3.3 Cellular motivic category as spherical sheaves

$\mathrm{Sp}_{\mathbb{C}}$ – stable motivic homotopy category / $\mathbb{C}$

$\cup\!\mid$

$\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$ – cellular motivic homotopy category / $\mathbb{C}$
(smallest subcategory containing all spheres which is closed under colimits)

GOAL Expressing $\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$ as spherical sheaves of spectra:

$$\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}} \simeq \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$$

the site of finite MGL-projective motivic spectra

TODAY: everything inherent to motivic theory, but looking like the stuff from before

$\rightsquigarrow$ this will lead the way to a comparison adjunction $\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}} \rightleftarrows \mathrm{Syn}_{\mathrm{MU}}^{\mathrm{ev}}$ (with $\perp$) which induces an equivalence of p-complete objects at each prime p

We construct a functor $\Upsilon : \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}} \longrightarrow \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$, where we recall $\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$ = finite MGL-projective motivic spectra, ie. $M \in \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$ st. M finite and $\mathrm{MGL}_{*,*}M$ free $\mathrm{MGL}_{*,*}$-module, finitely generated with generators in Chow degree zero. $((s,w) \rightsquigarrow s-2w = 0 \Leftrightarrow s = 2w)$

Def 7.16 $X \in \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$, $\Upsilon X := \mathrm{map}(-,X) : (\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})^{\mathrm{op}} \longrightarrow \mathrm{Sp}$
(presheaf of spectra).
mapping spectrum in the stable ∞-cat $\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$
(Corresponds to the spectral Yoneda embedding, rather than the synthetic analogue.)

Lem 7.17 For $X \in \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$, ΥX is a spherical sheaf of spectra on $\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$ wrt. the $\mathrm{MGL}_{*,*}$-surjection topology. If X is MGL-local, then ΥX is hypercomplete.

it sends hypercovers $U : \Delta_{s,+}^{\mathrm{op}} \longrightarrow \mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$ to a limit diagram in Sp $\overset{A.23}{\Longleftrightarrow}$ a local object wrt. all ∞-connected morphisms ("morphisms whose htpy fibres have vanishing homotopy")

Proof: Motivic analogue of 3.24. $\mathrm{Map}(-,X)$ preserves products, so ΥX is spherical. Sheaf property: use recognition principle 2.8.
$F \longrightarrow M \longrightarrow N$ fibre sequence in $\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$ with second map an $\mathrm{MGL}_{*,*}$-surjection.
Fibres in $\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$ along $\mathrm{MGL}_{*,*}$ surjections are computed in spectra (see 3.22), so sequence is cofibre
$\rightsquigarrow \mathrm{Map}(N,X) \longrightarrow \mathrm{Map}(M,X) \longrightarrow \mathrm{Map}(F,X)$ is a fibre sequence of spectra.
Hypercompleteness: need to show that if $U : \Delta_{s,+}^{\mathrm{op}} \longrightarrow \mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{op}}$ is a hypercover ("matching maps" $U_n \to M_n(U)$ covers),
finite ordinals and injective order-preserving maps
$\mathrm{Map}(U_{-1},X) \longrightarrow \mathrm{Map}(U_0,X) \rightrightarrows \dots$
is a limit diagram of spectra.
As X assumed MGL-local, enough to show that $\operatorname{colim}_{k \in \Delta_s} U_k \longrightarrow U_{-1}$ is an MGL-local equivalence, ie. an iso on MGL-homology, since all motivic spectra here are cellular. This can be verified using the homology of geometric realisation spectral sequence (as in the last part of 3.24). □

$\rightsquigarrow$ Get a functor $\Upsilon : \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}} \longrightarrow \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$

By 2.16, this admits a t-structure where coconnectivity is measured levelwise, and

$$\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})^{\heartsuit} \simeq \mathrm{Sh}_{\Sigma}^{\mathrm{Set}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}) \overset{7.15}{\simeq} \mathrm{Comod}_{\mathrm{MU}_*\mathrm{MU}}^{\mathrm{ev}}.$$

even $\mathrm{MU}_*\mathrm{MU}$ comodules

Now: describe homotopy groups of ΥX (motivic analogue of 4.21), using $(\mathrm{MGL}_{*,*}\mathrm{MGL})_0 \cong \mathrm{MU}_*\mathrm{MU}$ iso. of Hopf algebroids
follows from Hopkins–Morel–Hoyois (7.4)

Lem 7.18 $X \in \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$. Then $\pi_k^{\heartsuit}(\Upsilon X) \cong (\mathrm{MGL}_{*,*}\Sigma^{-k}X)_0 \cong (\mathrm{MGL}_{*,*}X)_k$ iso of even graded abelian groups for any $k \in \mathbb{Z}$.

t-structure homotopy groups $\pi_k^{\heartsuit} = \tau_{\geq 0} \circ \tau_{\leq 0} \circ \Omega^k$

Chow degree k part of MGL-homology

Proof: Υ commutes with finite limits, so is exact, so consider $k = 0$
$(\pi_k^{\heartsuit}(\Upsilon X) \cong \pi_0^{\heartsuit}(\Upsilon(X[-k])) \overset{!}{\cong} (\mathrm{MGL}_{*,*}X[-k])_0) \cong (\mathrm{MGL}_{*,*}X)_k)$.
Choose M_α filtered diagram of finite MGL-projectives with $\operatorname{colim}_\alpha M_\alpha \simeq \mathrm{MGL}$. The equivalence

$$\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})^{\heartsuit} \simeq \mathrm{Comod}_{\mathrm{MU}_*\mathrm{MU}}^{\mathrm{ev}} \quad (7.15)$$

is induced by $\mathrm{MU}_* \circ \mathrm{Re} : \mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}} \longrightarrow \mathrm{Comod}_{\mathrm{MU}_*\mathrm{MU}}^{\mathrm{fpe}}$.
Then $\mathrm{MU}_*\mathrm{Re}(\Sigma^{2l,l}M_\alpha)$ is a filtered diagram of dualisable even comodules with $\operatorname{colim}_\alpha \mathrm{MU}_*\mathrm{Re}(\Sigma^{2l,l}M_\alpha) \simeq \mathrm{MU}_*\mathrm{MU}[2l]$.

$$\rightsquigarrow (\pi_0^{\heartsuit}\Upsilon X)_{2l} \overset{(3.3)}{\simeq} \operatorname{colim}_\alpha \underbrace{(\pi_0^{\heartsuit}\Upsilon X)}_{} (D(\mathrm{MU}_*\mathrm{Re}(\Sigma^{2l,l}M_\alpha)))$$

viewed as a spherical sheaf of sets on fpe $\mathrm{MU}_*\mathrm{MU}$ comodules

$$\simeq \operatorname{colim}_\alpha \underbrace{(\pi_0^{\heartsuit}\Upsilon X)}_{} (\Sigma^{2l,l}DM_\alpha), \quad (*)$$

viewed as a spherical sheaf of sets on fp MGL-spectra

$$\begin{aligned} &\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})^{\heartsuit} \\ \simeq\ &\mathrm{Sh}_{\Sigma}^{\mathrm{Set}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}) \xleftarrow{(\mathrm{MU}_*\mathrm{Re})_*} \\ \simeq\ &\mathrm{Sh}_{\Sigma}^{\mathrm{Set}}(\mathrm{Sp}_{\mathrm{MU}}^{\mathrm{fpe}}) \\ \simeq\ &\mathrm{Sh}_{\Sigma}^{\mathrm{Set}}(\mathrm{Comod}_{\mathrm{MU}_*\mathrm{MU}}^{\mathrm{fpe}}) \\ \simeq\ &\mathrm{Comod}_{\mathrm{MU}_*\mathrm{MU}}^{\mathrm{ev}} \end{aligned}$$

where $DM_\alpha = F(M_\alpha, S^{0,0})$ motivic Spanier–Whitehead dual.
Now $\pi_0^{\heartsuit}\Upsilon X$ sheaf of sets associated to the presheaf
$\pi_0\mathrm{Map}(-,X) \simeq [-,X] : (\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})^{\mathrm{op}} \longrightarrow \mathrm{Set}$.
As in top. case (3.25), value computed by filtered colimit $(*)$ is unchanged by sheafification, so

$$\begin{aligned} (\pi_0^{\heartsuit}\Upsilon X)_{2l} &\simeq \operatorname{colim}(\pi_0^{\heartsuit}\Upsilon X)(\Sigma^{2l,l}DM_\alpha) \\ &\simeq \operatorname{colim}[\Sigma^{2l,l}DM_\alpha, X] && (\text{def } \Upsilon X) \\ &\simeq \operatorname{colim}[S^{2l,l}, M_\alpha \otimes X] \\ &\simeq [S^{2l,l}, \mathrm{MGL} \otimes X] \\ &\simeq \mathrm{MGL}_{2l,l}X. \end{aligned}$$

□

Recall: Have an adjunction $\mathrm{Sh}_{\Sigma}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}) \underset{\Omega^\infty}{\overset{\Sigma_+^\infty}{\rightleftarrows}} \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$ (with $\perp$)
fully faithful
computed levelwise
which by 2.19 restricts to an equivalence

$$\mathrm{Sh}_{\Sigma}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}) \simeq \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})_{\geq 0}$$

sheaves of anima :) connective sheaves of spectra

Lem 7.19 $X \in \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$ st. $\mathrm{MGL}_{*,*}X$ concentrated in ≥ 0-Chow degrees.
$(s,w) \rightsquigarrow s-2w$
Then $\Upsilon X \simeq \Sigma_+^\infty y(X)$, where $y(X) \in \mathrm{Sh}_{\Sigma}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$ sheaf of spaces represented by X.

Proof: ΥX is connective in the natural t-structure on $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$
$\Longleftrightarrow \pi_k^{\heartsuit}\Upsilon X$ vanishes for $k < 0$
$\overset{(7.18)}{\Longleftrightarrow} \mathrm{MGL}_{*,*}X$ concentrated in ≥ 0-Chow degree, which we assumed. ← TYPO IN PAPER
So $\Upsilon X \simeq \Sigma_+^\infty\Omega^\infty\Upsilon X$, but $(\Omega^\infty\Upsilon X)(M) \simeq \Omega^\infty\mathrm{Map}(M,X)$
$\simeq \mathrm{map}(M,X) \Rightarrow \Omega^\infty\Upsilon X \simeq y(X) \Rightarrow \Upsilon X \simeq \Sigma_+^\infty y(X)$. □

Thm 7.20 The functor $\Upsilon : \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}} \longrightarrow \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$ is an equivalence.

Proof: Υ exact functor between stable ∞-cats $\Rightarrow$ preserves finite colimits. So for cocontinuity, need to show it preserves filtered colimits. But any $M \in \mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$ is a finite motivic spectrum, so a compact object of $\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$ $\Rightarrow$ Υ maps filtered colimits to levelwise filtered colimits, and filtered colimits in spherical sheaves are computed levelwise. ← TYPO IN PAPER
Υ fully faithful: $\mathrm{map}(X,Y) \longrightarrow \mathrm{map}(\Upsilon X, \Upsilon Y)$ is an equivalence for $X, Y \in \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$. Fix Y and let X vary.
Can assume $X \in \mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$, as $S^{2k,k} \in \mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$, so topological (de)suspensions of objects of $\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$ generate $\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$ under colimits, and Υ, $\mathrm{map}(-,Y)$ & $\mathrm{map}(-,\Upsilon(Y))$ commute with colimits.
Proof for $M \in \mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$: Claim: $\Upsilon M \simeq \Sigma_+^\infty y(M)$. By 7.19, enough to show $\mathrm{MGL}_{*,*}M$ vanishes in <0-Chow degree. Since $\mathrm{MGL}_{*,*}M$ assumed free on generators in Chow-degree 0, this follows from the case $\mathrm{MGL}_{*,*}$, which is 7.5.
It follows:

$$\begin{aligned} \mathrm{map}(\Upsilon M, \Upsilon Y) &\simeq \mathrm{map}(\Sigma_+^\infty y(M), \Upsilon Y) \\ &\simeq \mathrm{map}(y(M), \Omega^\infty\Upsilon Y) \\ &\simeq \Omega^\infty\Upsilon Y(M) \\ &\simeq \Omega^\infty\mathrm{Map}(M,Y) \\ &\simeq \mathrm{map}(M,Y), \end{aligned}$$

TYPO IN PAPER

so Υ is fully faithful.
Υ essentially surjective: essential image closed under suspensions ($\Sigma\mathrm{Map}(-,X) \simeq \mathrm{Map}(-,\Sigma X)$) and colimits (seen before). Since it contains $\Upsilon M \simeq \Sigma_+^\infty y(M)$ for $M \in \mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$, it must be all of $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$
gen. by $\Sigma_+^\infty y(M)$ under suspensions and colimits □

Cor 7.21 There exist a right complete t-structure on $\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$ with $X \in \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$ connected $\Leftrightarrow$ $\mathrm{MGL}_{*,*}X$ concentrated in ≥ 0-Chow degrees. There is an equivalence $\mathrm{Sp}_{\mathbb{C}}^{\heartsuit} \simeq \mathrm{Comod}_{\mathrm{MU}_*\mathrm{MU}}^{\mathrm{ev}}$.

Proof: t-structure with these properties exists on $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$ by 2.16, 7.15. By 7.20, $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}) \simeq \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$.
Characterisation of connectivity follows from 7.18. □

This t-structure can be extended to all of $\mathrm{Sp}_{\mathbb{C}}$ by keeping the connective part: $(\mathrm{Sp}_{\mathbb{C}})_{\geq 0} := (\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}})_{\geq 0}$.
Then $(\mathrm{Sp}_{\mathbb{C}})_{\leq 0} = \{X \in \mathrm{Sp}_{\mathbb{C}} \mid \underbrace{[M,X]_k}_{\pi_0\mathrm{Map}(M,\Sigma^{-k}X)} = 0 \ \forall M \in \mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}, k \geq 1\}$

Rem 7.22 Plausible (but not shown in the paper): coconnectivity condition in induced t-structure on $\mathrm{Sp}_{\mathbb{C}}$ can be checked only on $M \simeq S^{2k,k}$, rather than all $M \in \mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$. In this case,
X coconnective $\Longleftrightarrow \pi_0\mathrm{Map}(S^{2n,n}, \Sigma^{-k}X) = 0 \ \forall n, k \geq 1$
$= \pi_{2n+k,n}(X) = 0$ (Chow degree $2n+k-2n = k$)
$\Longleftrightarrow \pi_{*,*}(X)$ concentrated in ≤ 0-Chow degree.

Existence of a t-structure on $\mathrm{Sp}_{\mathbb{C}}$ related to $\mathrm{MU}_*\mathrm{MU}$-comodules not that surprising, although the construction of $\mathrm{Sp}_{\mathbb{C}}$ does not involve MU as a homology theory:
Levine showed that the spectral sequence obtained by applying Betti realisation to the slice filtration of $S^{0,0}$ is, up to reindexing, the classical Adams-Novikov spectral sequence.

Rem 7.23 The equivalence $\Upsilon : \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}} \xrightarrow{\sim} \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$ is symmetric monoidal:
usual tensor product; Day convolution from $\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$

The inverse $\Upsilon^{-1} : \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}) \longrightarrow \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$ is a cocontinuous functor of stable ∞-categories, and $\Upsilon M \simeq \Sigma_+^\infty y(M)$ for $M \in \mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$, so have:

$$\begin{tikzcd} \mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}} \arrow[r, "y"] \arrow[rrr, bend right, "="', "\text{incl. of full subcat.}"] & \mathrm{Sh}_{\Sigma}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}) \arrow[r, "\Sigma_+^\infty"] & \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}) \arrow[r, "\Upsilon^{-1}"] & \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}} \end{tikzcd}$$

By 2.31: cocontinuous sym. mon. functor $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}) \to \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$
$\Longleftrightarrow$ cocontinuous sym. mon. functor $\mathrm{Sh}_{\Sigma}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}) \to \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$
$\Longleftrightarrow$ sym. mon. functor $\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}} \longrightarrow \mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}}$ whose left Kan extension factors through $\mathrm{Sh}_{\Sigma}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$,
which in this case is just the inclusion.
$\rightsquigarrow \Upsilon^{-1}$ and so Υ are symmetric monoidal.

Rem 7.24 The equivalence $\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}} \simeq \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$ restricts to an equivalence $(\mathrm{Sp}_{\mathbb{C}}^{\mathrm{cell}})_{\mathrm{MGL}} \simeq \widehat{\mathrm{Sh}}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$
MGL-local spectra; hypercomplete sheaves
(7.17: X MGL-local $\Rightarrow$ ΥX hypercomplete, 7.18: both sides are localisations at the same class of maps)