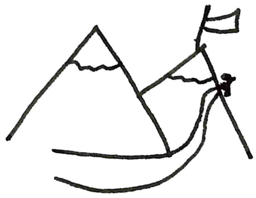
AN ARITHMETIC EXPEDITION INTO K3 SURFACES



André Weil (1958)

K3 - Kummer, Kähler, Kodaira

K2 in Cachemir

- 2^{nd} tallest mountain in the world
- ~ 400 people
- $1/5$ mortality rate

1- WHAT IS A K3 SURFACE?

- Most natural generalisation of elliptic curves

  → Abelian surfaces (because of their group law)

- What are special features of elliptic curves that can be generalised to higher dimensions?

$\Omega^1_E = \overline{K}(C)$-vector space generated by symbols of the form dx for $x \in \overline{K}(C)$ satisfying differential properties

Let $C: y^2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$

To every differential we can associate a divisor

$$\mathrm{div}(\omega) = \sum_{P \in C} \mathrm{ord}_P(\omega)\, P$$

Then, in elliptic curves

$$\mathrm{div}(dx) = (\alpha_1, 0) + (\alpha_2, 0) + (\alpha_3, 0) - 3\infty = \mathrm{div}(y)$$

$$\mathrm{div}\left(\frac{dx}{y}\right) = 0$$

Turns out, given any $\underset{\omega'}{\omega} \in \Omega_C$ $\quad \mathrm{div}(\omega) \sim \mathrm{div}(\omega')$

To this special divisor is the canonical divisor K_C

As we have seen $K_C \sim 0$ $\quad K_C = 0$ in $\mathrm{Pic}\, C$

But also, $\ell(D) - \ell(K_C - D) = \deg D - g + 1$.

If $K_C = 0$, $D = 0 \Rightarrow g = 1$ This condition completely determines elliptic curves.

Let X be an smooth projective algebraic variety of dimension n.

$$\Omega^n_X = \underbrace{\Omega_X^1 \wedge \Omega_X^1 \wedge \ldots \wedge \Omega_X^1}_{n\text{-times}}$$

$\Rightarrow$ We can also associate a canonical divisor K_X.

Suppose we have X surface. What are the surfaces of trivial canonical divisor?

WAYS OF STUDYING ALGEBRAIC VARIETIES OVER $\mathbb{C}$

From its cohomology over $\mathbb{C}$. $h^{p,q} = H^q(X, \Omega^p_X)$

Smooth curves of genus g

$$\begin{matrix} & 1 & \\ g & & g \\ & 1 & \end{matrix}$$

HODGE DIAMOND

$$\begin{matrix} & & h^{0,0} & & \\ & h^{0,1} & & h^{1,0} & \\ h^{0,2} & & h^{1,1} & & h^{2,0} \\ & & \vdots & & \end{matrix}$$

Which surfaces we have?

$$\begin{matrix} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{matrix} \qquad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

ABELIAN SURFACES $\qquad$ K3 SURFACES

$p_g - p_a$

$q(X) = 0$
$p_g(X) = 1$
$\chi(X) = 2$
$e(X) = 24$

Why is the second more natural generalisation?
Have some geometric and algebraic genus

<u>Definition</u> $\to$ A K3 surface is a smooth surface X with $K_X = 0$ and $h^1(X, \mathcal{O}_X) = 0$

<u>Examples</u>

- $\to X_4$ (smooth) in $\mathbb{P}^3$
- $\to X_{2,3}$ in $\mathbb{P}^4$
- $\to X_{2,2,2}$ in $\mathbb{P}^5$

$\to$ There are 95 families of K3 surfaces described as hypersurfaces in w.p.s $\mathbb{P}(a_1, a_2, a_3, a_4)$

e.g. $X_6 = \mathbb{P}(1,1,1,3)$ double cover of a smooth sextic.

<u>Kummer surfaces</u> → Let A be an abelian surface. Then, for every $P \in A$ we have an inverse $-P$.
There is an involution $i: P \mapsto -P$, we can consider the quotient $Kum(A) = A/\langle i \rangle$. The points in $A[2]$ are fixed points in the action ⇒ singularities

② THE ARITHMETIC OF K3 SURFACES

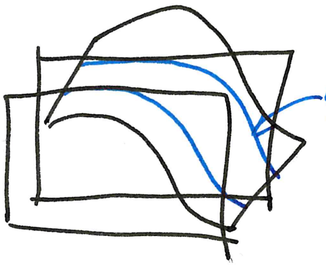
- Counting points? $x^3y + xy^2w + zyw^2 + z^3w = 0$

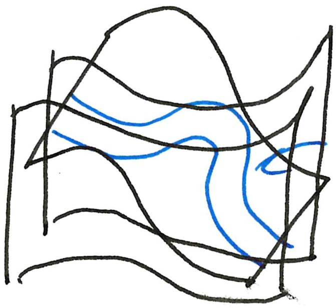
$w = 0 \quad x^3y = 0$ lines

$w \neq 0 \quad y = 0 \Rightarrow z^3 = 0$ lines

$w \neq 0 \quad y \neq 0 \Rightarrow$ smooth quartic curves ⇒ genus 3 curves

- <u>Observation</u> → Genus 0 and 1 curves provide infinite points whereas $g > 1$ don't.





For K3 surfaces we cannot find infinite genus 0 curves in them.
Sometimes we can get infinite genus 1 curves (elliptic fibration).

<u>Definition</u> → A genus 1 fibration is a surjective proper morphism to a variety Z $f: X \to Z$ s.t ~~for~~ all fibers except for finitely many are smooth curves of genus 1.
Perfect for algebraic geometers but not for us.

A section of a map as before is a morphism $\sigma : Z \to X$ s.t $f \circ \sigma : Z \to Z$ is the identity.

An elliptic surface over K is a genus 1 fibration $X \to C$ with a section over K

$$sy^2z = x(x-z)(s^7x - t^7z) \subset \mathbb{P}^2_{x,y,z} \times \mathbb{P}^1_{s,t} \xrightarrow{f} \mathbb{P}^1$$
$$([x:y:z],[s:t]) \longmapsto [s:t].$$

Section $\mathbb{P}^1 \longmapsto \mathbb{P}^2 \times \mathbb{P}^1$
$[s:t] \longmapsto ([0:1:0],[s:t])$

We can easily see by dehomogenising that this is $y^2 = x(x-1)(x-t^7)$

$$X \to C \text{ elliptic fibration over } K \longleftrightarrow E(K(C))$$

MORDELL-WEIL FOR FUNCTION FIELDS

Let $X \to C$ be an elliptic surface over K. If $X \to C$ does not split $\Rightarrow E(K(C))$ is finitely generated.

$X \to C$ splits if $X \cong E_0 \times C$ for some E_0/K
$\Leftrightarrow \Delta(E(K(C)))$ is not constant

There are singular fibers | ADE singularities in surfaces
behaviour of singular fibers

Do K3s have elliptic fibrations?

Yes & no. When they do, it is over $\mathbb{P}^1$

Elliptic fibered K3 surfaces$/K \Rightarrow$ Elliptic curves over $K(t)$

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

Not a $\Leftrightarrow$ $v_t(a_i(t)) \le 2i$ $\forall i$ but $v_t(a_i(t)) > i$ for some

Interestingly enough there is connections between the rank of $MW(E(K(t)))$ and the geometry of K3 surfaces.

In nice surfaces we can define a product of curves

We can identify curves that behave the same under this product. $C_1 \equiv C_2$ if $C_1 \cdot D = C_2 \cdot D \ \forall D \in Div(X)$.

In K3 surfaces, $Num(X) = NS(X)$ $\quad Num(X) = Pic(X)$

and also turns out $NS(X)$ is finitely generated $\cong \mathbb{Z}^r$ for some $r = \rho(X)$.

$$\begin{cases} \rho(X) \le b_2(X) = 22 \\ \text{char } 0 \\ \rho(X) \le 20 \end{cases}$$

There is a connection between elliptic fibrations and this Picard rank

$$\rho(X) = \underbrace{r}_{\text{rank}(E(K(t)))} + 2 + \sum_{\text{singular fibers}} (m_v - 1)$$

In general, in characteristic 0 $\Rightarrow$ $\text{rank}(E(K(t))) \le 18$

Previous example $\quad y^2 = X^3 + (2t^3+2t)X^2 + t^4 X$

$$E(K(t)) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

$P_1 = (\frac{1}{3}t(2t^2-1), 0)$

$P_2 = (\frac{1}{3}t(-t^2+2), 0)$

$P_3 = (\frac{1}{3}t(-t^2-1), 0)$

Interactions between $NS(X)$ and Elliptic Fibrations

Point counting
| Upper bounds on $NS(X)$
Understand lattice to find things.

Elliptic fibrations
| Lower bounds on $NS(X)$
Understand complexity of varieties

Elkies