From formal groups to elliptic cohomology

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November 2023

Abstract

Give me a ring and a formal group law over it, then I'll give you a cohomology theory! Although... I have to admit, your ring should be Landweber flat as an $MU^*(pt)$-module. In this talk we will find out what all that means and how complex cobordism is able to classify cohomology theories which have Euler classes for complex vector bundles. This is a celebrated result of Daniel Quillen which opened up the world of chromatic homotopy theory. We will use it to rephrase topological K-theory in terms of complex cobordism and see how it can be used to get new cohomology theories from elliptic curves.

Contents

1 Formal group laws

Definition 1.1. Let R be a commutative ring and let $F \in R[\![x,y]\!]$ be a power series in two variables with coefficients in R. Then F is a *(commutative, one-dimensional) formal group law*, if is satisfies

$$\begin{aligned} F(0,x) = F(x,0) &= x && \text{(neutral element)}, \\ F(x,\, F(y,z)) &= F(F(x,y),\, z) && \text{(associativity)}, \\ F(x,y) &= F(y,x). && \text{(symmetry)}. \end{aligned}$$

Can construct a graded ring $\mathcal{L}$ called *Lazard ring*, which has a universal formal group law $F_{\text{uni}}(x,y) = \sum_{i,j\geq 0} l_{ij} x^i y^j \in \mathcal{L}[\![x,y]\!]$ over it:

$$\mathcal{L} \cong \mathbb{Z}[l_{ij} \mid i,j \geq 0]/(\text{relations (1), (2), (3)})$$

with relations imposed by the formal group law:

(1) $l_{10} = l_{01} = 1$,

(2) $l_{ij} = l_{ji}$ for all i, j and $l_{i0} = 0$ for $i \neq 1$,

(3) the relations imposed by the associativity law.

The Lazard ring has the following universal property: for any ring R with formal group law $G(x, y) \in R[\![x, y]\!]$, there is a unique ring homomorphism $\delta : \mathcal{L} \to R$ mapping F_{uni} to G. This gives a bijection

$$\mathrm{Hom}_{\mathrm{Rng}}(\mathcal{L}, R) \overset{\sim}{\longleftrightarrow} \{\text{Formal group laws on } R\}.$$

In fact, $\mathcal{L}$ has a simpler structure as the presentation above might suggest:

Theorem 1.2 (Lazard, 1955). *$\mathcal{L} \cong \mathbb{Z}[x_1, x_2, x_3, \ldots]$, where x_i has grading $-2i$.*

2 Complex oriented cohomology

Let us consider a contravariant functor on the finite CW-pairs (think pairs of topological spaces)

$$h^* : \mathbf{CW}_{\mathrm{f}}^{\mathrm{op}} \to \mathbf{AbGp}_\bullet$$

First we assume it to be a generalized cohomology theory, so it satisfies:

- Homotopy invariance (homotopic maps induce same homomorphism on cohomology)
- Exactness (CW-pairs induce a long exact sequence on cohomology)
- Excision (excising subspaces induces an isomorphism on cohomology)
- Additivity (cohomology of disjoint union is product of cohomology)

It should furthermore satisfy:

- Multiplicativity (each graded cohomology group is compatibly equipped with the structure of a graded ring $\rightsquigarrow$ can replace $\mathbf{AbGp}_\bullet$ by $\mathbf{Rng}_\bullet$)
- Complex orientation, i.e., we have a theory of Euler classes: for every complex line bundle $L \to X$ over a finite CW-complex X there is an Euler class $e^h(L) \in h^2(L)$ s.t.
  - e^h is natural for bundle maps
  - $h^*(\mathbb{CP}^n) \cong h^*(pt)[z]/z^{n+1}$, where $z = e^h(\gamma_{\mathbb{C}}^{1,n+1})$, for all $n \geq 0$, where $\gamma_{\mathbb{C}}^{1,n+1} \to \mathbb{CP}^n$ is the tautological line bundle

Such a cohomology theory is said to be *complex oriented.*

Proposition 2.1. *If $h^* : \boldsymbol{CW}_{\mathrm{f}}^{\mathrm{op}} \to \boldsymbol{Rng}_\bullet$ is a complex oriented cohomology theory, then there is a formal group law $F_h(x, y)$ over $h^*(pt)$ satisfying*

$$e^h(L_1 \otimes L_2) = F_h(e^h(L_1), e^h(L_2)).$$

for any two line bundles $L_1, L_2 \to X$.

The proof relies on the fact that $\mathbb{CP}^\infty$ is classifying for line bundles and the properties $L \otimes \underline{\mathbb{C}}^1 \cong L,\ L_1 \otimes L_2 \cong L_2 \otimes L_1,\ (L_1 \otimes L_2) \otimes L_3 \cong L_1 \otimes (L_2 \otimes L_3)$ of the tensor product of line bundles. This formal group law F_h is referred to as the group law of the theory h.

Example 2.2. (i) Integral cohomology: $h^* = H^*(-;\mathbb{Z})$, $H^*(pt;\mathbb{Z}) = \mathbb{Z}$, $e^H(L_1 \otimes L_2) = e^H(L_1) + e^H(L_2)$ for line bundles $L_1, L_2 \to X \rightsquigarrow F_H(x,y) = x + y$ over $\mathbb{Z}$.

(ii) Complex K-theory $h^* = K^*(-)$, $K^*(pt) = \mathbb{Z}[\beta, \beta^{-1}]$, $|\beta| = -2$, $e^K(L) = 1 - \overline{L}$ for a line bundle $L \to X$, $e^K(L_1 \otimes L_2) = e^K(L_1) + e^K(L_2) - e^K(L_1)e^K(L_2)$ for two line bundles $L_1, L_2 \to X \rightsquigarrow F_K(x,y) = x + y - xy$ in the $\mathbb{Z}/2$-graded case, $x + y - \beta xy$ in the $\mathbb{Z}$-graded case.

3 Complex cobordism

There is a complex oriented cohomology theory called MU, complex cobordism. It is represented by a CW-spectrum which is constructed as follows. For $\xi \to X$ a complex vector bundle with Hermitian inner product over a CW-complex X, we can consider its disc bundle $\mathbb{D}(\xi)$ and sphere bundle $\mathbb{S}(\xi)$ to from the Thom space $\mathbb{T}(\xi) \coloneqq \mathbb{D}(\xi)/\mathbb{S}(\xi)$. Consider the classifying space

$$BU(n) \coloneqq \mathrm{Gr}_n(H) = \{V \leq H \mid \dim(V) = n\}$$

of $U(n)$, where H an infinite-dimensional complex Hilbert space, together with its tautological bundle

$$\gamma^n \to BU(n).$$

It has the property that any complex vector bundle with Hermitian inner product over a paracompact space X is the pullback of γ^n by a map $X \to BU(n)$, unique up to homotopy. Then setting $MU(n) \coloneqq \mathbb{T}(\gamma^n)$ gives the complex cobordism spectrum (structure maps omitted).

$$\widetilde{MU}^n(X) \coloneqq [\Sigma^{-n}X, MU] = \varinjlim_{k\to\infty} [\Sigma^k X, MU(n+k)]$$

defines a reduced cohomology theory, which belongs to the generalized cohomology theory $MU^* : \mathbf{CW}^{\mathrm{op}} \to \mathbf{AbGp}$ complex cobordism. There is a geometric interpretation for manifolds which are smooth and embeddable into Euclidean space as a smooth, closed submanifold:

Proposition 3.1. *For such a manifold X,*

$$MU^q(X) \cong \left\{ f : Z \to X \;\middle|\; \begin{array}{c} f \text{ smooth, proper, complex oriented,} \\ \text{of dimension -}q \end{array} \right\} \Big/ \text{cobordism.}$$

In fact MU is a ring spectrum which make the theory multiplicative, and it has Gysin maps and a Euler classes for complex vector bundles which make it complex oriented. So there is a complex cobordism formal group law $F_{MU}(x,y)$ over the ring $MU^*(pt)$.

Theorem 3.2 (Quillen, 1971)**.** *The map $\delta : \mathcal{L} \to MU^*(pt)$ sending $F_{\mathrm{uni}} \mapsto F_{MU}$ from the universal property of the Lazard ring is a ring isomorphism. Consequently,*

$$MU^*(pt) \cong \mathbb{Z}[x_1, x_2, x_3, \ldots], \text{ with } |x_i| = -2i.$$

Proof sketch. (1) For $C = \langle c_{ij} \mid i, j \geq 0 \rangle \leq MU^*(pt)$ the subring generated by the coefficients of the formal group F_{MU}, a structure theorem for $MU^*(X)$ shows that $MU^{\text{even}}(pt) \cong C$ and $MU^{\text{odd}}(pt) = 0$. The map δ sends $F_{\text{uni}} \mapsto F_{MU}$, so $l_{ij} \mapsto c_{ij}$. As $MU^*(pt)$ is generated by the c_{ij}, δ is surjective.

(2) There is a cohomology operation $\beta : MU^*(-) \to H^*(-)[t_1, t_2, \ldots]$ called the *Boradman map* which on a point gives a homomorphism $\beta : MU^*(pt) \to \mathbb{Z}[t_1, t_2, \ldots]$. We show that $\beta \circ \delta : \mathcal{L} \to \mathbb{Z}[\mathbf{t}]$ induces an isomorphism of rationalizations

$$\mathbb{Q} \otimes \mathcal{L} \to \mathbb{Q} \otimes \mathbb{Z}[t_1, t_2, \ldots] \cong \mathbb{Q}[t_1, t_2, \ldots].$$

(3) A homomorphism $u : \mathbb{Z}[t_1, t_2, \ldots] \to \mathbb{Q}[t_1, t_2, \ldots]$ can be identified with the (invertible) power series $\theta_u(z) := \sum_{j \geq 0} u(t_j) z^{j+1},\ t_0 = 1$, as u is uniquely determined by the images of the $t_j, j \geq 1$. Under the bijection

$$\mathrm{Hom}_{\mathrm{Rng}}(\mathcal{L}, \mathbb{Q}[t_1, t_2, \ldots]) \xleftrightarrow{\sim} \{\text{Formal group laws on } \mathbb{Q}[t_1, t_2, \ldots]\}$$

the composite $u \circ \beta \circ \delta : \mathcal{L} \to \mathbb{Q}[t_1, t_2, \ldots]$ corresponds to the formal group law $u\beta\delta(F_{\text{uni}}) = \theta_u^*(x + y)$ over $\mathbb{Q}[t_1, t_2, \ldots]$ (conjugation of the additive group law by the power series θ_u).

(4) Now any formal group law over $\mathbb{Q}[t_1, t_2, \ldots]$ is of the form $\theta_a^*(x+y)$ for a unique power series $\theta_a = \sum_j a_j z^j$, which is called the *logarithm* of the group law. So the assignment $(u : \mathbb{Z}[\mathbf{t}] \to \mathbb{Q}[t_1, t_2, \ldots]) \mapsto \theta_u^*$ is bijective.

(5) Using the correspondence above again, precomposing with $\beta \circ \delta$ induces a bijection

$$\mathrm{Hom}_{\mathbf{Rng}}(\mathbb{Z}[t_1, t_2, \ldots], \mathbb{Q}[t_1, t_2, \ldots]) \xleftrightarrow{\sim} \mathrm{Hom}_{\mathbf{Rng}}(\mathcal{L}, \mathbb{Q}[t_1, t_2, \ldots]).$$

Hence, $\mathrm{id}_{\mathbb{Q}} \otimes (\beta \circ \delta) : \mathbb{Q} \otimes \mathcal{L} \to \mathbb{Q} \otimes \mathbb{Z}[t_1, t_2, \ldots] \cong \mathbb{Q}[t_1, t_2, \ldots]$ is an isomorphism. By the structure theorem for the Lazard ring we saw before, $\mathcal{L}$ is torsion free, so $\beta \circ \delta$ is injective. Therefore, δ is also injective, hence an isomorphism.

$\square$

With this we can now do the following: given a ring R and a formal group law $G \in R[\![x, y]\!]$, there is a unique ring homomorphism $\varphi : MU^*(pt) \to R$ sending the universal formal group law F_{MU} over $MU^*(pt)$ to G. In particular, φ makes R into an $MU^*(pt)$-module. Then we can consider the functor

$$R^*(-) := MU^*(-) \otimes_{MU^*(pt)} R : \ \mathbf{CW}^{\mathrm{op}} \to \mathbf{AbGp}.$$

It clearly is a homotopy invariant functor and satisfies excision, its coefficient ring is

$$R^*(pt) = MU^*(pt) \otimes_{MU^*(pt)} R \cong R,$$

which again carries G as a formal group law over it. But in general, the functor $R^*(-)$ is not a generalized cohomology theory, as one can not expect that tensoring with an arbitrary ring preserves exact sequences. The assumption that R is flat over $MU^*(pt)$ would make $R^*(pt)$ into a cohomology theory, but this assumption turns out to be too strong in practise. With the *Landweber exact functor theorem* will give a more refined condition on R.

4 Rephrasing topological K-theory

Topological K-theory is a complex oriented cohomology theory which uses the vector bundles a space (best behaved for compact Hausdorff spaces) admits: $K^0(X)$ is the group completion of the abelian monoid $(\text{Vect}(X), \oplus, \underline{\mathbb{C}}^0)$, where

$$\text{Vect}(X) := \{\text{vector bundles } E \to X\}/\text{isomorphism}.$$

Then one sets $\widetilde{K}^{-i}(X) := \widetilde{K}^0(\Sigma^i X)$ for $i \geq 0$, and using Bott periodicity, one shows that there are in fact only two distinct K-groups K^0 and K^{-1}. For K-theory, the above holds:

Theorem 4.1 (Conner, Floyd, 1966)**.** *For every finite CW-pair (X, A), there is an isomorphism of $\mathbb{Z}$-graded rings*

$$K^*(X, A) \cong MU^*(X, A) \otimes_{MU^*(pt)} K^*(pt).$$

The $\mathbb{Z}/2$-graded version of this theorem would be $K^*(X, A) \cong MU^*(X, A) \otimes_{MU^*(pt)} \mathbb{Z}$, where the isomorphism id one of $\mathbb{Z}/2$-graded rings. Here we use $K^0(pt) = \mathbb{Z}$ and $K^{-1}(pt) = 0$. The proof heavily relies on the existence of a cohomology operation $\mu : MU^* \to K^*$, for which it can be shown that the map $\mu : MU^*(pt) \to K^*(pt)$ is the one the universal property of the Lazard ring induces, so the theorem really comes in the proposed fashion.
Given the result for K-theory from above one could expect that the situation is different to the one with an arbitrary $MU^*(pt)$-module when you start with a multiplicative cohomology theory $h^* : \mathbf{CW}_{\mathrm{f}}^{\mathrm{op}} \to \mathbf{AbGp}_\bullet$ and form the functor $MU^*(-) \otimes_{MU^*(pt)} h^*(pt)$ using a given homomorphism $MU^*(pt) \to h^*(pt)$. Although MU^* and h^* might be related by a cohomology operation in the first place, $h^*(X) \cong MU^*(X) \otimes_{MU^*(pt)} h^*(pt)$ might not hold, and furthermore $MU^*(-) \otimes_{MU^*(pt)} h^*(pt)$ need not be a cohomology theory.

5 The Landweber exact functor theorem

We can fix generators $x_i \in MU^{-2i}(pt)$ of $MU^*(pt) \cong \mathbb{Z}[x_1, x_2, x_3, \ldots]$ such that for the corresponding generators $x_i \in MU_{2i}(pt)$ of the complex bordism ring $MU_*(pt) \cong \mathbb{Z}[x_1, x_2, x_3, \ldots]$ satsify: for each prime p, all Chern numbers of x_{p^n-1} are divisible by p. Then define the prime ideals

$$I(p, n) := (p, x_{p-1}, \ldots, x_{p^{n-1}-1}) \trianglelefteq MU^*(pt)$$

for all primes p and $n \geq 0$ (the quotient $MU^*(pt)/I(p, n)$ is isomorphic to a polynomial ring over $\mathbb{Z}/p$ which is an integral domain, so the ideals are indeed prime).

Definition 5.1. An $MU^*(pt)$-module R is *Landweber flat* if for each prime p and each integer $n > 0$, multiplication by p on R and by x_{p^n-1} on $R/I(p, n)R$ is injective (i.e., these elements are not zero divisors in the respective rings).

Theorem 5.2 (Landweber exact functor theorem)**.** *Let R be an $MU^*(pt)$-module. Then the functor*

$$R^*(-) = MU^*(-) \otimes_{MU^*(pt)} R : \boldsymbol{CW}_{\mathrm{f}}^{\mathrm{op}} \to \boldsymbol{AbGp}_\bullet$$

defines a generalized cohomology theory if and only if R is Landweber flat.

We can apply this in the following way: it can be shown that the complex Thom spectrum MU is initial within the CW-spectra E defining complex oriented cohomology theories. So if for such E the cohomology ring $E^*(pt)$ happens to be Landweber flat, then there is an isomorphism of cohomology theories

$$MU^*(-) \otimes_{MU^*(pt)} E^*(pt) \xrightarrow{\sim} E^*(-)$$

which holds on the category of finite CW-complexes.

Example 5.3. Consider $R = \mathbb{Q}$ with additive formal group law $G(x,y) = x + y \in \mathbb{Q}[\![x,y]\!]$, which is the formal group law corresponding to rational (co-)homology. The associated homomorphism making $\mathbb{Q}$ an $MU^*(pt)$-module is the map

$$\varphi : MU^*(pt) \cong \mathbb{Z}[x_1, x_2, \dots] \to \mathbb{Q},\ x_i \mapsto 0.$$

Multiplication by p on $\mathbb{Q}$ is injective, and with $I(p,n) = \mathbb{Q}$ for $n > 0$ multiplication by $x_{p^n-1} = 0$ is injective on $\mathbb{Q}/I(p,n) = 0$. So $MU^*(-) \otimes_{MU^*(pt)} \mathbb{Q}$ is a cohomology theory, and there is an isomorphism

$$MU^*(X) \otimes_{MU^*(pt)} \mathbb{Q} \cong H^*(X;\mathbb{Q})$$

on finite CW-complexes.

Example 5.4. Now consider $R = \mathbb{Z}$ with additive formal group law $G(x,y) = x + y \in \mathbb{Z}[\![x,y]\!]$, corresponding to integral (co-)homology. The associated homomorphism is

$$\varphi : MU^*(pt) \cong \mathbb{Z}[x_1, x_2, \dots] \to \mathbb{Z},\ x_i \mapsto 0.$$

Then multiplication by $x_{p-1} = 0 \in \mathbb{Z}$ on $\mathbb{Z}/I(p,1) = \mathbb{Z}/(p)$ is certainly not injective, so $\mathbb{Z}$ (with this $MU^*(pt)$-module structure) is not Landweber flat. It can be shown that the functor $MU^*(-) \otimes_{MU^*(pt)} \mathbb{Z}$ does not satisfy exactness, and so does not define a cohomology theory. One can explicitly construct CW-complexes for which $MU^*(X) \otimes_{MU^*(pt)} \mathbb{Z} \cong H^*(X;\mathbb{Z})$ fails.

Example 5.5. We can also show that the cobordism interpretation of K-theory is a consequence of the Landweber exact functor theorem. Consider $R = \mathbb{Z}$ with the multiplicative formal group law $G(x,y) = x+y-xy \in \mathbb{Z}[\![x,y]\!]$, corresponding to ($\mathbb{Z}/2$-graded) K-theory. Then the associated homomorphism (up to a sign) is $\varphi : MU^*(pt) \to \mathbb{Z}$ in the case of complex bordism is given by the Todd genus and by the choice of generators we made before sends x_{p-1} to 1. Multiplication by p on $\mathbb{Z}$ is injective, multiplication by $x_{p-1} = 1 \in \mathbb{Z}$ on $\mathbb{Z}/(p) = \mathbb{Z}/I(p,1)$ is injective, and with $I(p,n) = \mathbb{Z}$ for $n \geq 2$ it follows that multiplication by x_{p^n-1} on $\mathbb{Z}/I(p,1) = 0$ is also injective. So with this module structure, $\mathbb{Z}$ is in fact Landweber flat.
For the transformation $\mu : MU^* \to K^*$ Conner and Floyd used to show $MU^*(X) \otimes_{MU^*(pt)} \mathbb{Z} \cong K^*(X)$ from scratch, $\mu_{pt} : MU_*(pt) \to K_*(pt)$ agrees (up to a sign) with $\varphi : MU_*(pt) \to \mathbb{Z}$, and the isomorphism one gets from the Landweber exact functor theorem agrees with the one constructed by Conner and Floyd.

Example 5.6. Let $\delta, \varepsilon \in \mathbb{C}$ with discriminant $\Delta := \varepsilon(\delta^2 - \varepsilon)^2 \neq 0$. Then we can consider an elliptic curve $C \subseteq \mathbb{CP}^2$ given by (homogenizing) the Jacobi quartic equation

$$y^2 = R(x) = 1 - 2\delta x^2 + \varepsilon x^4.$$

It comes with a formal group law $F_R(t_1,t_2) = \frac{t_1\sqrt{R(t_1)}+t_2\sqrt{R(t_2)}}{1-\varepsilon t_1^2t_2^2}$, which can be extended as a power series over the ring $\mathbb{Z}[\frac{1}{2},\delta,\varepsilon,\Delta^{-1}]$. It is Landweber flat. We then get the cohomology theory

$$E^*(-) \coloneqq MU^*(-) \otimes_{MU^*(pt)} \mathbb{Z}[\frac{1}{2},\delta,\varepsilon,\Delta^{-1}]$$

associated to the elliptic curve C on finite CW-complexes. More generally, given an elliptic curve C with homogenized Weierstrass form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

by Hensel's lemma there is a unique power series $w \in \mathbb{Z}[a_1,\dots,a_6][\![t]\!]$ so that $(t,w(t))$ gives a parametrization of C on the affine piece $Y \neq 0$. Another application of Hensel's lemma then gives a power series $F \in \mathbb{Z}[a_1,\dots,a_6][\![t_1,t_2]\!]$ with

$$F(0,0) = 0 \text{ and } (t_1,w(t_1)) + (t_2,w(t_2)) = (F(t_1,t_2),w(F(t_1,t_2))).$$

The group law of the elliptic curve C ensures that F is a formal group law, called the formal group law *associated to* C.
This motivates to consider cohomology theories arising from group laws over elliptic curves in more generality, which are called *elliptic cohomology theories*. They consist of

(i) a multiplicative cohomology theory E which is even periodic, i.e. $E^{2n+1}(pt) = 0$ for all $n \in \mathbb{Z}$ with an element $\beta \in E^{-2}(pt)$ invertible in the ring $E^*(pt)$,

(ii) an elliptic curve C over a commutative ring R, and

(iii) an isomorphism $E^*(pt) \cong R$ as well as an isomorphism of the formal group law from E and the one associated to C over $E^*(pt) \cong R$.

It is then of interest to know if there is some kind of universal elliptic cohomology theory, which would belong to some universal elliptic curve. As there is no canonical choice of Weierstrass equation, this question can not be answered with a single elliptic curve. This search leads to the consideration of *topological modular forms*, where one considers not a single elliptic curve, but a whole *moduli stack* of those.

For a formal group law $F \in R[\![x,y]\!]$ over a ring R and $n \in \mathbb{Z}_{\geq 0}$, we define the n-series $[n]_F(t) \in R[\![t]\!]$ recursively by $[1]_F(t) \coloneqq t$ and $[n]_F(t) \coloneqq F(t,[n-1]_F(t))$. It gives a homomorphism of formal group laws from F to itself:

$$F([n]_F(x),[n]_F(y)) = [n]_F(F(x,y)).$$

Then one shows that if R is a commutative ring in which a prime number p is zero (i.e., R is a $\mathbb{Z}/p$-algebra), then either $[p]_F(t) = 0$, or $[p]_F(t) = \lambda t^{p^n}$ + terms of degree $\geq p^n + 1$, for some $n > 0$ and $\lambda \in R$.
Fixing a prime number p and letting v_n denote the coefficient of t^{n^p} in the p-series $[p]_F(t)$.

- F has *height* $\geq n$ if $v_i = 0$ for $i < n$
- F has *height* n if it has height $\geq n$ and $v_n \in R$ is invertible.

- F has *height* ∞ if it has height $\geq n$ for all $n \geq 0$.

Then the height of formal group laws can be used to organize complex oriented cohomology theories by their formal groups, which is the *chromatic* point of view in stable homotopy theory. To illustrate this, we restrict to perfect fields.

- First assume that k is an algebraically closed field. Then if $\mathrm{char}(k) = 0$, it can be shown that every formal group law is isomorphic to the additive group law $F(x, y) = x + y$ over k, so in this case the situation is not very interesting.

- If $\mathrm{char}(k) > 0$, then formal group laws are in fact determined by their height (which is a positive integer or ∞). For example, the multiplicative formal group law $F(x, y) = x + y + xy$ over k has p-series $[p]_F(t) = (1 + t)^p - 1 = t^p$, so it has height 1. The additive formal group law $F(x, y) = x + y$ over k has p-series $[p]_F(t) = 0$, so it has height ∞.

- Now if $F(x, y) \in k[\![x, y]\!]$ is a formal group law of height $0 < n < \infty$ over a perfect field k, then by a theorem of Morava it comes from an even periodic cohomology theory $K(n)$, which is called *Morava K-theory* and said to *lie on the n-th chromatic level.*

- For instance, if $k = \mathbb{F}_p$ and $F(x, y) = x + y + xy$ is the multiplicative group law which has height 1, then $K(1)$ can be taken as the mod p reduction of complex K-theory.

- Modern research in chromatic homotopy theory tries to understand the behaviour of these chromatic levels and to make use of them.