Even synthetic spectra based on MU

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Abstract

These are informal notes for my talk in the reading group on computing the stable homotopy groups of spheres. The goal is to discuss the subcategory Syn_E^{ev} of Syn_E and some results that hold in the case $E = \mathrm{MU}$.

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1 Even synthetic spectra

In the construction of synthetic spectra, if we replace the indexing ∞-category of finite E-projective spectra with even projective spectra, we obtain the notion of even synthetic spectra $Syn_E^{ev} = Sh_\Sigma^{Sp}(Sp_E^{fpe})$. In view of theorem 1.2 we can think of Syn_E^{ev} as a full subcategory of Syn_E. The main reason to understand this construction is that when $E = \mathrm{MU}$, the ∞-category of even synthetic spectra is strongly related to the cellular motivic category.

Definition 1. *A spectrum P is finite even projective if it is finite and E_*P is finitely generated projective and concentrated in even degrees. We denote the ∞-category of finite even projective spectra by Sp_E^{fpe}. An Adams-type homology theory is said to be even Adams if E can be written as a filtered colimit $E \simeq \varinjlim E_\alpha$ of finite even projective spectra.*

From the inclusion $Sp_E^{fpe} \hookrightarrow Sp_E^{fp}$, Sp_E^{fpe} inherits a topology and a symmetric monoidal structure making it an excellent ∞-site. Explicitly, a map $P \to Q$ of finite even projective spectra is a covering if it is an E_* surjection and the symmetric monoidal structure is given by the tensor product of spectra.

Definition 2. *An even synthetic spectrum X is a spherical sheaf of spectra on the site Sp_E^{fpe}. We denote the ∞-category of even synthetic spectra by Syn_E^{ev}.*

By general facts about sheaves of spectra on excellent ∞-sites, we obtain that Syn_E^{ev} is a presentable, stable ∞-category with a symmetric monoidal structure which is cocontinuous in each variable. Furthermore, it admits a right complete t-structure compatible with filtered colimits such that $Sh_\Sigma(Ab) \simeq (Syn_E^{ev})^\heartsuit$.

We now show that there exists a natural embedding $Syn_E^{ev} \hookrightarrow Syn_E$, for that we need the following lemma.

Lemma 1.1. *The inclusion $i : Sp_E^{fpe} \to Sp_E^{fp}$ is a morphism of excellent ∞-sites with the covering lifting property.*

Proof. By construction i is a morphism of excellent ∞-sites, to show that it has the covering lifting property, pick an E_*-surjection $Q \to P$ where P is finite even projective and Q is finite projective. We have to show that there exists R finite even projective with a map $R \to Q$ such that the composition $R \to P$ is E_*-surjective. Consider the Spanier-Whitehed dual of $Q \to P$ and the diagram

$$\begin{array}{ccc} DP & \longrightarrow & DQ \\ \downarrow & \swarrow & \\ E \otimes DP & & \end{array}$$

Where the vertical map is induced by the unit of E. The dashed arrow making the diagram commutative exists by the universal coefficient theorem, which implies that $E^*DQ \simeq \mathrm{Hom}_{E_*}(E_*DQ, E_*)$ (see [1] III.13). Finally, we can write $E \simeq \varinjlim E_\alpha$ where E_α is finite even projective and since DQ is finite, the map $DQ \to E \otimes DP$ factors through one of the $E_\alpha \otimes DP$. Then we can take $R = DE_\alpha \otimes P$ with the map into Q the dual of the factorization. ∎

Theorem 1.2. *The inclusion $i : Sp_E^{fpe} \hookrightarrow Sp_E^{fp}$ induces a cocontinuous, symmetric monoidal embedding $Syn_E^{ev} \hookrightarrow Syn_E$ whose image is the full subcategory generated under colimits and suspensions by νP, where P is finite even projective.*

Proof. By [7] proposition 2.22, there is an induced adjunction $i^* \dashv i_* : Syn_E^{ev} \rightleftarrows Syn_E$ on the ∞-category of spherical sheaves of spectra, where i_* is given by precomposition and it is cocontinuous. Being i^* the only cocontinuous functor such that $i^*(\Sigma_+^\infty y(c)) \simeq \Sigma_+^\infty y(i(c))$, by the universal property of Day convolution (see [5] 4.8.1) i^* has a canonical symmetric monoidal structure induced from the one of i. Now we prove that $i^* : Syn_E^{ev} \to Syn_E$ is a fully faithful embedding. Since both i_* and i^* are cocontinuous, it's enough to show that the unit $\nu(P) \to i_* i^* \nu(P)$ is an equivalence for any $P \in Sp_E^{fpe}$. Now by definition of synthetic analogue, $\nu(P)$ is the sheafification of the presheaf $F(-, P)_{\geq 0} : Sp_E^{fp} \to Sp$ restricted along i. By [7] proposition 4, i_* commutes with sheafification, therefore we can identify also $i_* i^* \nu(P)$ with the sheafification of the presheaf $F(-, P)_{\geq 0} : Sp_E^{fp} \to Sp$. ∎

2 Synthetic spectra based on MU

In this section, we prove some results for synthetic spectra based on MU.

2.1 Cellularity

The goal of this subsection is to show that the ∞-category Syn_{MU} is generated under colimits by the bigraded spheres $\mathbb{S}^{k,l}$

Lemma 2.1. *Any graded projective module over $MU_* \simeq \mathbb{Z}[a_1, a_2, ...]$-module is free*

Proof. See [2], proposition 3.2. ∎

Theorem 2.2. *The ∞-category Syn_{MU} is cellular i.e. it is generated under colimits by the bigraded spheres $\mathbb{S}^{k,l}$*

Proof. Let C be the smallest subcategory of Syn_{MU} containing the bigraded spheres and closed under colimits. Since the bigraded spheres are closed under suspension, so is C. Since Syn_{MU} is generated under colimits by suspensions of $\nu(P)$ with $P \in Sp^{fp}_{\mathrm{MU}}$, it is enough to show that $\nu(P) \in C$. By lemma 2.1, MU_*P is free and finitely generated, then the same is true for the integral homology, in fact, the spectral sequence $Tor^{\mathrm{MU}_*}_{*,*}(\mathrm{MU}_*P, H_*) \Rightarrow H_*P$ collapses to

$$\mathrm{MU}_*P \otimes_{\mathrm{MU}_*} H_* \cong H_*P$$

Moreover, MU_*P and $H_*(P,\mathbb{Z})$ must be of the same rank as MU_*-module and H_*-module respectively. We prove the result by induction on the rank of $H_*(P,\mathbb{Z})$. Let this rank be $k \geq 1$ and assume the result for all $Q \in Sp^{fp}_{\mathrm{MU}}$ with $\mathrm{rk}(H_*(Q,\mathbb{Z})) < k$. Let $H_i(P,\mathbb{Z})$ be the lowest non-zero homology group, by Hurewicz we have $\pi_i P \simeq H_i(P,\mathbb{Z})$, hence there is a map $S^i \to P$ corresponding to the inclusion of a free summand of $H_i(P,\mathbb{Z})$. Consider the cofibre sequence

$$S^i \to P \to P'$$

since the first map is injective in integral homology by construction, we obtain a short exact sequence in integral homology, therefore $H_*(P',\mathbb{Z})$ is free of rank $k-1$. Because of [2] lemma 3.1, this implies that MU_*P' is also free of the same rank. We deduce that, since $H_*(P,\mathbb{Z}) \to H_*(P',\mathbb{Z})$ is surjective, so is $\mathrm{MU}_*P \to \mathrm{MU}_*P'$ (see [6]). By [7] lemma 4.23, $\nu\mathbb{S}^l \to \nu P \to \nu P'$ is a fibre sequence of synthetic spectra, but now $\nu\mathbb{S}^l \simeq \mathbb{S}^{l,l}$ and $\nu P'$ belongs to C by inductive hypothesis. Because Syn_{MU} is stable, that fibre sequence is also a cofibre sequence, so after a rotation we see that νP can be written as a colimit of $\mathbb{S}^{l,l}$ and $\Sigma\nu P'$. ∎

2.2 The synthetic dual Steenrod algebra

In this section we give an example of calculation in Syn_{MU} by computing the synthetic dual Steenrod algebra.

Definition 3. *let H be the Eilemberg-MacLane spectrum $H\mathbb{Z}/p$, we call νH the synthetic Eilemberg-MacLane spectrum.*

Observe that H is a commutative ring spectrum and $\nu : Sp^{fp}_{\mathrm{MU}} \to Syn_{\mathrm{MU}}$ is lax symmetric monoidal, so νH is a commutative algebra in synthetic spectra. As a consequence, $\nu H_{*,*}$ is a bigraded commutative ring and $\nu H_{*,*}X$ is a module over it for any synthetic spectrum X. Since H is an MU-module, by [7] proposition 4.60, $\nu H_{*,*} \simeq \mathbb{F}_p[\tau]$. To compute the dual Steenrod algebra $\nu H_{*,*}\nu H$ we consider the MU-algebra given by BP, the Brown-Peterson spectrum. It is a well known fact that $BP_* \simeq \mathbb{Z}_{(p)}[v_1, v_2, ...]$ and, by [4], the spectrum $BP/(v_0, ..., v_k)$ admits the structure of an MU-algebra such that the quotient maps $BP/(v_0, ..., v_k) \to BP/(v_0, ..., v_{k+1})$ are algebra morphisms. Looking at the homotopy groups, we see that $H \simeq \varinjlim BP/(v_0, ..., v_k)$, then since $BP_*BP \simeq BP_*[b_1, b_2, ...]$, it follows that $H_*BP \simeq \mathbb{F}_p[b_1, b_2, ...]$. Consider the cofibre sequence of $BP/(v_0, ..., v_{k-1})$-modules.

$$\Sigma^{2p^k-2}BP/(v_0, ..., v_{k-1}) \to BP/(v_0, ..., v_k) \to BP/(v_0, ..., v_k)$$

where the first map is multiplication by v_k. We obtain a short exact sequence of $H_*BP/(v_0, ..., v_{k-1})$-modules passing in homology

$$H_*BP/(v_0, ..., v_{k-1}) \to H_*BP/(v_0, ..., v_k) \to H_*BP/(v_0, ..., v_{k-1})[2p^k - 1]$$

Fix lifts τ_k of the unit of $H_*BP/(v_0, ..., v_{k-1})[2p^k - 1]$, together with b_i they additively generate the Steenrod algebra. For odd p, necessarily $\tau_k^2 = 0$ by commutativity, so we have an induced map $\mathbb{F}_p[b_1, b_2, ...] \otimes E[\tau_0, \tau_1, ...] \to H_*H$. When $p = 2$ we recall the calculations in [3] which show that we can obtain the relations $\tau_k^2 = b_{k+1}$, so in this case there is an induced map $(\mathbb{F}_2[b_1, b_2, ...] \otimes \mathbb{F}_2[\tau_0, \tau_1, ...])/(\tau_k^2 = b_{k+1}) \to H_*H$. In both cases the maps are injective by dimension count, therefore they are isomorphisms and we have determined the additive structure of H_*H. Now we try to apply similar techniques to the synthetic version of H, so we compute $\nu H_{*,*}\nu H$. Since BP is a direct summand of $\mathrm{MU}_{(p)}$, it is a filtered colimit of projectives, so we are in the hypothesis of [7] proposition 4.24, and we obtain $\nu BP \otimes \nu X \simeq \nu(BP \otimes X)$ for any X. It follows that

$$\nu H_{*,*}\nu BP \simeq \pi_{*,*}\nu(H \otimes BP) \simeq \mathbb{F}_p[b_1, b_2, ...][\tau]$$

with $|b_i| = (2p^i - 2, 2p^i - 2)$ and $|\tau| = (0, -1)$, where the last equivalence follows from [7] proposition 4.60.

Lemma 2.3. *For any $k \geq 0$, the cofibre sequence of $BP/(v_0, ..., v_{k-1})$-modules.*

$$\Sigma^{2p^k-2}BP/(v_0, ..., v_{k-1}) \to BP/(v_0, ..., v_k) \to BP/(v_0, ..., v_k)$$

induce a short exact sequence

$$\nu H_{*,*}\nu BP/(v_0, ..., v_{k-1}) \to \nu H_{*,*}\nu BP/(v_0, ..., v_k) \to \nu H_{*,*}\nu BP/(v_0, ..., v_{k-1})[2p^k-1, 2p^k-2]$$

of $\nu H_{*,*}\nu BP/(v_0, ..., v_{k-1})$-modules.

Proof. The starting short exact sequence induces a short exact sequence in BP, and hence MU homology, therefore by [7] proposition 4.24

$$\nu\Sigma^{2p^k-2}BP/(...,v_{k-1}) \to \nu BP/(...,v_{k-1}) \to \nu BP/(...,v_k)$$

is a cofibre sequence of synthetic spectra. Taking $\nu H_{*,*}$ we obtain a long exact sequence, we show that this splits in short exact sequences as v_k acts by zero on $\nu H_{*,*}\nu BP/(...,v_k)$. First observe that, since multiplication by v_k induces a map of $\nu H_{*,*}\nu BP/(...,v_{k-1})$-modules, it is sufficient to show that the unit of $\nu H_{*,*}\nu BP/(...,v_{k-1})$ is sent to zero. this follows from the fact that the unit is in the image of $\nu H_{*,*}\nu BP \simeq H_*BP[\tau] \simeq \mathbb{F}_p[b_1,...][\tau]$ on which v_k acts by zero. ■

For each k let τ_k be a lift to $\nu H_{2p^k-1,2p^k-2}\nu BP/(...,v_k)$ of the unit of $\nu H_{2p^k-1,2p^k}\nu BP/(...,v_{k-1})$ in the short exact sequence of lemma 2.3. After τ-inversion these τ_k reduce to $\tau^{top} \in H_{2p^k-1}H$ in the topological Steenrod algebra.

Corollary 2.3.1. *The synthetic dual Steenrod algebra $\nu H_{*,*}\nu H$ is generated by elements $b_k \in \nu H_{2p^k-2,2p^k-2}$ in the image of $\nu H_{*,*}\nu BP$ with $k \geq 1$, the elements $\tau_l \in \nu H_{2p^l-1,2p^l-2}\nu H$ with $l \geq 0$ and the element $\tau \in \nu H_{0,-1}\nu H$*

Proof. We show that $\nu H_{*,*}\nu BP/(...,v_k)$ is generated by the b_i, τ and τ_i for $i \leq k$ and then the result follows since $\nu H_{*,*}\nu H \simeq \varinjlim \nu H_{*,*}\nu BP/(...,v_k)$. We work by induction, the base case, follows from $\nu H_{*,*}\nu BP \simeq \mathbb{F}_p[b_1,b_2,...][\tau]$. For $k \geq 0$, we use the short exact sequence from lemma 2.3 we have the short exact sequence

$$\nu H_{*,*}\nu BP/(v_0,...,v_{k-1}) \to \nu H_{*,*}\nu BP/(v_0,...,v_k) \to \nu H_{*,*}\nu BP/(v_0,...,v_{k-1})[2p^k-1,2p^k-2]$$

of $\nu H_{*,*}\nu BP/(v_0,...,v_{k-1})$-modules. Therefore, $\nu H_{*,*}\nu BP/(v_0,...,v_k)$ is an extension of cyclic $\nu H_{*,*}\nu BP/(v_0,...,v_{k-1})$-modules and the generators of both of these are respectively the unit and τ_k. ■

Corollary 2.3.2. *The element τ acts injectively on the synthetic dual Steenrod algebra $\nu H_{*,*}\nu H$*

Proof. Since $\nu H_{*,*}\nu H \simeq \varinjlim \nu H_{*,*}\nu BP/(...,v_k)$ and the transition maps in the diagram are injective, it's enough to show that τ acts injectively on every $H_{*,*}\nu BP/(...,v_k)$, but this follows by induction on k using the short exact sequence from lemma 2.3. ■

We can finally prove the structural theorems for the synthetic dual Steenrod algebra.

Theorem 2.4. *Let p be an odd prime and $\nu H_{*,*}\nu H$ the corresponding synthetic dual Steenrod algebra. Then $\nu H_{*,*}\nu H \cong \mathbb{F}_p[b_1,...][\tau] \otimes_{\mathbb{F}_p} E[\tau_0,...]$ as a bigraded algebra. In particular $\nu H_{*,*}\nu H \cong H_*H \otimes_{\mathbb{F}_p} \mathbb{F}_p[\tau]$.*

Proof. We have seen that the algebra is generated by the b_i, τ_k and τ and the map $\nu H_{*,*}\nu BP \to \nu H_{*,*}\nu H$ induces an inclusion $\mathbb{F}_p[b_1, ...][\tau]$. Since $|\tau_k| = (2p^k - 1, 2p^k - 2)$ are of odd topological degree, by commutativity we have that $\tau_k^2 = 0$. We deduce that there is an induced map

$$\mathbb{F}_p[b_1, b_2, ..., \tau] \otimes_{\mathbb{F}_p} E(\tau_0, \tau_1, ...) \to \nu H_{*,*}\nu H$$

Contemplating the short exact sequence of lemma 2.3, it is easy to see that this map is also an isomorphism of $\mathbb{F}_p$-vector spaces. Therefore it is an isomorphism of algebras. ■

Theorem 2.5. *Let $p = 2$ and $\nu H_{*,*}\nu H$ be the corresponding synthetic dual Steenrod algebra. Then there is an isomorphism of bigraded algebras*

$$\nu H_{*,*}\nu H \cong (\mathbb{F}_2[b_1, b_2, ..., \tau, \tau_0, \tau_1, ...])/(\tau_k^2 = \tau^2 b_{k+1})$$

Proof. We can repeat the argument of the previous lemma with the difference that this time we want the relation $\tau_k^2 = \tau^2 b_{k+1}$. After τ-inversion b_{k+1}, τ_k, reduce to the usual elements $b_{k+1}^{top}, \tau_k^{top} \in H_*H$. Here the relation $b_{k+1}^{top} = (\tau_k^{top})^2$ is classical, we deduce that b_{k+1} and τ_k coincides after multiplying both sides by sufficiently large powers of τ. Since $|b_{k+1}| = (2p^{k+1} - 2, 2p^{k+1} - 2)$ and $|\tau_k| = (2p^k - 1, 2p^k - 2)$ (with $p = 2$), the elements $\tau^2 b_{k+1}$ and τ_k^2 are in the same degree, so we must have $\tau^l(\tau^2 b_{k+1} - \tau_k^2) = 0$. However, by corollary 2.3.2, τ acts injectively on $\nu H_{*,*}\nu H$ hence $\tau^2 b_{k+1} = \tau_k^2$. ■

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