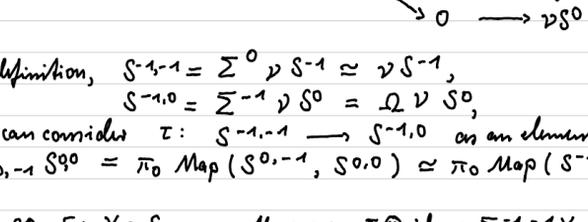


2.6 The generic fibre

§1 Cofiber of τ

Def 4.27 We denote by $\tau: \mathcal{S}^{-1} \rightarrow \Omega(\mathcal{S}^0)$ the canonical limit comparison map induced by $\Omega \mathcal{S}^0 \simeq \mathcal{S}^{-1}$.



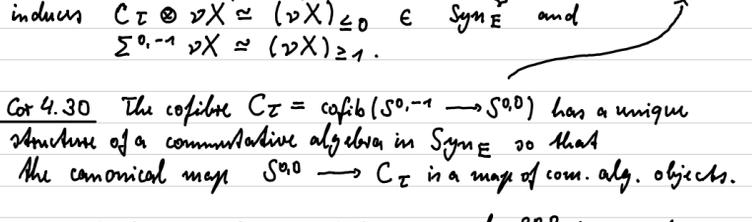
By definition, $\mathcal{S}^{-1,-1} = \Sigma^0 \mathcal{P}\mathcal{S}^{-1} \simeq \mathcal{P}\mathcal{S}^{-1}$,
 $\mathcal{S}^{-1,0} = \Sigma^{-1} \mathcal{P}\mathcal{S}^0 = \Omega \mathcal{P}\mathcal{S}^0$,
 so can consider $\tau: \mathcal{S}^{-1,-1} \rightarrow \mathcal{S}^{-1,0}$ as an element of
 $\pi_{0,-1} \mathcal{S}^0 = \pi_0 \text{Map}(\mathcal{S}^{0,-1}, \mathcal{S}^{0,0}) \simeq \pi_0 \text{Map}(\mathcal{S}^{-1,-1}, \mathcal{S}^{-1,0})$.

Prop 4.28 For $X \in \text{Sym}_E$, the map $\tau \otimes \text{id}_X: \Sigma^{-1,-1} X \rightarrow \Sigma^{-1,0} X$ can be identified with the canonical colimit-to-limit comparison map $X(\Sigma P) \rightarrow \Omega X(P)$, where P runs through Sp_E^{fp} .

Proof: (This uses $(\Sigma^{k,l} X)(P) \simeq \Sigma^{k-l} X(\Sigma^{-l} P)$ (4.26)).
 τ is induced by the diagram $\mathcal{P}\mathcal{S}^{-1} \rightarrow 0$,
 $\downarrow \qquad \downarrow$
 $0 \rightarrow \mathcal{P}\mathcal{S}^0$

so $\tau \otimes X$ is induced by the diagram $\mathcal{P}\mathcal{S}^{-1} \otimes X \rightarrow 0$.
 $\downarrow \qquad \downarrow$
 $0 \rightarrow \mathcal{P}\mathcal{S}^0 \otimes X$

By the tensor structure on Sym_E , for $Q \in \text{Sp}_E^{\text{fp}}$:
 $(\mathcal{P}Q \otimes X)(P) \simeq X(Q^* \otimes P)$ (2.27)
 So $\tau \otimes \text{id}_X$ over $P \in \text{Sp}_E^{\text{fp}}$ induced by applying X to

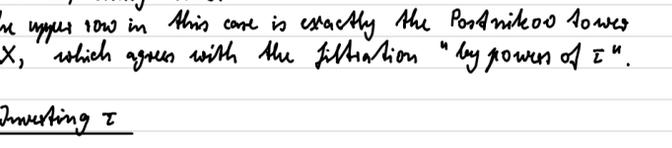


Lemma 4.29 $X \in \text{Sp} \Rightarrow$ cofiber sequence $\Sigma^{0,-1} \mathcal{P}X \xrightarrow{\tau \otimes \text{id}_X} \mathcal{P}X \rightarrow C_{\tau} \otimes \mathcal{P}X$ induces $C_{\tau} \otimes \mathcal{P}X \simeq (\mathcal{P}X)_{\leq 0} \in \text{Sym}_E$ and $\Sigma^{0,-1} \mathcal{P}X \simeq (\mathcal{P}X)_{\geq 1}$.

Cor 4.30 The cofiber $C_{\tau} = \text{cofib}(\mathcal{S}^{0,-1} \rightarrow \mathcal{S}^{0,0})$ has a unique structure of a commutative algebra in Sym_E so that the canonical map $\mathcal{S}^{0,0} \rightarrow C_{\tau}$ is a map of comm. alg. objects.

Proof: By 4.29, $C_{\tau} \simeq (\mathcal{S}^{0,0})_{\leq 0}$ and $\mathcal{S}^{0,0}$ is connective, so its τ -structure truncations are canonically commutative algebra objects ($\tau_{\leq 0}: (\text{Sym}_E)_{\geq 0} \rightarrow \text{Sym}_E^{\text{fp}}$ is lax monoidal.) \square

Rem 4.31 Using the canonical map $\mathcal{S}^{0,0} \rightarrow C_{\tau}$ we can build for any $Y_0 \in \text{Sym}_E$ a tower of "cofibers of powers of τ ":



with each $Y_{n+1} \rightarrow Y_n \rightarrow C_{\tau} \otimes Y_n$ fibre/cofibre.
 If $X \in \text{Sp}$, $Y_0 = \mathcal{P}X$, then $C_{\tau} \otimes Y_0 \simeq (\mathcal{P}X)_{\leq 0}$,
 $Y_1 \simeq \text{cofib}(\mathcal{P}X \rightarrow (\mathcal{P}X)_{\leq 0}) \simeq (\mathcal{P}X)_{\geq 1} \simeq \Sigma^{0,-1} \mathcal{P}X$,
 $C_{\tau} \otimes Y_1 \simeq \Sigma^{0,-1} (\mathcal{P}X)_{\leq 0} \simeq (\mathcal{P}X)_{\leq 1}$,
 $Y_2 \simeq \text{cofib}((\mathcal{P}X)_{\geq 1} \rightarrow (\mathcal{P}X)_{\leq 1}) \simeq (\mathcal{P}X)_{\geq 2}$,
 and so on, using 4.29.

So the upper row in this case is exactly the Postnikov tower of $\mathcal{P}X$, which agrees with the filtration "by powers of τ ".

§2 Inverting τ

Def 4.32 $X \in \text{Sym}_E$ is τ -invertible if the map $\tau: \Sigma^{0,-1} X \rightarrow X$ ($= \tau \otimes \text{id}_X: \mathcal{S}^{0,-1} X \rightarrow \mathcal{S}^{0,0} X$) is an equivalence. $\text{Sym}_E(\tau^{-1}) \subseteq \text{Sym}_E$ the full sub- ∞ -cat. on τ -invertible symmetric spectra.

Goal: $\text{Sym}_E(\tau^{-1}) \simeq \text{Sp}$.

Prop 4.33 The inclusion $\text{Sym}_E(\tau^{-1}) \hookrightarrow \text{Sym}_E$ has a left adjoint $\tau^{-1}: \text{Sym}_E \rightarrow \text{Sym}_E(\tau^{-1})$, and $\text{Sym}_E(\tau^{-1})$ is closed under colimits of Sym_E .

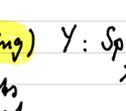
"Slogan: $\text{Sym}_E \rightarrow \text{Sym}_E(\tau^{-1})$ is a smashing localization."

Proof: Explicitly, for $X \in \text{Sym}_E$, let $\tau^{-1} X := \text{colim}(X \rightarrow \Sigma^{0,1} X \rightarrow \Sigma^{0,2} X \rightarrow \dots)$ colimit over the poset \mathbb{N} , $\Sigma^{0,k} X \rightarrow \Sigma^{0,k+1} X$ given by $\Sigma^{0,k+1}(\tau \otimes \text{id}_X)$.

Then for $Y \in \text{Sym}_E(\tau^{-1})$, have $\text{Map}(\tau^{-1} X, Y) \simeq \text{Map}(\text{colim } \Sigma^{0,k} X, Y) \simeq \lim \text{Map}(\Sigma^{0,k} X, Y) \simeq \lim \text{Map}(X, \Sigma^{0,-k} Y) \simeq \text{Map}(X, Y)$

using that all maps $\Sigma^{0,-k-1} Y \rightarrow \Sigma^{0,-k} Y$ are equivalences as Y assumed τ -invertible

implicitly using that τ is self-dual (up to a shift), on the diagram $\mathcal{S}^{-1} \rightarrow 0$ used to define it



witnesses $\mathcal{S}^{-1} \simeq \Omega \mathcal{S}^0$ and its dual witnesses $\mathcal{S}^0 \simeq \Omega \mathcal{S}^{-1}$

Now $X \in \text{Sym}_E$ is in $\text{Sym}_E(\tau^{-1}) \Leftrightarrow \mathcal{S}^{0,-1} X \xrightarrow{\tau} \mathcal{S}^{0,0} X$ (" τ is invertible")
 $\Leftrightarrow \text{cofib}(\mathcal{S}^{0,-1} X \rightarrow \mathcal{S}^{0,0} X) \simeq 0$
 $\Leftrightarrow C_{\tau} \otimes X \simeq 0$, a condition closed under colimits as \otimes commutes with colimits in both variables.
 So localization is smashing. \square

Cor 4.34 $\tau^{-1} \mathcal{S}^{0,0} \in \text{CAlg}(\text{Sym}_E)$, $\text{Sym}_E(\tau^{-1}) \simeq \text{Mod}_{\tau^{-1} \mathcal{S}^{0,0}}(\text{Sym}_E)$.

Proof: For $X \in \text{Sym}_E$, $\tau^{-1} X \simeq \text{colim}(X \otimes \mathcal{S}^{0,k}) \simeq X \otimes (\text{colim } \mathcal{S}^{0,k}) \simeq X \otimes \tau^{-1} \mathcal{S}^{0,0}$ (smashing localization)

In particular, $\tau^{-1} \mathcal{S}^{0,0} \otimes \tau^{-1} \mathcal{S}^{0,0} \simeq \tau^{-1} \mathcal{S}^{0,0}$ and $X \in \text{Sym}_E(\tau^{-1}) \Leftrightarrow \tau^{-1} X \simeq X \Leftrightarrow X \otimes \tau^{-1} \mathcal{S}^{0,0} \simeq X$. \square

Def (Spectral Yoneda embedding) $Y: \text{Sp} \rightarrow \text{Sym}_E$
 $X \mapsto (\text{Sp}_E^{\text{fp}} \ni P \mapsto F(P, X) \in \text{Sp})$
 $(Y(X))$ preserves finite products \Rightarrow spectral, and is a sheaf using the recognition principle 2.8) mapping spectrum $\Omega^{\infty} F(P, X) \simeq \text{Map}(P, X)$

\exists canonical map $\mathcal{P}X \rightarrow Y(X)$ which is a ...
 sheafification of $F(-, X)_{\geq 0}$

τ -inversion: induces $\tau^{-1} \mathcal{P}X \simeq Y(X)$ (4.36)
 connective covers: induces $\mathcal{P}X \simeq (Y(X))_{\geq 0}$

Thm 4.37 The spectral Yoneda $Y: \text{Sp} \rightarrow \text{Sym}_E$ is fully faithful and has essential image $\text{Sym}_E(\tau^{-1})$. The induced equivalences $\text{Sp} \simeq \text{Sym}_E(\tau^{-1}) \simeq \text{Mod}_{\tau^{-1} \mathcal{S}^{0,0}}(\text{Sym}_E)$ are symmetric monoidal.

Proof: $Y: \text{Sp} \rightarrow \text{Sym}_E$ is continuous \Rightarrow exact functor of stable ∞ -cat. egories
 preserving all small limits of the domain \Rightarrow preserves finite colimits

Also preserves filtered colimits, as since each $P \in \text{Sp}_E^{\text{fp}}$ is finite, Y takes filtered colimits to laxwise filtered colimits:
 $Y(\text{colim}_{i \in I} X_i)(P) = F(P, \text{colim}_{i \in I} X_i) \simeq \text{colim}_{i \in I} F(P, X_i)$
 $\uparrow \qquad \uparrow$
 compact filtered
 $= \text{colim}_{i \in I} (Y(X_i)(P))$
 laxwise colimit diagram of sheaves in a colimit diagram $\Rightarrow Y(\text{colim } X_i) \simeq \text{colim}(Y(X_i))$.

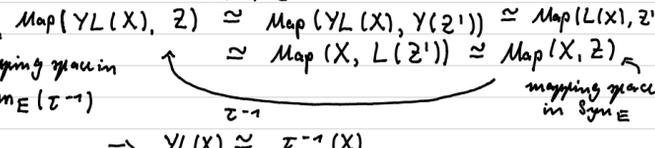
So Y preserves finite and filtered colimits, so all small colimits.

$Y(X) \simeq \tau^{-1} \mathcal{P}X \Rightarrow$ image of Y contained in $\text{Sym}_E(\tau^{-1})$; since the latter is a smashing localization of Sym_E , $Y: \text{Sp} \rightarrow \text{Sym}_E(\tau^{-1})$ also (co)continuous.

fully faithful: fix $B \in \text{Sp}$ and consider the full subcat. C of the $A \in \text{Sp}$ for which $\text{Map}_{\text{Sp}}(A, B) \xrightarrow{\sim} \text{Map}_{\text{Sym}_E}(Y(A), Y(B))$ is an equivalence. Y cocontinuous $\Rightarrow C$ closed under colimits:
 $\text{Map}_{\text{Sp}}(\text{colim } A_i, B) \xrightarrow{\sim} \text{Map}_{\text{Sym}_E}(Y(\text{colim } A_i), Y(B))$
 $\downarrow \qquad \downarrow$
 $\text{colim } \text{Map}_{\text{Sp}}(A_i, B) \xrightarrow{\sim} \text{colim } \text{Map}_{\text{Sym}_E}(Y(A_i), Y(B))$

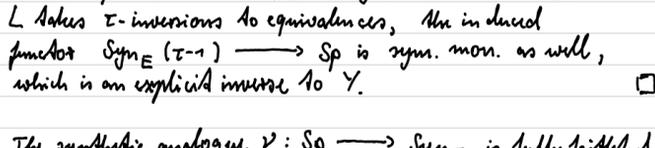
So suffices to show: $S^k \in C \forall k \in \mathbb{Z}$
 $\text{Map}(Y(S^k), Y(B)) \simeq \text{Map}(\mathcal{P}S^k, Y(B)) \simeq \tau^{-1} \mathcal{P}S^k \simeq \Omega^{\infty} Y(B)(S^k) = F(S^k, B) \simeq \text{Map}(S^k, B)$ (4.36) (4.11)

essential image is $\text{Sym}_E(\tau^{-1})$: $R: \text{Sym}_E(\tau^{-1}) \rightarrow \text{Sp}$ right adjoint of Y (exists as Y cocontinuous, ∞ -cats presentable).
 Suffices to show: $RX \simeq 0 \Rightarrow X \simeq 0$ for $X \in \text{Sym}_E(\tau^{-1})$, as $A \in Y$ is f.f., it is onto $\text{Sym}_E(\tau^{-1}) \Leftrightarrow \text{colim } YR \xrightarrow{\sim} \text{id}$

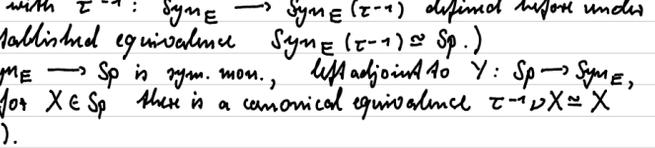


$RX \simeq 0 \Leftrightarrow 0 \simeq \text{Map}(P, 0) \simeq \text{Map}(P, RX) \simeq \text{Map}(Y(P), X) \simeq \text{Map}(\mathcal{P}P, X) \simeq \Omega^{\infty} X(P)$
 holds for all $P \in \text{Sp}_E^{\text{fp}}$, in particular for S^k , $k \geq 0$
 So X (-1)-cocommutative $\stackrel{4.35}{\Leftrightarrow} X \simeq \tau^{-1} X \simeq 0$.
 \uparrow X assumed τ -invertible

symmetric monoidal: know Y induces $\text{Sp} \simeq \text{Sym}_E(\tau^{-1})$. Y cocontinuous \Rightarrow admits left adjoint $L: \text{Sym}_E \rightarrow \text{Sp}$ which under $\text{Sp} \simeq \text{Sym}_E(\tau^{-1})$ corresponds to $\tau^{-1}: \text{Sym}_E \rightarrow \text{Sym}_E(\tau^{-1})$:



$\text{Map}(YL(X), Z) \simeq \text{Map}(YL(X), Y(Z)) \simeq \text{Map}(L(X), Z) \simeq \text{Map}(X, L(Z)) \simeq \text{Map}(X, Z)$
 mapping space in $\text{Sym}_E(\tau^{-1})$ $\xrightarrow{\tau^{-1}}$ mapping space in Sym_E
 $\Rightarrow YL(X) \simeq \tau^{-1}(X)$.



Day convolution on Sym_E is the unique symmetric monoidal structure on Sym_E which is continuous in each variable and makes $\mathcal{P}: \text{Sp}_E^{\text{fp}} \rightarrow \text{Sp}$ sym. mon. $\Rightarrow L$ acquires a canonical sym. mon. structure. L takes τ -inversions to equivalences, the induced functor $\text{Sym}_E(\tau^{-1}) \rightarrow \text{Sp}$ is sym. mon. as well, which is an explicit inverse to Y . \square

Cor 4.38 The synthetic analogue $\mathcal{P}: \text{Sp} \rightarrow \text{Sym}_E$ is fully faithful.

Def 4.39 $\tau^{-1}: \text{Sym}_E \rightarrow \text{Sp}$, $X \mapsto \text{colim } \Sigma^{-n} X(S^{-n})$ τ -inversion / underlying spectrum functor.

(Agrees with $\tau^{-1}: \text{Sym}_E \rightarrow \text{Sym}_E(\tau^{-1})$ defined before under the established equivalence $\text{Sym}_E(\tau^{-1}) \simeq \text{Sp}$.)
 $\tau^{-1}: \text{Sym}_E \rightarrow \text{Sp}$ is sym. mon., left adjoint to $Y: \text{Sp} \rightarrow \text{Sym}_E$, and for $X \in \text{Sp}$ there is a canonical equivalence $\tau^{-1} \mathcal{P}X \simeq X$ (4.40).

established generic fibre " τ^{-1} " special mod. talk! fibre