Constructing Synthetic Spectra

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Abstract

These are informal notes for my talk on synthetic spectra in the reading group after five talks on computing the stable homotopy groups of spheres. The goal was to motivate and sketch the construction of synthetic spectra. Disclaimer: there might be mistakes in these notes as my understanding of the subject is limited. Any comments and corrections are very welcome.

Contents

1 Where Do Spectral Sequences Come From?

We begin with the question of where spectral sequences come from. For the purpose of today, the answer is that they come from filtered spaces/spectra. We illustrate this through the following two examples.

Consider a finite spectrum $X \in \mathcal{S}p^{\text{fin}}$ (I called it a space in the title to distinguish it from the spectrum that represents a cohomology theory in the next section). In order to understand the cohomology of this space/spectrum, a standard technique is to construct a cell structure, hence a skeletal filtration of X.

$$\begin{array}{ccccccccc} X_0 & \hookrightarrow & X_1 & \hookrightarrow & X_2 & \hookrightarrow & \cdots & \hookrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & & & \\ c_0 & & c_1 & & c_2 & & & & \end{array}$$

Since X is a finite spectrum, the cofibres are wedges of spheres which are easy to understand cohomologically. Then the hope is that the cohomology of the cofibres would tell us something about the cohomology of the space X. With mild conditions on the spectrum representing the cohomology theory (for instance satisfied by KU), this is indeed the case and is expressed in terms of the spectral sequence of filtered spectra

$$E_1^{*,*} \cong \mathrm{KU}^*(c_\bullet) \Longrightarrow \mathrm{KU}^* X.$$

Instead of filtering the space/spectra, we may attempt to cut up the cohomology theory instead. This is done by constructing the Postnikov Tower

$$\begin{array}{ccccccccc} \mathrm{KU} & \longrightarrow & \cdots & \longrightarrow & \tau_{\leq 2n}\mathrm{KU} & \longrightarrow & \tau_{\leq 2n+2}\mathrm{KU} & \longrightarrow & \cdots \\ & & & & \uparrow & & \uparrow & & \\ & & & & \Sigma^{2n+2}\mathrm{H}\mathbb{Z} & & \Sigma^{2n}\mathrm{H}\mathbb{Z} & & \end{array}$$

which gives rise to the spectral sequence

$$E_1^{*,*} \cong [X, \Sigma^* \mathrm{H}\mathbb{Z}]_* \Longrightarrow \mathrm{KU}^*(X).$$

(I have heard slogans of how the latter approach has the advantage that the differentials bear geometric interpretation, though I have not been able to understand/find a reference to this.)

The (generalised) Adams spectral sequence arises by resolving a space/spectrum by an E_*-Adams resolution. It has E_2-page given by

$$E_2^{*,*} \cong \mathrm{Ext}_{E_*E}(E_*X) \Longrightarrow \pi_*(\hat{X})$$

where the (suggestive) notation $\mathrm{Ext}_{E_*E}(-)$ denotes $\mathrm{Ext}_{E_*E}(E_*, E_*(-))$, which gives the Ext-groups computed in the abelian category $\mathcal{C}\mathrm{omod}_{E_*E}$. Then a natural question is can we take the approaches towards the Adams spectral sequence? In other words, can we treat $\mathrm{Ext}_{E_*E}(-)$ as a cohomology theory and cut it up? Or can we find a filtration of X such that the cofibre is E_*X (this question doesn't even make sense in this context but it would in $\mathcal{S}yn_E$)? To this end, we need a category/homotopy theory where $\mathrm{Ext}_{E_*E}(-)$ is representable in the sense of a spectrum representing a cohomology theory by Brown's representability theorem. In other words, we need to enlarge the category $\mathcal{S}p$ to something perhaps akin to "derived category of spectra".

(In the end, we will see that $\mathrm{Ext}_{E_*E}(-)$ is represented by the synthetic sphere in $\mathcal{S}yn_E$, though I have not been able to understand/find the theory of how we cut up $\mathrm{Ext}_{E_*E}(-)$ as in subsection 1.2.)

2 Derived Categories

There are usually two steps in constructing the derived category of an abelian category: first we freely adjoint all filtered colimits and geometric realisation/derived equaliser to obtain chain complexes. Then we might need to stabilise the resulting category or enforce the exactness of some short exact sequences. For this reason, we start by reviewing how to freely adjoint colimits.

2.1 Presheaf Completion

The story begins with the following significant consequence of the Yoneda Lemma.

Theorem 2.1. *[Lur09, Theorem 5.1.5.6] Let $\mathcal{C}$ be a small category, the Yoneda embedding exhibits the presheaf category as the free cocompletion of $\mathcal{C}$. That is, given any cocomplete category $\mathcal{D}$, precomposing with the Yoneda embedding gives an equivalence of categories*

$$\mathrm{Fun}(\mathcal{C},\mathcal{D}) \longrightarrow \mathrm{Fun}^{L}(P(\mathcal{C}),\mathcal{D})$$

where $\mathrm{Fun}^{L}(P(\mathcal{C}),\mathcal{D})$ is the subcategory of colimit preserving functors.

Proof. We will sketch the proof for 1-categories here for illustration. I believe the proof for ∞-categories follows a somewhat similar idea. Recall that any presheaf is the colimit of representables. Then every colimit preserving functor $f\colon P(\mathcal{C}) \to \mathcal{D}$ is uniquely determined by its restrictions to $f'\colon \mathcal{C} \to \mathcal{D}$. Conversely, given a functor $f'\colon \mathcal{C} \to \mathcal{D}$, the assignment of $H \mapsto \mathrm{colim}_{c\in\mathcal{C}/H} f'(c)$ defines a functor. □

It is often the case that we only wish to adjoin a certain collection of colimits (e.g. filtered colimits, sifted colimits). In these cases, we simply restrict to the subcategory of $P(\mathcal{C})$ generated under these collections of colimits of representables. To be precise, let $\mathcal{K}$ be a collection of simplicial sets, we define $P^{\mathcal{K}}(\mathcal{C})$ to be the subcategory of $P(\mathcal{C})$ whose objects are the $\mathcal{K}$-indexed colimits of representables. Then, universal properties analogous to 2.1 hold in that precomposing with the Yoneda embedding gives an equivalence of categories

$$\mathrm{Fun}(\mathcal{C},\mathcal{D}) \longrightarrow \mathrm{Fun}^{\mathcal{K}}(P^{\mathcal{K}}(\mathcal{C}),\mathcal{D})$$

where $\mathrm{Fun}^{\mathcal{K}}$ is the subcategory of functors that preserves $\mathcal{K}$-indexed colimits.

The trouble is, however, that there are no general criteria to determine whether a presheaf is a $\mathcal{K}$-indexed colimits of representables, which makes it difficult to prove categorical properties about $P^{\mathcal{K}}(\mathcal{C})$ such as presentability. So our next step would be to express such presheaves as the presheaves that preserve certain kinds of limits.

Theorem 2.2. *[Lur09, Corollary 5.3.5.4] Let $\mathcal{C}$ be a small category with finite colimits and $\mathcal{K}$ be the collection of all filtered simplicial sets, then $P^{\mathcal{K}}(\mathcal{C})$ is the subcategory of finite limit preserving presheaves. Moreover, the Yoneda embedding factors through $\mathcal{C} \to P^{\mathcal{K}}(\mathcal{C})$ which also preserves finite limits.*

This is the Ind-completion of $\mathcal{C}$. To further add geometric realisations, we adjoint all colimits over the diagram $\mathrm{N}(\Delta^{\mathrm{op}})$.

Theorem 2.3. *[Lur09, Lemma 5.5.8.14, Proposition 5.5.8.10] Let $\mathcal{C}$ be a small category with finite coproducts and $\mathcal{K}$ be the collection of all filtered simplicial sets and $\mathrm{N}(\Delta^{\mathrm{op}})$, then $P^{\mathcal{K}}(\mathcal{C})$ is the subcategory of finite product preserving presheaves. Moreover, the Yoneda embedding factors through $\mathcal{C} \to P^{\mathcal{K}}(\mathcal{C})$ which also preserves finite product. In this case, we denote $P^{\mathcal{K}}(\mathcal{C})$ by $P_{\Sigma}(\mathcal{C})$.*

Remark 2.4. For $\mathcal{C}$ a small additive category with finite coproducts as in the theorem, product preserving presheaves in fact deloops to valuing in $\mathcal{S}p_{\geq 0}$:

$$P_{\Sigma}(\mathcal{C}) \coloneqq \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}},\mathcal{S}) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}},\mathrm{CGrp}(\mathcal{S}))$$

where $\mathrm{CGrp}(\mathcal{S})$ denotes the commutative group objects in spaces, which are precisely the connective spectra by May's recognition principle.

Our goal is to build the category of synthetic spectra out of some modification of the presheaf completion of $\mathcal{S}p$ at the end. Again, the presheaf completion is to add some colimits and geometric realisations. But notice that $\mathcal{S}p$ is presentable, which means that it is built by freely adjoining filtered colimits to its subcategory of compact generators (i.e. $\mathcal{S}p \simeq \mathrm{Ind}(\mathcal{S}p^{\omega})$). So we might as well start with the subcategory $\mathcal{S}p^{\omega}$. This is reminiscent of the following examples of constructing the derived category from an abelian category with a collection of compact generators.

2.2 The Grothendieck Abelian Case

For this section, we narrow our attention to when $\mathcal{C}$ is a Grothendieck category (i.e. abelian, presentable and that all filtered colimits are exact), which is the closer analogue to the situation we are in with $\mathcal{S}p$. See Section 2.5 of [Pst22] and Section 6.4 of [PP23] for reference.

Let $\mathcal{P} \hookrightarrow \mathcal{C}$ denote a system of compact generators. If all objects in $\mathcal{P}$ are projective, we have the following description of the derived categories:

$$P_\Sigma(\mathcal{P}) \simeq \mathcal{D}_{\geq 0}(\mathcal{C}), \qquad P_\Sigma^{\mathcal{S}p}(\mathcal{P}) \simeq \mathcal{D}(\mathcal{C})$$

where $\mathcal{P}$ is the full subcategory of projective compact generators in $\mathcal{C}$. (Note that $P_\Sigma(\mathcal{P}) \simeq \mathcal{D}_{\geq 0}(\mathcal{C})$ is pre-stable, and this should not be too surprising thanks to Remark 2.4 which tells us that these presheaves in fact values is connective spectra.) Heuristically, the first equivalence is due to the fact that objects in $P_\Sigma(\mathcal{P})$ are equivalently the simplicial objects in the filtered colimits of $\mathcal{P}$. (This is formally expressed as the underlying ∞-category of the simplicial model category $P^{\mathrm{sSet}}(\mathcal{P})$ is $P_\Sigma(\mathcal{P})$.) By a theorem of Goerss–Hopkins we get $P_\Sigma^{\mathrm{Set}}(\mathcal{P}) \simeq \mathcal{C}$, and by the Dold–Kan correspondence, $P_\Sigma(\mathcal{P})$ is equivalent to the category of bounded below chain complexes of projectives which are the fibrant-cofibrant objects in $\mathcal{C}$, and hence the equivalence $P_\Sigma(\mathcal{P}) \simeq \mathcal{D}_{\geq 0}(\mathcal{C})$. Then the equivalence of $P_\Sigma^{\mathcal{S}p}(\mathcal{P}) \simeq \mathcal{D}(\mathcal{C})$ is acquired by simply stabilising both sides.

Note that requiring $\mathcal{C}$ to have a system of compact generators that are all projective is quite restrictive. On the other hand, we have the following results for categories with a set of compact generators with no reference towards projectivity:

$$\widehat{Shv}_\Sigma^{\mathcal{S}p}(\mathcal{P}) \simeq \mathcal{D}(\mathcal{C})$$

where $\widehat{Shv}_\Sigma^{\mathcal{S}p}(\mathcal{P})$ is the subcategory of presheaves that satisfies the sheaf condition with respect to the epimorphism Grothendieck topology. If we neglect the hyper-completeness condition and focus on the heuristics for a moment, Theorem 2.8 of [Pst22] tells us that given any additive site, a presheaf is a sheaf precisely when it takes those fibre sequences whose second map is covering to fibre sequences. In our situation, this says a presheaf is a sheaf if it treats all compact generators as if they were projectives.

3 Synthetic Spectra

Following the above discussion, we will construct synthetic spectra as some sort of sheaves on spectra. We start with the category $P_\Sigma(\mathcal{S}p^\omega)$. There are two technical modifications that we need to make. To start with, we want the category at the end to be stable. This can be done by simply stabilising $P_\Sigma(\mathcal{S}p^\omega)$, which is equivalent to $P_\Sigma^{\mathcal{S}p}(\mathcal{S}p^\omega)$ via the identification $\mathcal{S}p(\mathrm{Fun}(\mathcal{C}, \mathcal{S}_*)) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{S}p)$. Note that this stabilisation is not to produce the derived category with respect to a homology theory E that we want, but more of a technicality in order to end up with a stable category. On the other hand, the effect of this does show up in, for instance, the synthetic sphere being bigraded: one for this stabilisation and one for taking the "derived category" with respect to a homology theory E that we will do next.

Let E be an Adams-type homology theory (roughly those with Universal coefficient theorem and Kunneth formula), we wish to endow $P_\Sigma^{\mathcal{S}p}(\mathcal{S}p^\omega)$ with the E_*-surjection Grothendieck topology. Unfortunately, E_*-surjection is not closed under products which would result in the category we end up with not being symmetric monoidal. So we replace $\mathcal{S}p^\omega$ with $\mathcal{S}p_E^{fp}$ the full subcategory of $\mathcal{S}p$ such that E_*P is a projective E_*-module for all $P \in \mathcal{S}p_E^{fp}$. Then the Kunneth formula tells us that $E_*(P \otimes Y) \simeq E_*P \otimes E_*Y$, so $\mathcal{S}p_E^{fp}$ is closed under products and E_*-surjections are closed under products in this category. As a side note, replacing $\mathcal{S}p^\omega$ with $\mathcal{S}p_E^{fp}$ is not such a big switch. In fact, the two are equivalent for $E = H\mathbb{F}_p$.

Definition 3.1. Endow $P_\Sigma^{\mathcal{S}p}(\mathcal{S}p_E^{fp})$ with the E_*-epimorphism Grothendieck topology: $\{Q_i \to P\}$ is a covering family if and only if it is a set of one element, and $Q_i \to P$ is a E_*-surjection (degree-wise). The category of synthetic spectra

$$\mathcal{S}yn_E \coloneqq Shv_\Sigma^{\mathcal{S}p}(\mathcal{S}p_E^{fp})$$

is defined to be the full subcategory of sheaves in $P_\Sigma^{\mathcal{S}p}(\mathcal{S}p_E^{fp})$ with respect to the E_*-epimorphism Grothendieck topology.

Remark 3.2. One might wonder why we don't take only the hypercomplete sheaves. But it is a fact that the category of hypercomplete sheaves in this context is a Bousfield localisation of $\mathcal{S}yn_E$, we don't lose anything studying the larger category.

Given any additive site $\mathcal{C}$, we can express the sheafification via the following adjunction

$$Shv_\Sigma^{\mathcal{S}p}(\mathcal{C}) \hookleftarrow P_\Sigma^{\mathcal{S}p}(\mathcal{C})$$

which says that the category of product-preserving sheaves on $\mathcal{C}$ is a left exact, accessible Bousfield localisation of the presheaf category. In particular, $\mathcal{S}yn_E$ is presentable.

Proposition 3.3. *The category $\mathcal{S}yn_E$ has the following properties:*

1. *It is stable, presentable, closed symmetric monoidal (with Day convolution product).*
2. *It has a canonical t-structure whose heart is equivalent to $\mathcal{C}omod_{E_*E}$.*
3. *Taking E_*-homology factors uniquely through*

$$\mathcal{S}p \xrightarrow{\nu} (\mathcal{S}yn_E)_{\geq 0} \xrightarrow{\pi_0^\heartsuit} \mathcal{C}omod_{E_*E}, \quad E_* : \mathcal{S}p \to \mathcal{C}omod_{E_*E}$$

   where ν is the synthetic analogue functor built out of the Yoneda embedding. Moreover, ν is symmetric monoidal and $\nu\mathbb{S}$ is the unit in $\mathcal{S}yn_E$.

Remark 3.4. The synthetic analogue $\nu\colon \mathcal{S}p \to \mathcal{S}yn_E$ is defined by the concrete formula

$$\nu X(P) = Shv(\tau_{\geq 0} \operatorname{map}_{\mathcal{S}p}(P, X))$$

where Shv denotes the sheafification. The choice of this definition is by no means random and will be explained in subsequent talks.

There is a bigraded suspension functor in $\mathcal{S}yn_E$ given by

$$\Sigma^{t,w} X(P) \coloneqq \Sigma^{t-w} X(\Sigma^{-w} P)$$

for all $X \in \mathcal{S}yn_E$. we denote by $S^{t,w}$ the synthetic sphere $\Sigma^{t,w}\nu\mathbb{S}$.

4 The map τ and Adams Spectral Sequence

As promised in the introduction, the functor $\operatorname{Ext}_{E_*E}(-)$ becomes representable in $\mathcal{S}yn_E$. To this end, we first construct the map τ.

Apply any presheaf X to the suspension pushout square

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma P \end{array}$$

gives us a commutative diagram

$$\begin{array}{ccc} X(\Sigma P) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X(P) \end{array}$$

Then the Universal property of Pullback gives us a map $X(\Sigma P) \to \Omega X(P)$, whose adjoint is $\Sigma X(\Sigma P) \to X(P)$. Take $X = S^{0,0}$ and we have the τ that we are looking for

$$\tau \colon S^{-1,0} \longrightarrow S^{0,0}.$$

In fact, for any synthetic spectrum X, the map obtained by the previous pushout and pullback squares is given by

$$\tau \otimes \mathrm{id}_X \colon S^{-1,0} \otimes X \simeq \Sigma X(\Sigma P) \to X(P)$$

which we will also denote by τ.

Proposition 4.1. *[Pst22] The cofibre of τ, denoted by $C\tau$ has a unique structure of a commutative algebra object. Furthermore, the cofibre $C\tau \otimes \nu Y$ of $\tau \colon \Sigma^{0,-1}\nu X \to \nu X$ lives in $\mathcal{S}yn_E^{\heartsuit}$, and we have*

$$[\Sigma^{*,*}\nu X, C\tau \otimes \nu Y] \simeq \mathrm{Ext}^{*,*}_{E_*E}(E_*X, E_*Y)$$

(up to some re-indexing).

This is the representability result for Ext that we were promised. On the computational side, the category $\mathcal{S}yn_E$ encodes the E_*-Adams spectral sequence as follows (see Section 5.4 of [PP23]). Let X be a spectrum and νX be its synthetic analogue, we can construct it Postnikov/Whitehead tower

$$\cdots \xrightarrow{\tau} \Sigma^{0,-1}\nu X \xrightarrow{\tau} \nu X \xrightarrow{\tau} \Sigma^{0,1}\nu X \xrightarrow{\tau} \Sigma^{0,2}\nu X \xrightarrow{\tau} \cdots$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$C\tau \otimes \Sigma^{0,-1}\nu X \qquad C\tau \otimes \nu X \qquad C\tau \otimes \Sigma^{0,1}\nu X \qquad C\tau \otimes \Sigma^{0,2}\nu X$$

Then Applying $[\nu\mathbb{S}, -]$ to the cofibre gives a spectral sequence with

$$E_1^{*,*} \cong [\nu\mathbb{S}, \tau \otimes \Sigma^{*,*}\nu X] \cong \mathrm{Ext}^{*,*}_{E_*E}(E_*, E_*X)$$

which is equivalent to the E_2-page of the Adams spectral sequence up to some re-indexing.

On the conceptual side, τ exhibits $\mathcal{S}yn_E$ as the deformation of $\mathcal{S}p$ into quasi-coherent sheaves over a certain algebraic stack:

$$\begin{array}{ccc} & \mathcal{S}yn_E & \\ \swarrow^{\text{kill } \tau} & & \searrow^{\text{invert } \tau} \\ \mathcal{S}\mathrm{table}_{E_*E} \hookleftarrow \mathcal{M}\mathrm{od}_{C\tau} & & \mathcal{M}\mathrm{od}_{\tau^{-1}\nu\mathbb{S}} \simeq \mathcal{S}p \end{array}$$

where $\mathcal{M}\mathrm{od}_{C\tau}$ is equivalent to $\mathcal{S}\mathrm{table}_{E_*E}$ whenever E has plenty of finite projectives (e.g. whenever E is Landweber exact).

Remark 4.2. The full derived category $\mathcal{D}(\mathcal{C}\mathrm{omod}_{E_*E})$ can be identified with the subcategory of hypercomplete sheaves in $\mathcal{M}\mathrm{od}_{C\tau}$.

References

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