3 Thick tensor ideals, localisations, rigidity

Recall : $2\mathrm{CAlg} := \mathrm{CAlg}^{\times}(\mathrm{Cat}_\infty^{\mathrm{perf}})$ \quad idempotent complete
\hspace{6cm} stably sym. mon.
\hspace{6.5cm} ∞ - categories

$$\wr$$

$$\mathrm{CAlg}^{\otimes}(\mathrm{Pr}^L_{\omega, \mathrm{st}})$$ \quad stable compactly gen.
\hspace{5.5cm} presentably sym. mon.
\hspace{5.7cm} ∞ - categories

admits a sym. mon.
$\wr$ enhancement

$$\mathrm{Sp}[-] : \mathrm{Cat}_\infty \; \underset{\longleftarrow}{\longrightarrow} \; \mathrm{Pr}^L_{\omega, \mathrm{st}} \; : \; (-)^\omega$$

$$\mathcal{C} \; \longmapsto \; \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$$
$$\simeq \mathrm{Fun}^L(\mathcal{P}(\mathcal{C}^{\mathrm{op}}), \mathrm{Sp})$$
$$\simeq \mathrm{Fun}^R(\mathcal{P}(\mathcal{C})^{\mathrm{op}}, \mathrm{Sp})$$
$$\simeq \mathrm{Sp} \otimes \mathcal{P}(\mathcal{C})$$

$\rightsquigarrow \quad \mathrm{Sp}[-] : \; \mathrm{CAlg}(\mathrm{Cat}_\infty) \; \underset{\longleftarrow}{\longrightarrow} \; \mathrm{CAlg}(\mathrm{Pr}^L_{\omega, \mathrm{st}}) \; : \; (-)^\omega$

$$\mathcal{C} \; \longmapsto \; \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$$
\hspace{3cm} equipped with Day convolution
\hspace{3cm} sym. mon. structure

Free 2-ring functor: $\; \mathrm{Sp}^\omega[-] : \; \mathrm{CAlg}(\mathrm{Cat}_\infty) \xrightarrow{\mathrm{Sp}[-]} \mathrm{CAlg}(\mathrm{Pr}^L_{\omega, \mathrm{st}}) \xrightarrow{(-)^\omega} 2\mathrm{CAlg}$

Free - forgetful adjunction : $\; \mathrm{Sp}^\omega[-] : \mathrm{CAlg}(\mathrm{Cat}_\infty) \; \underset{\longleftarrow}{\longrightarrow} \; 2\mathrm{CAlg} : \mathrm{fgt}$
\hspace{10.5cm} $\wr$
unit: $\Sigma^\infty_+ \circ \iota : \mathcal{C} \longrightarrow \mathrm{Sp}^\omega[\mathcal{C}] \subseteq \mathrm{Sp}[\mathcal{C}]$ \hspace{2.5cm} forgets stability

$\mathrm{Fin}^{\simeq}$ with cocartesian sym. mon. structure is the free sym. mon.
∞ - cat. on a single generator. \hspace{1.5cm} $\mathrm{Sp}^\omega\{x\} := \mathrm{Sp}^\omega[\mathrm{Fin}^{\simeq}] \in 2\mathrm{CAlg}$
$\rightsquigarrow \; \mathrm{Fun}^{\mathrm{ex}, \otimes}(\mathrm{Sp}^\omega\{x\}, \mathcal{K}) \xrightarrow[\sim]{\mathrm{ev}_{\pm(x)}} \mathcal{K}$ \hspace{1cm} for \; $\mathcal{K} \in 2\mathrm{CAlg}$

Thick tensor ideals and localisation theory

Def $X \in 2\text{CAlg}$. Full rigid. subcat. $J \subseteq X$ is a thick tensor
ideal (tt-ideal) if
(a) $J = \text{Thick}(J)$, ie. closed under cofibres, shifts, retracts
(b) $\forall x \in X$, $x' \in J$, $x \otimes x' \in J$.
$\leadsto$ $\text{Idl}(X)$ poset of tt-ideals ordered by inclusion $\qquad$ finite limits and finite colimits

Not $S \subseteq X \leadsto \langle S \rangle$ tt ideal gen. by S, $\langle x \rangle = \langle \{x\} \rangle$ for $x \in X$.

Def A tt-ideal $J \subseteq X$ is
(a) principal, if $J = \langle x \rangle$ for some $x \in X$
(b) radical, if $x^{\otimes n} \in J$ for $x \in X$, $n \geq 1 \implies x \in J$.
$\leadsto$ $\text{Prin}(X)$, $\text{Rad}(X) \subseteq \text{Idl}(X)$

Not $S \subseteq X \leadsto \sqrt{S}$ smallest radical tt-ideal containing S

Lem $\sqrt{x \otimes y} = \sqrt{x} \cap \sqrt{y}$.

Ex $\langle 0 \rangle$, $\langle \mathbb{1} \rangle = X \in \text{Idl}(X)$.
$\underset{\text{smallest}}{\qquad} \underset{\text{largest}}{\qquad}$

$f \in X \longrightarrow \mathcal{2} \in 2\text{CAlg} \leadsto \ker f := X \times_{\mathcal{2}} \{0\} \in \text{Idl}(X)$
$\leadsto$ $\ker(-): 2\text{CAlg}_{X/} \longrightarrow \text{Idl}(X) \qquad$ (lattice is the same with finite)

Def A complete lattice is a poset admitting arbitrary colimits
(or equivalently admitting arbitrary limits).
Eg. closed bounded subsets of the reals under $\leq$ (usual sup + inf)
lattice of opens of a topological space (sup $= \cup$, inf $= \text{int}(\cap)$)
A frame is a complete lattice in which finite limits distribute
over arbitrary colimits. A frame is coherent if the compact
objects are closed under finite limits and every object is a
colimit of compacts. $\qquad$ } say this together

Prop For $X \in 2\text{CAlg}$,
(a) $\text{Idl}(X)$ is a complete lattice with $\text{Idl}(X)^\omega = \text{Prin}(X)$
(b) $\text{Rad}(X)$ is a coherent frame with $\text{Rad}(X)^\omega = \{\sqrt{x} \mid x \in X\}$

Def $\mathcal{2} \in \text{CAlg}^{rigid}_{\mathcal{2}}$, $J \subseteq \mathcal{2}$ rigid. stable subcat closed under retracts,
$W = \{f \in \text{Arr}(X) \mid \text{cofib}(f) \in J\}$.
Karoubi quotient of X by J: $X/J := (X[W^{-1}])^\natural$
$\qquad\qquad \uparrow L_J$
idempotent completion of $\qquad$ X
Verdier quotient

· X/J stable, $L_J: X \longrightarrow X/J$ exact
· $\text{mod}(X) \longrightarrow \text{mod}(X/J)$ is a localisation, ie. it has a
fully faithful right adjoint with kernel $\text{mod}(J)$
· if $J \in \text{Idl}(X)$ then X/J and L_J admit sym. mon. enhancements,
$\text{Fun}^{\text{ex}, \otimes}_\infty(X/J, \mathcal{2}) \xrightarrow{L^*_J} \text{Fun}^{\text{ex}, \otimes}_{J \to 0}(X, \mathcal{2}) \qquad$ for $\mathcal{2} \in 2\text{CAlg}$

Prop For $X \in 2\text{CAlg}$, there is an adjunction
$$X/- : \; \text{Idl}(X) \rightleftarrows 2\text{CAlg}_{X/} : \ker(-)$$
with left adjoint fully faithful, right adjoint preserving
filtered colimits and $J \in \text{Idl}(X)$ satisfies $X/J \in (2\text{CAlg}_{X/})^\omega$
iff $J \in \text{Prin}(X)$.

Proof: · $J \in \text{Idl}(X)$, $f: X \longrightarrow \mathcal{2} \in 2\text{CAlg}$.

$$\text{Fun}^{\otimes, \text{ex}}_{X/}(X/J, \mathcal{2}) \xrightarrow{\quad\perp\quad} \text{Fun}^{\otimes, \text{ex}}(X/J, \mathcal{2})$$
$$\downarrow \qquad\qquad\qquad \downarrow L^*_J$$
$$\Delta^0 \xrightarrow{\quad f \quad} \text{Fun}^{\otimes, \text{ex}}(X, \mathcal{2})$$

$\leadsto \text{Fun}^{\otimes, \text{ex}}_{X/}(X/J, \mathcal{2}) \cong \{^*_g, \; J \subseteq \ker(f) \atop \emptyset, \; \text{o/w}\} \cong \text{Map}_{\text{Idl}(X)}(J, \ker(f))$.

$\ker(X \longrightarrow X/J) \cong J \; \leadsto \; X/- \text{ fully faithful}$
(for $x \in J$, $0 \to x$ is in $W \implies J \subseteq \ker$. $x \in \ker$
$\implies 0 \to x \in W \implies x \cong \text{cofib}(0 \to x) \in J$)

· $\ker(-): 2\text{CAlg}_{X/} \longrightarrow \text{Idl}(X)$ preserves filtered colimits:
$F: I \longrightarrow 2\text{CAlg}_{X/}$ filtered diagram
$\leadsto f = \text{colim } f_i: X \longrightarrow \text{colim } F$, need $\ker(f) \cong \text{colim } \ker(f_i)$
(colimits in overcategories are computed underlying) $\qquad$ union of ideals
$\text{colim}_I \ker(f_i) \subseteq \ker(f)$ clear. $\qquad\qquad X$
let $x \in \ker(f)$, suffices to $\qquad f_i \nearrow \qquad \searrow f_j$
show $\exists i \in I$ st. $x \in \ker(f_i)$ $\qquad F(i) \longrightarrow F(j)$
$\iff \exists i \in I$ st. f_i sends the
homology class of the
automorphism id_x to the zero map. $\qquad\qquad$ colim F

$x \xrightarrow{\text{id}} x \longrightarrow 0$, $f(x) \cong 0 \iff f(\text{id}_x) \cong 0$
$f(\text{id}_x) \atop f(x) \xrightarrow{f(\text{id}_x)} f(x) \xrightarrow{\cong} 0$

$2\text{CAlg} \xrightarrow{\text{fgt}} \text{Cat}_\infty \xrightarrow{(-)^\cong} \text{Grp}_\infty$ both preserve filtered colimits
$\leadsto \text{BAut}(f(x)) \cong \text{colim}_I \text{BAut}_{F(i)}(f_i(x))$
$\wedge \wedge \qquad\qquad\qquad \wedge \wedge$
$(\text{colim } F)^\cong \qquad\qquad F(i)^\cong$

Hence $[\text{id}_x]$ must map to 0 in some $F(i)$ since homology
groups map commute with filtered colimits (being 0 in
BAut can be checked on homology groups).
· $J \in \text{Idl}(X)$ satisfies $X/J \in (2\text{CAlg}_{X/})^\omega$
$\iff \text{Fun}^{\otimes, \text{ex}}_{X/}(X/J, -)$ commutes with fil. colim.
$\qquad\qquad\qquad 12$
$\implies \text{follows} \atop \text{from } X/- \text{ fully faithful}$ $\xrightarrow{\text{Map}_{\text{Idl}(X)}(J, \ker(-))}$ commutes with fil. colim.
$\iff J \in \text{Idl}(X)^\omega = \text{Prin}(X)$ $\qquad\qquad \square$

Cor $f: X \longrightarrow \mathcal{2}$ in 2CAlg, $J \in \text{Idl}(X)$. $\quad X \longrightarrow \mathcal{2}$
$$\downarrow \qquad \ulcorner \downarrow$$
$$X/J \longrightarrow \mathcal{2}/\langle f(J) \rangle$$

Cor $X \in 2\text{CAlg}$, $a \in X$. $\quad \text{Sp}^\omega\{x\} \longrightarrow \text{Sp}^\omega$
$$\downarrow \qquad\qquad \downarrow$$
$$X \longrightarrow X/\langle a \rangle$$

Cor Karoubi quotients in 2CAlg are closed under compositions,
retracts and left cancellations.

Rigidity

Def A symmetric monoidal ∞-cat is rigid if every object is dualisable
$\leadsto$ $2\mathrm{CAlg}^{rig} \subseteq 2\mathrm{CAlg}$ $\quad$ ($\mathcal{X} \in 2\mathrm{CAlg}$ is rigid $\Leftrightarrow$ $h\mathcal{X}$ rigid)

Lem $\mathcal{X} \in 2\mathrm{CAlg}$, $\quad S \subseteq \mathcal{X}^{dbl}$ $\quad \Rightarrow$ Thick $(S) \subseteq \mathcal{X}^{dbl}$
$\leadsto \mathcal{X}^{dbl} \in 2\mathrm{CAlg}$ $\qquad\qquad\qquad \overset{\scriptstyle\frown}{\text{finite limits, colimits}}$ and retracts

Prop incl : $2\mathrm{CAlg}^{rig} \rightleftarrows 2\mathrm{CAlg} : (-)^{dbl}$

Proof : $\mathcal{X} \in 2\mathrm{CAlg}^{rig}$, $\mathcal{Z} \in 2\mathrm{CAlg}$. Map $_{2\mathrm{CAlg}}(\mathcal{X}, \mathcal{Z}) \simeq \mathrm{Map}_{2\mathrm{CAlg}}(\mathcal{X}, \mathcal{Z}^{dbl})$.

Lem $\mathcal{X} \in 2\mathrm{CAlg}^{rig}$, then $\mathrm{Ind}(\mathcal{X})^\omega = \mathrm{Ind}(\mathcal{X})^{dbl} \simeq \mathcal{X}$.

Proof : $\mathrm{Ind}(\mathcal{X})^\omega \simeq \mathcal{X}^{\natural} \simeq \mathcal{X}$ $\quad$ since $\mathcal{X}$ has finite colimits.
$\leadsto \mathcal{X} \simeq \mathrm{Ind}(\mathcal{X})^\omega \subseteq \mathrm{Ind}(\mathcal{X})^{dbl}$ $\quad$ since $\mathcal{X}$ is rigid.
$\mathbb{1}_{\mathrm{Ind}(\mathcal{X})} \in \mathrm{Ind}(\mathcal{X})^\omega$ implies $\quad \mathrm{Ind}(\mathcal{X})^{dbl} \subseteq \mathrm{Ind}(\mathcal{X})^\omega$:
$X \in \mathrm{Ind}(\mathcal{X})^{dbl}$, then $\quad$ Map $(X, \mathrm{colim}_i \, Y_i)$
$\simeq$ Map $(\mathbb{1}, X^\vee \otimes \mathrm{colim}_i \, Y_i)$
$\simeq$ Map $(\mathbb{1}, \mathrm{colim}_i X^\vee \otimes Y_i)$
$\simeq \mathrm{colim}_i$ Map $(\mathbb{1}, X^\vee \otimes Y_i)$
$\simeq \mathrm{colim}_i$ Map (X, Y_i). $\qquad \square$

Def $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\omega, st})$ is rigidly compactly generated if $\mathcal{C}^\omega = \mathcal{C}^{dbl}$
$\leadsto \mathrm{CAlg}(\mathrm{Pr}^L_{\omega, st})^{rig} \subseteq \mathrm{CAlg}(\mathrm{Pr}^L_{\omega, st})$

($2\mathrm{CAlg}^{rig} \simeq \mathrm{CAlg}(\mathrm{Pr}^L_{\omega, st})^{rig}$)

Localisation theory in the rigid setting

Lem $\mathcal{X} \in 2CAlg^{rig}$, then $\mathrm{Rad}(\mathcal{X}) = \mathrm{Idl}(\mathcal{X})$.

Proof: $J \in \mathrm{Idl}(\mathcal{X})$. Let $x \in \mathcal{X}$ s.t. $x^{\otimes 2} \in J$ $\Rightarrow$ $x \otimes x \otimes x^{\vee} \in J$
and $x \xrightarrow{\mathrm{coev} \otimes x} x \otimes \check{x} \otimes x \xrightarrow{x \otimes \mathrm{ev}} x$, hence x is

$$\text{id}$$

a retract of $x \otimes x \otimes x^{\vee}$ $\Rightarrow$ $x \in J$. $\qquad \square$

Cor If $\mathcal{X} \in 2CAlg^{rig}$, $J \in \mathrm{Idl}$, $\mathcal{X}/J \in 2CAlg^{rig}$.
There is an adjunction

$$\mathcal{X}/- : \underbrace{\mathrm{Idl}(\mathcal{X})}_{\overset{"}{\mathrm{Rad}(\mathcal{X})}} \rightleftarrows 2CAlg^{rig}_{\mathcal{X}/} : \ker(-)$$

where the left adjoint is fully faithful, the right adjoint preserves filtered colimits, and $J \in \mathrm{Idl}(\mathcal{X})$ satisfies $\mathcal{X}/J \in (2CAlg^{rig}_{\mathcal{X}/})^{\omega}$
iff $J \in \mathrm{Prin}(\mathcal{X})$.

$$(2CAlg^{rig} \longrightarrow 2CAlg \text{ is left adjoint})$$

Def $\mathcal{C}$ sym. mon. ∞-cat.

(a) An idempotent algebra of $\mathcal{C}$ is an object $A \in \mathcal{C}_{A/}$ s.t. $A \otimes (\mathbb{1} \to A)$
is an equivalence. (Such an object is uniquely a commutative
algebra: $\mathrm{Idem}(\mathcal{C}) \subseteq CAlg(\mathcal{C}) \simeq CAlg(\mathcal{C})_{\mathbb{1}/} \longrightarrow \mathcal{C}_{\mathbb{1}/}$

$$\underset{\text{multiplication eq.}}{\overset{\frown}{A \otimes A \overset{\sim}{\to} A}}$$

is fully faithful with essential image the maps $\mathbb{1} \to A$
that exhibit A as an idempotent object of $\mathcal{C}$.)
$\mathrm{Idem}(\mathcal{C})$ is a poset, since for $A \in \mathrm{Idem}(\mathcal{C})$, $\mathrm{Map}_{CAlg(\mathcal{C})}(A, -)$
takes values in $\mathrm{Spc}_{\leq -1}$. $f,g : A \longrightarrow B$

In this case forgetful functor
$\mathrm{Mod}_A(\mathcal{C}) \longrightarrow \mathcal{C}$ fully faithful!

$$
\begin{array}{ccc}
A & \xrightarrow{f \cdot g} & B \\
{\scriptstyle \text{id} \otimes \eta} \downarrow {\scriptstyle \eta \overset{?}{\otimes} \text{id}} & & \uparrow \mu \\
A \otimes A & \xrightarrow{f \otimes g} & B \otimes B
\end{array}
$$

(b) A smashing localisation of $\mathcal{C}$ is a sym. mon. localisation
$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ witnessing $\mathcal{D} \simeq \mathrm{Mod}_A(\mathcal{C})$ for $A \in \mathrm{Idem}(\mathcal{C})$.
$\hookleftarrow$ a localisation is smashing $\iff$ the projection formula holds

Prop $\mathcal{X} \in 2CAlg^{rig}$, $J \in \mathrm{Idl}(\mathcal{X})$. The localisation

$$L_J : \mathrm{Ind}(\mathcal{X}) \rightleftarrows \mathrm{Ind}(\mathcal{X}/J) : R_J$$

is smashing with tensor
idempotent $R_J I_J (\mathbb{1})$.

Def Such idempotent algebras are called finite ($\mathcal{X}$ not necessarily rigid).
$\rightsquigarrow$ $\mathrm{Idem}^{fin}_{\mathcal{X}} \subseteq \mathrm{Idem}_{\mathcal{X}} := \mathrm{Idem}(\mathrm{Ind}(\mathcal{X}))$.

Cor $\mathcal{X} \in 2CAlg^{rig}$.

$$
\begin{array}{ccccc}
CAlg(\mathrm{Ind}(\mathcal{X})) & \xleftarrow{\mathrm{Mod}_{(-)}(\mathrm{Ind}(\mathcal{X}))} & CAlg(\mathcal{Pr}^L_{w.st})_{\mathrm{Ind}(\mathcal{X})/} & \xrightarrow{(-)^{\omega}} & 2CAlg^{rig}_{\mathcal{X}/} \\
\cup| & & & & \\
\mathrm{Idem}_{\mathcal{X}} & & \mathrm{Mod}^{\omega} & & \mathcal{X}/- \quad \left[\begin{array}{c} -| \\ \end{array}\right] \ker(-) \\
\cup| & & & & \\
\mathrm{Idem}^{fin}_{\mathcal{X}} & \xrightarrow[\quad\ker(-) \circ \mathrm{Mod}^{\omega}\quad]{\sim} & & & \mathrm{Idl}(\mathcal{X})
\end{array}
$$