2.6 The generic fibre

§1 Cofibre of τ

Def 4.27 We denote by $\tau : \nu S^{-1} \longrightarrow \Omega(\nu S^0)$ the canonical limit comparison map induced by $\Omega S^0 \simeq S^{-1}$.

$$\begin{tikzcd}
\Omega S^0 \simeq S^{-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \arrow[r, squiggly, "\nu(-)"] & \nu S^{-1} \arrow[dr, dashed, "\tau"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
0 \arrow[r] & S^0 & & \Omega(\nu S^{-1}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
& & & 0 \arrow[r] & \nu S^0
\end{tikzcd}$$

By definition, $S^{-1,-1} = \Sigma^0 \nu S^{-1} \simeq \nu S^{-1}$,
$S^{-1,0} = \Sigma^{-1} \nu S^0 = \Omega \nu S^0$,
so can consider $\tau : S^{-1,-1} \longrightarrow S^{-1,0}$ as an element of
$\pi_{0,-1} S^{0,0} = \pi_0 \mathrm{Map}(S^{0,-1}, S^{0,0}) \simeq \pi_0 \mathrm{Map}(S^{-1,-1}, S^{-1,0})$.

Prop 4.28 For $X \in \mathrm{Syn}_E$, the map $\tau \otimes \mathrm{id}_X : \Sigma^{-1,-1} X \longrightarrow \Sigma^{-1,0} X$ can be identified with the canonical colimit-to-limit comparison map $X(\Sigma P) \longrightarrow \Omega X(P)$, where P runs through Sp_E^{fp}.

Proof: (This uses $(\Sigma^{k,l} X)(P) \simeq \Sigma^{k-l} X(\Sigma^{-l} P)$ (4.26)).
τ is induced by the diagram

$$\begin{tikzcd}
\nu S^{-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \nu S^0
\end{tikzcd},$$

so $\tau \otimes X$ is induced by the diagram

$$\begin{tikzcd}
\nu S^{-1} \otimes X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \nu S^0 \otimes X
\end{tikzcd}.$$

By the tensor structure on Syn_E, for $Q \in \mathrm{Sp}_E^{fp}$:
$(\nu Q \otimes X)(P) \simeq X(Q^* \otimes P)$ (2.27)
So $\tau \otimes \mathrm{id}_X$ over $P \in \mathrm{Sp}_E^{fp}$ induced by applying X to

S^1 is the dual of S^{-1} →

$$\begin{tikzcd}
S^1 \otimes P & 0 \arrow[l] \\
0 \arrow[u] \arrow[ur, phantom, "\lrcorner", very near end] & P \arrow[l] \arrow[u]
\end{tikzcd}$$

obtained by dualizing the pullback square witnessing $S^{-1} \simeq \Omega^0$ and applying $- \otimes P$.

Pushout witnessing $S^1 \otimes P \simeq \Sigma P$, so applying X gives canonical map $X(\Sigma P) \longrightarrow \Omega X(P)$. □

Lem 4.29 $X \in \mathrm{Sp} \Rightarrow$ cofiber sequence $\Sigma^{0,-1} \nu X \xrightarrow{\tau \otimes \mathrm{id}_{\nu X}} \nu X \longrightarrow C\tau \otimes \nu X$ induces $C\tau \otimes \nu X \simeq (\nu X)_{\leq 0} \in \mathrm{Syn}_E^{\heartsuit}$ and $\Sigma^{0,-1} \nu X \simeq (\nu X)_{\geq 1}$.

Cor 4.30 The cofibre $C\tau = \mathrm{cofib}(S^{0,-1} \longrightarrow S^{0,0})$ has a unique structure of a commutative algebra in Syn_E so that the canonical map $S^{0,0} \longrightarrow C\tau$ is a map of com. alg. objects.

Proof: By 4.29, $C\tau \simeq (S^{0,0})_{\leq 0}$ and $S^{0,0}$ is connective, so its t-structure truncations are canonically commutative algebra objects ($\tau_{\leq 0} : (\mathrm{Syn}_E)_{\geq 0} \longrightarrow \mathrm{Syn}_E^{\heartsuit}$ is lax monoidal.) □

Rem 4.31 Using the canonical map $S^{0,0} \longrightarrow C\tau$ we can build for any $Y_0 \in \mathrm{Syn}_E$ a tower of "cofibres of powers of τ":

$$\begin{tikzcd}
\cdots \arrow[r] & Y_1 \arrow[r] \arrow[d] & Y_0 \arrow[d] \\
& C\tau \otimes Y_1 & C\tau \otimes Y_0
\end{tikzcd}$$

Syn_E stable

with each $Y_{n+1} \longrightarrow Y_n \longrightarrow C\tau \otimes Y_n$ fibre/cofibre.
If $X \in \mathrm{Sp}$, $Y_0 = \nu X$, then $C\tau \otimes Y_0 \simeq (\nu X)_{\leq 0}$,
$Y_1 \simeq \mathrm{cofib}(\nu X \longrightarrow (\nu X)_{\leq 0}) \simeq (\nu X)_{\geq 1} \simeq \Sigma^{0,-1} \nu X$,
$C\tau \otimes Y_1 \simeq \Sigma^{0,-1}(\nu X)_{\leq 0} \simeq (\nu X)_{\leq 1}$,
$Y_2 \simeq \mathrm{cofib}((\nu X)_{\geq 1} \longrightarrow (\nu X)_{\leq 1}) \simeq (\nu X)_{\geq 2}$,
and so on, using 4.29.
So the upper row in this case is exactly the Postnikov tower of νX, which agrees with the filtration "by powers of τ".

§2 Inverting τ

Def 4.32 $X \in \mathrm{Syn}_E$ is τ-invertible if the map $\tau : \Sigma^{0,-1} X \longrightarrow X$ ($= \tau \otimes \mathrm{id}_X : S^{0,-1} \otimes X \longrightarrow S^{0,0} \otimes X$) is an equivalence. $\mathrm{Syn}_E(\tau^{-1}) \subseteq \mathrm{Syn}_E$ the full sub-∞-cat. on τ-invertible synthetic spectra.

$$\boxed{\text{Goal: } \mathrm{Syn}_E(\tau^{-1}) \simeq \mathrm{Sp}.}$$

Prop 4.33 The inclusion $\mathrm{Syn}_E(\tau^{-1}) \hookrightarrow \mathrm{Syn}_E$ has a left adjoint $\tau^{-1} : \mathrm{Syn}_E \longrightarrow \mathrm{Syn}_E(\tau^{-1})$, and $\mathrm{Syn}_E(\tau^{-1})$ is closed under colimits of Syn_E.
"Slogan: $\mathrm{Syn}_E \longrightarrow \mathrm{Syn}_E(\tau^{-1})$ is a smashing localization."

Proof: Explicitly, for $X \in \mathrm{Syn}_E$, let
$\tau^{-1} X := \mathrm{colim}(X \to \Sigma^{0,1} X \to \Sigma^{0,2} X \to \dots)$
colimit over the poset $\mathbb{N}$, $\Sigma^{0,k} X \longrightarrow \Sigma^{0,k+1} X$ given by $\Sigma^{0,k+1}(\tau \otimes \mathrm{id}_X)$.
Then for $Y \in \mathrm{Syn}_E(\tau^{-1})$, have

$$\begin{aligned}
\mathrm{Map}(\tau^{-1} X, Y) &\simeq \mathrm{Map}(\mathrm{colim}\, \Sigma^{0,k} X, Y) \\
&\simeq \lim \mathrm{Map}(\Sigma^{0,k} X, Y) \\
&\simeq \lim \mathrm{Map}(X, \Sigma^{0,-k} Y) \\
&\simeq \mathrm{Map}(X, Y)
\end{aligned}$$

using that all maps $\Sigma^{0,-k-1} Y \longrightarrow \Sigma^{0,-k} Y$ are equivalences as Y assumed τ-invertible

implicitly using that τ is self-dual (up to a shift), as the diagram

$$\begin{tikzcd}
S^{-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & S^0
\end{tikzcd}$$

used to define it witnesses $S^{-1} \simeq \Omega S^0$ and its dual witnesses $S^0 \simeq \Omega S^1$

Now $X \in \mathrm{Syn}_E$ is in $\mathrm{Syn}_E(\tau^{-1})$
$\Leftrightarrow S^{0,-1} \otimes X \xrightarrow{\sim} S^{0,0} \otimes X$ ("τ is invertible")
$\Leftrightarrow \mathrm{cofib}(S^{0,-1} \otimes X \longrightarrow S^{0,0} \otimes X) \simeq 0$
$\Leftrightarrow C\tau \otimes X \simeq 0$, a condition closed under colimits as $\otimes$ commutes with colimits in both variables.
So localization is smashing. □

Cor 4.34 $\tau^{-1} S^{0,0} \in \mathrm{CAlg}(\mathrm{Syn}_E)$, $\mathrm{Syn}_E(\tau^{-1}) \simeq \mathrm{Mod}_{\tau^{-1} S^{0,0}}(\mathrm{Syn}_E)$.

Proof: For $X \in \mathrm{Syn}_E$,

$$\begin{aligned}
\tau^{-1} X &\simeq \mathrm{colim}(X \otimes S^{0,k}) \\
&\simeq X \otimes (\mathrm{colim}\, S^{0,k}) \\
&\simeq X \otimes \tau^{-1} S^{0,0}.
\end{aligned}$$

smashing localization

In particular, $\tau^{-1} S^{0,0} \otimes \tau^{-1} S^{0,0} \simeq \tau^{-1} S^{0,0}$.
and $X \in \mathrm{Syn}_E(\tau^{-1}) \Leftrightarrow \tau^{-1} X \simeq X$
$\Leftrightarrow X \otimes \tau^{-1} S^{0,0} \simeq X$. □

Def (Spectral Yoneda embedding) $Y : \mathrm{Sp} \longrightarrow \mathrm{Syn}_E$
$X \longmapsto (\mathrm{Sp}_E^{fp} \ni P \mapsto F(P, X) \in \mathrm{Sp})$

mapping spectrum $\Omega^\infty F(P, X) \simeq \mathrm{Map}(P, X)$

($Y(X)$ preserves finite products $\Rightarrow$ spherical, and is a sheaf using the recognition principle 2.8)

$\exists$ canonical map $\nu X \longrightarrow Y(X)$ which is a ...

νX = sheafification of $F(-, X)_{\geq 0}$; $Y(X)$ = $F(-, X)$

- τ-inversion: induces $\tau^{-1} \nu X \simeq Y(X)$
- connective covers: induces $\nu X \simeq (Y(X))_{\geq 0}$ (4.36)

Thm 4.37 The spectral Yoneda $Y : \mathrm{Sp} \longrightarrow \mathrm{Syn}_E$ is fully faithful and has essential image $\mathrm{Syn}_E(\tau^{-1})$. The induced equivalences $\mathrm{Sp} \simeq \mathrm{Syn}_E(\tau^{-1}) \simeq \mathrm{Mod}_{\tau^{-1} S^{0,0}}(\mathrm{Syn}_E)$ are symmetric monoidal.

Proof:
- $Y : \mathrm{Sp} \longrightarrow \mathrm{Syn}_E$ is continuous $\Rightarrow$ exact functor of stable ∞-categories $\Rightarrow$ preserves finite colimits

  continuous: preserving all small limits of the domain

  It also preserves filtered colimits, as since each $P \in \mathrm{Sp}_E^{fp}$ is finite, Y takes filtered colimits to levelwise filtered colimits:

  $$\begin{aligned}
  Y(\mathrm{colim}_{i \in I} X_i)(P) &= F(P, \mathrm{colim}_{i \in I} X_i) \simeq \mathrm{colim}_{i \in I} F(P, X_i) \\
  &= \mathrm{colim}_{i \in I}(Y(X_i)(P))
  \end{aligned}$$

  compact, filtered

  levelwise colimit diagram of sheaves is a colimit diagram

  $\Rightarrow Y(\mathrm{colim}_{i \in I} X_i) \simeq \mathrm{colim}_{i \in I}(Y(X_i))$.

  So Y preserves finite and filtered colimits, so all small colimits.
- $Y(X) \simeq \tau^{-1} \nu X \Rightarrow$ image of Y contained in $\mathrm{Syn}_E(\tau^{-1})$; since the latter is a smashing localization of Syn_E, $Y : \mathrm{Sp} \longrightarrow \mathrm{Syn}_E(\tau^{-1})$ also (co)continuous.
- fully faithful: fix $B \in \mathrm{Sp}$ and consider the full subcat. C of the $A \in \mathrm{Sp}$ for which $\mathrm{Map}_{\mathrm{Sp}}(A, B) \longrightarrow \mathrm{Map}_{\mathrm{Syn}_E}(Y(A), Y(B))$ is an equivalence. Y cocontinuous $\Rightarrow$ C closed under colimits:

  $$\begin{tikzcd}
  \mathrm{Map}_{\mathrm{Sp}}(\mathrm{colim}_i A_i, B) \arrow[r] \arrow[d, "\wr"] & \mathrm{Map}_{\mathrm{Syn}_E}(Y(\mathrm{colim}_i A_i), Y(B)) \arrow[d, "\wr"] \\
  \mathrm{colim}_i \mathrm{Map}_{\mathrm{Sp}}(A_i, B) \arrow[r, "\sim"] & \mathrm{colim}_i \mathrm{Map}_{\mathrm{Syn}_E}(Y(A_i), Y(B))
  \end{tikzcd}$$

  So suffices to show: $S^k \in C$ $\forall k \in \mathbb{Z}$

  $$\begin{aligned}
  \mathrm{Map}(\underbrace{Y(S^k)}, Y(B)) &\simeq \mathrm{Map}(\nu S^k, Y(B)) && (4.36) \\
  &\simeq \Omega^\infty \underbrace{Y(B)(S^k)}_{= F(S^k, B)} && (4.11) \\
  &\simeq \mathrm{Map}(S^k, B)
  \end{aligned}$$

  $Y(S^k) \simeq \tau^{-1} \nu S^k = \tau^{-1} S^{0,k} \simeq S^{0,k} = \nu S^k$
- essential image is $\mathrm{Syn}_E(\tau^{-1})$: $R : \mathrm{Syn}_E(\tau^{-1}) \longrightarrow \mathrm{Sp}$ right adjoint of Y (exists as Y cocontinuous, ∞-cats presentable).
  Suffices to show: $RX \simeq 0 \Rightarrow X \simeq 0$ for $X \in \mathrm{Syn}_E(\tau^{-1})$, as
  As Y is f.f., it is onto $\mathrm{Syn}_E(\tau^{-1}) \Leftrightarrow$ counit $YR \xrightarrow{\sim} \mathrm{id}$

  $$\left(\begin{tikzcd}
  R(X) \arrow[r, "\eta_{R(X)}", "\sim"'] \arrow[dr, "\mathrm{id}_{R(X)}"'] & RYR(X) \arrow[d, "R\varepsilon_X"] \arrow[dl, phantom, "//"] \\
  & R(X)
  \end{tikzcd}\right)$$

  $\Leftrightarrow$ R conservative
  $\Leftrightarrow$ $(RX \simeq 0 \Rightarrow X \simeq 0)$

  $\left(\begin{aligned} & RX \xrightarrow{\sim} R \\ \Leftrightarrow\ & \mathrm{fib}(RX \to RZ) \simeq 0 \\ & \quad = R(\mathrm{fib}(X \to Z)) \end{aligned}\right)$

  $$\begin{aligned}
  RX \simeq 0 \Rightarrow 0 \simeq \mathrm{Map}(P, 0) &\simeq \mathrm{Map}(P, RX) \simeq \mathrm{Map}(Y(P), X) \\
  &\simeq \mathrm{Map}(\nu P, X) \simeq \Omega^\infty X(P)
  \end{aligned}$$

  holds for all $P \in \mathrm{Sp}_E^{fp}$, in particular for S^k, $k \geq 0$
  So X (-1)-coconnective $\overset{4.35}{\Longrightarrow}$ $X \simeq \tau^{-1} X \simeq 0$.
  X assumed τ-invertible
- symmetric monoidal: know Y induces $\mathrm{Sp} \simeq \mathrm{Syn}_E(\tau^{-1})$.
  Y cocontinuous $\Rightarrow$ admits left adjoint $L : \mathrm{Syn}_E \longrightarrow \mathrm{Sp}$ which under $\mathrm{Sp} \simeq \mathrm{Syn}_E(\tau^{-1})$ corresponds to $\tau^{-1} : \mathrm{Syn}_E \longrightarrow \mathrm{Syn}_E(\tau^{-1})$:

  $$\begin{tikzcd}
  \mathrm{Syn}_E \arrow[r, "L", shift left] \arrow[r, phantom, "\perp"] \arrow[dr, "\tau^{-1}"'] & \mathrm{Sp} \arrow[l, "Y", shift left] \arrow[d, "Y", "\wr"'] & Z' \arrow[d, mapsto] \\
  & \mathrm{Syn}_E(\tau^{-1}) & Z
  \end{tikzcd}$$

  $Y(Z') = Z$

  $$\begin{aligned}
  \mathrm{Map}(YL(X), Z) &\simeq \mathrm{Map}(YL(X), Y(Z')) \simeq \mathrm{Map}(L(X), Z') \\
  &\simeq \mathrm{Map}(X, L(Z')) \simeq \mathrm{Map}(X, Z)
  \end{aligned}$$

  mapping space in $\mathrm{Syn}_E(\tau^{-1})$

  τ^{-1}

  mapping space in Syn_E

  $\Rightarrow YL(X) \simeq \tau^{-1}(X)$.

  $$\begin{tikzcd}
  \mathrm{Sp}_E^{fp} \arrow[r, "\nu"] \arrow[rr, bend right, "//"'] & \mathrm{Syn}_E \arrow[r, "L"] & \mathrm{Sp} \simeq \mathrm{Syn}_E(\tau^{-1}) \\
  P \arrow[r, mapsto] & \nu P \arrow[r, mapsto] & L\nu P
  \end{tikzcd}$$

  usual inclusion, sym. mon.

  $YL\nu P \simeq \tau^{-1} \nu P \simeq YP \Rightarrow L\nu P \simeq P$

  Day convolution on Syn_E is the unique symmetric monoidal structure on Syn_E which is continuous in each variable and makes $\nu|_{\mathrm{Sp}_E^{fp}} : \mathrm{Sp}_E^{fp} \longrightarrow \mathrm{Sp}$ sym. mon.
  $\Rightarrow L$ aquires a canonical sym. mon. structure.
  L takes τ-inversions to equivalences, the induced functor $\mathrm{Syn}_E(\tau^{-1}) \longrightarrow \mathrm{Sp}$ is sym. mon. as well, which is an explicit inverse to Y. □

Cor 4.38 The synthetic analogue $\nu : \mathrm{Sp} \longrightarrow \mathrm{Syn}_E$ is fully faithful.

Def 4.39 $\tau^{-1} : \mathrm{Syn}_E \longrightarrow \mathrm{Sp}$, $X \mapsto \mathrm{colim}_n \Sigma^{-n} X(S^{-n})$
τ-inversion / underlying spectrum functor.

(Agrees with $\tau^{-1} : \mathrm{Syn}_E \longrightarrow \mathrm{Syn}_E(\tau^{-1})$ defined before under the established equivalence $\mathrm{Syn}_E(\tau^{-1}) \simeq \mathrm{Sp}$.)
$\tau^{-1} : \mathrm{Syn}_E \longrightarrow \mathrm{Sp}$ is sym. mon., left adjoint to $Y : \mathrm{Sp} \to \mathrm{Syn}_E$, and for $X \in \mathrm{Sp}$ there is a canonical equivalence $\tau^{-1} \nu X \simeq X$ (4.40).

established — generic fibre — next talk! — special fibre

$$\begin{tikzcd}
& \mathrm{Syn}_E \arrow[dl, "\text{“}\tau = 1\text{”}"'] \arrow[dr, "\text{“}\tau = 0\text{”}"] & \\
\mathrm{Sp} \simeq \mathrm{Syn}_E(\tau^{-1}) \simeq \mathrm{Mod}_{\tau^{-1} S^{0,0}}(\mathrm{Syn}_E) & & \mathrm{Mod}_{C\tau}(\mathrm{Syn}_E)
\end{tikzcd}$$