some prime p and some $n \geq 0$.

*(2) Each object M of **MU** has a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

*in **MU**, so that $M_i/M_{i-1} \cong MU_*(pt)/\mathfrak{p}_i$ in **MU** after possibly a shift of grading for some prime ideal $\mathfrak{p}_i \trianglelefteq MU_*(pt)$ of the kind from (1), for each $1 \leq i \leq n$.*

For the proof, we refer to [Lan73a, Corollary 2.2, p. 274] for (1) and to [Lan73b, Theorem 3.3', p. 48] for (2). We will also need the following:

Proposition 4.7. *For each $M \in$ ob* ***MU****, there exists an $F \in$ ob* ***MU*** *which is a free module over $MU_*(pt)$ and admits an epimorphism $F \twoheadrightarrow M$.*

The proof can be found in [Lan76, Proposition 2.4, p. 594]. Using these results about the structure of **MU**, we can prove Theorem 4.2.

Proof of Theorem 4.2. For better readability, write MU_* for $MU_*(pt)$. Consider the short exact sequences

$$0 \longrightarrow MU_* \xrightarrow{\cdot p} MU_* \longrightarrow MU_*/(p) \longrightarrow 0$$

for each prime p, and

$$0 \longrightarrow MU_*/I(p,n) \xrightarrow{\cdot x_{p^n-1}} MU_*/I(p,n) \longrightarrow MU_*/I(p,n+1) \longrightarrow 0$$

for every prime p and $n > 0$. The first sequence is a free resolution of $MU_*/(p)$ by MU_*-modules, so for an MU_*-module R the multiplication by p map $R \to R$ is injective if and only if the map $p \otimes \mathrm{id}_R : MU_* \otimes_{MU_*} R \to MU_* \otimes_{MU_*} R$ is injective, and this is equivalent to $\mathrm{Tor}_1^{MU_*}(MU_*/(p), R) = 0$. The second sequence may not be a projective resolution, but it induces a long exact sequence containing

$$\longrightarrow \mathrm{Tor}_1^{MU_*}(MU_*/I(p,n), R) \longrightarrow \mathrm{Tor}_1^{MU_*}(MU_*/I(p,n+1), R) \longrightarrow$$
$$\longrightarrow MU_*/I(P,n) \otimes_{MU_*} R \longrightarrow MU_*/I(p,n) \otimes_{MU_*} R \longrightarrow$$

Now multiplication by x_{p^n-1} on $R/I(p,n)$ is injective if and only if the map on tensor products $x_{p^n-1} \otimes \mathrm{id}_R : MU_*/I(p,n) \otimes_{MU_*} R \to MU_*/I(p,n) \otimes_{MU_*} R$ is injective, which by the above is equivalent to

$$\mathrm{Tor}_1^{MU_*}(MU_*/I(p,n), R) \to \mathrm{Tor}_1^{MU_*}(MU_*/I(p,n+1), R)$$

being surjective, and in this case the right-hand side is zero if the left is. So the module R is Landweber flat if and only if $\mathrm{Tor}_1^{MU_*}(MU_*/I(p,n), R) = 0$ for all n and p (where we write $I(p,1) = (p)$). Then for each $M \in$ ob **MU**, Theorem 4.6 gives a filtration

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

in **MU** for which $M_i/M_{i-1} \cong MU_*/\mathfrak{p}$ holds after a shift of grading, where $\mathfrak{p} = (p)$ or $\mathfrak{p} = I(p,n)$ for some p and n, so in the Landweber flat case we have $\mathrm{Tor}_1^{MU_*}(M_i/M_{i-1}, R) = 0$ for $1 \leq i \leq n$, which inductively shows $\mathrm{Tor}_1^{MU_*}(M_i, R) = 0$ for $0 \leq i \leq n$ by considering the long exact sequence of $\mathrm{Tor}_\bullet^{MU_*}(-, R)$ for each pair (M_i, M_{i-1}) in degree 1. This shows that Landweber flatness of R is equivalent to $\mathrm{Tor}_1^{MU_*}(M, R) = 0$ for all $M \in$ ob **MU**, which implies that $M \mapsto M \otimes_{MU_*} R$ is exact on **MU**.

Conversely, assuming this exactness, let $M \in$ ob **MU**, and using Proposition 4.7 choose a short exact sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$$

where F is a free MU_*-module (and in particular $\mathrm{Tor}_1^{MU_*}(F,R) = 0$). The induced long exact sequence reads

$$\longrightarrow \mathrm{Tor}_1^{MU_*}(N,R) \longrightarrow \underbrace{\mathrm{Tor}_1^{MU_*}(F,R)}_{=0} \longrightarrow \mathrm{Tor}_1^{MU_*}(M,R) \longrightarrow N \otimes_{MU_*} R \rightarrowtail F \otimes_{MU_*} R \longrightarrow M \otimes_{MU_*} R \longrightarrow 0$$

where exactness being preserved under tensoring gives the injection in the bottom row. So the connecting homomorphism is the zero map, and it follows $\mathrm{Tor}_1^{MU_*}(M,R) = 0$. We conclude that R is Landweber flat. □

4.2 Applications

In this last section, we show how Theorem 4.2 can be applied to form new (co-)homology theories, and also in which cases it can not be applied.

Example 4.8. Consider $R = \mathbb{Q}$ with additive formal group law $G(x,y) = x + y \in \mathbb{Q}[\![x,y]\!]$, which is the formal group law corresponding to rational (co-)homology. The associated homomorphism making $\mathbb{Q}$ an $MU_*(pt)$-module is the map

$$\varphi : MU_*(pt) \cong \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Q},\ x_i \mapsto 0.$$

Multiplication by p on $\mathbb{Q}$ is injective, and with $I(p,n) = \mathbb{Q}$ for $n > 0$ multiplication by $x_{p^n-1} = 0$ is injective on $\mathbb{Q}/I(p,n) = 0$. So Theorem 4.2 applies, $MU_*(-) \otimes_{MU_*(pt)} \mathbb{Q}$ is a homology theory, and with Remark 4.5 there is an isomorphism

$$MU_*(X) \otimes_{MU_*(pt)} \mathbb{Q} \cong H_*(X;\mathbb{Q})$$

on CW-complexes. The analogous results hold for cohomology on finite CW-complexes.

Example 4.9. Now consider $R = \mathbb{Z}$ with additive formal group law $G(x,y) = x + y \in \mathbb{Z}[\![x,y]\!]$, corresponding to integral (co-)homology. The associated homomorphism is

$$\varphi : MU_*(pt) \cong \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z},\ x_i \mapsto 0.$$

Then multiplication by $x_{p-1} = 0 \in \mathbb{Z}$ on $\mathbb{Z}/I(p,1) = \mathbb{Z}/(p)$ is certainly not injective, so $\mathbb{Z}$ (with this $MU_*(pt)$-module structure) is not Landweber flat. It can be shown that the functor $MU_*(-) \otimes_{MU_*(pt)} \mathbb{Z}$ does not satisfy exactness, and so does not define a homology theory. One can also explicitly construct CW-complexes for which $MU_*(X) \otimes_{MU_*(pt)} \mathbb{Z} \cong H_*(X;\mathbb{Z})$ fails[9].

Example 4.10. We can also show that the cobordism interpretation of K-theory we proved with Theorem 3.5 is a consequence of the Landweber exact functor theorem (Theorem 4.2). For this consider $R = \mathbb{Z}$ with the multiplicative formal group law $G(x,y) = x + y - xy \in \mathbb{Z}[\![x,y]\!]$,

[9] For an example, see https://mathoverflow.net/questions/346470/example-of-a-space-x-exhibiting-the-landweber-non-exactness-of-the-additive-form.

corresponding to ($\mathbb{Z}/2$-graded) K-homology. Then the associated homomorphism (up to a sign) is

$$\varphi : MU_*(pt) \to \mathbb{Z},\ [M] \mapsto Td[M],$$

the Todd genus of the manifold M (as defined in [Hir95, p. 93]). By the choice of generators we made before, $x_{p-1} = [\mathbb{CP}^{p-1}]$ has Todd genus 1. Multiplication by p on $\mathbb{Z}$ is injective, multiplication by $x_{p-1} = 1 \in \mathbb{Z}$ on $\mathbb{Z}/(p) = \mathbb{Z}/I(p,1)$ is injective, and with $I(p,n) = \mathbb{Z}$ for $n \geq 2$ it follows that multiplication by x_{p^n-1} on $\mathbb{Z}/I(p,1) = 0$ is also injective. So with this module structure, $\mathbb{Z}$ is in fact Landweber flat.
In Section 3 we introduced the transformation $\mu : MU^* \to K^*$, and it had an associated transformation $\hat{\mu} : MU^*(-) \otimes_{MU^*(pt)} \mathbb{Z} \to K^*(-)$ of $\mathbb{Z}/2\mathbb{Z}$-graded cohomology theories. Now $\mu_{pt} : MU_*(pt) \to K_*(pt)$ agrees (up to a sign) with $\varphi : MU_*(pt) \to \mathbb{Z}$. So Remark 4.5 gives an isomorphism

$$\hat{\mu}_X : MU^*(X) \otimes_{MU^*(pt)} \mathbb{Z} \xrightarrow{\sim} K^*(X).$$

Equivalently, we could consider the $\mathbb{Z}$-graded case by taking $R = K_*(pt) = \mathbb{Z}[\beta, \beta^{-1}]$, $|\beta| = 2$, with multiplicative formal group law $G(x,y) = x + y - \beta xy \in \mathbb{Z}[\![x,y]\!]$ corresponding to $\mathbb{Z}$-graded K-homology. Then the associated homomorphism is

$$\varphi : MU_*(pt) \to \mathbb{Z},\ [M^{2n}] \mapsto T[M^{2n}]\beta^n.$$

Analogously, $\mathbb{Z}[\beta, \beta^{-1}]$ is Landweber flat, $MU_*(-) \otimes_{MU_*(pt)} \mathbb{Z}[\beta, \beta^{-1}]$ is a homology theory, and we have an isomorphism

$$\hat{\mu}_X : MU^*(X) \otimes_{MU^*(pt)} \mathbb{Z} \xrightarrow{\sim} K^*(X)$$

of the $\mathbb{Z}$-graded cohomology theories on finite CW-complexes.

Example 4.11. The last example we consider is of a somewhat more exotic flavour: let $\delta, \varepsilon \in \mathbb{C}$ with discriminant $\Delta := \varepsilon(\delta^2 - \varepsilon)^2 \neq 0$. Then we can consider an elliptic curve $C \subseteq \mathbb{CP}^2$ given by (homogenizing) the Jacobi quartic equation

$$y^2 = R(x) = 1 - 2\delta x^2 + \varepsilon x^4.$$

It comes with a formal group law $F_R(t_1, t_2) = \frac{t_1\sqrt{R(t_1)} + t_2\sqrt{R(t_2)}}{1 - \varepsilon t_1^2 t_2^2}$, which can be extended as a power series over the ring $\mathbb{Z}[\frac{1}{2}, \delta, \varepsilon, \Delta^{-1}]$. Checking Landweber flatness is a bit more subtle in this situation, and we refer to [Fra92, p. 5]. We then get the cohomology theory

$$E^*(-) := MU^*(-) \otimes_{MU^*(pt)} \mathbb{Z}[\frac{1}{2}, \delta, \varepsilon, \Delta^{-1}]$$

associated to the elliptic curve C on finite CW-complexes. More generally, given an elliptic curve C with homogenized Weierstrass form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

by Hensel's lemma there is a unique power series $w \in \mathbb{Z}[a_1, \ldots, a_6][\![t]\!]$ so that $(t, w(t))$ gives a parametrization of C on the affine piece $Y \neq 0$. Another application of Hensel's lemma then gives a power series $F \in \mathbb{Z}[a_1, \ldots, a_6][\![t_1, t_2]\!]$ with

$$F(0,0) = 0 \text{ and } (t_1, w(t_1)) + (t_2, w(t_2)) = (F(t_1, t_2), w(F(t_1, t_2))).$$

The group law of the elliptic curve C ensures that F is a formal group law, called the formal group law *associated to* C.
This motivates to consider cohomology theories arising from group laws over elliptic curves in more generality, which are called *elliptic cohomology theories*. They consist of

(i) a multiplicative cohomology theory E which is even periodic, i.e. $E^{2n+1}(pt) = 0$ for all $n \in \mathbb{Z}$ with an element $\beta \in E^{-2}(pt)$ invertible in the ring $E^*(pt)$,

(ii) an elliptic curve C over a commutative ring R, and

(iii) an isomorphism $E^*(pt) \cong R$ as well as an isomorphism of the formal group law from E and the one associated to C over $E^*(pt) \cong R$.

It is then of interest to know if there is some kind of universal elliptic cohomology theory, which would belong to some universal elliptic curve. As there is no canonical choice of Weierstrass equation, this question can not be answered with a single elliptic curve. This search leads to the consideration of *topological modular forms*, where one considers not a single elliptic curve, but a whole *moduli stack* of those.

To conclude the essay, we briefly want to indicate how the procedure of Quillen which associates formal group laws to complex oriented cohomology theories opened up new areas of research.
Recall that for a formal group law $F \in R[\![x,y]\!]$ over a ring R and $n \in \mathbb{Z}_{\geq 0}$, we defined the n-series $[n]_F(t) \in R[\![t]\!]$ recursively by $[1]_F(t) := t$ and $[n]_F(t) := F(t, [n-1]_F(t))$. It gives a homomorphism of formal group laws from F to itself:

$$F([n]_F(x), [n]_F(y)) = [n]_F(F(x,y)).$$

Then one shows that if R is a commutative ring in which a prime number p is zero (i.e., R is a $\mathbb{Z}/p$-algebra), then either $[p]_F(t) = 0$, or $[p]_F(t) = \lambda t^{p^n}$ + terms of degree $\geq p^n + 1$, for some $n > 0$ and $\lambda \in R$.
Fixing a prime number p and letting v_n denote the coefficient of t^{n^p} in the p-series $[p]_F(t)$, we say F has *height* $\geq n$ if $v_i = 0$ for $i < n$, and *height* n if it has height $\geq n$ and $v_n \in R$ is invertible. Furthermore, F has *height* ∞ if it has height $\geq n$ for all $n \geq 0$. Then the height of formal group laws can be used to organize complex oriented cohomology theories by their formal groups, which is the *chromatic* point of view in stable homotopy theory.
To illustrate this, we restrict to perfect fields. First assume that k is an algebraically closed field. Then if $\mathrm{char}(k) = 0$, it can be shown that every formal group law is isomorphic to the additive group law $F(x,y) = x + y$ over k, so in this case the situation is not very interesting. If $\mathrm{char}(k) > 0$, then formal group laws are in fact determined by their height (which is a positive integer or ∞). For example, the multiplicative formal group law $F(x,y) = x + y + xy$ over k has p-series $[p]_F(t) = (1+t)^p - 1 = t^p$, so it has height 1. The additive formal group law $F(x,y) = x + y$ over k has p-series $[p]_F(t) = 0$, so it has height ∞.
Now if $F(x,y) \in k[\![x,y]\!]$ is a formal group law of height $0 < n < \infty$ over a perfect field k, then by a theorem of Morava it comes from an even periodic cohomology theory $K(n)$, which is called *Morava K-theory* and said to *lie on the n-th chromatic level*.
For instance, if $k = \mathbb{F}_p$ and $F(x,y) = x + y + xy$ is the multiplicative group law which has height 1, then $K(1)$ can be taken as the mod p reduction of complex K-theory.
Modern research in chromatic homotopy theory tries to understand the behaviour of these chromatic levels and to make use of them, e.g. in the form of chromatic towers of spectra to compute homotopy groups of spheres.

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