Curves, surfaces and singularities in characteristic p

0. Motivation

Why do we care about geometry in characteristic p:

- Birational geometry

  Mori (1982) $\rightarrow$ Let X smooth projective variety s.t. $-K_X$ is ample. Then X contains a rational curve (in fact through every point $\exists D$ rational with $0 < -(D \cdot K_X) \leq \dim X + 1$).

- Computation of cohomology groups X smooth/n.f. good reduction at p

$$Z_X(T) = \exp\left(\sum \#X(\mathbb{F}_q) \frac{T^n}{n}\right) = \frac{P_1(T) \cdots P_{2n-1}(T)}{P_0(T) \cdots P_{2n}(T)}$$

with $\deg(P_i) = b_i$ (Betti numbers).

What is so special about characteristic p?

Consequence of Galois theory

$$L/K \text{ Finite + Galois} \Rightarrow \text{Intermediate subfields} \xleftrightarrow{1:1} \text{Subgroups of } \mathrm{Gal}(L/K)$$

Galois $\downarrow$ normal + $\boxed{\text{separable}}$ $\hookrightarrow$ For free in char 0, not always true in char $p>0$

In char p we have Frob: $\alpha \longmapsto \alpha^p$.

When Frob is surjective, everything is good (perfect fields) but when we are not in that situation, we face problems when dealing with inseparable field extensions:

$$X^p - \beta \quad \text{with } \beta \neq \alpha^p - \alpha \text{ for some } \alpha$$

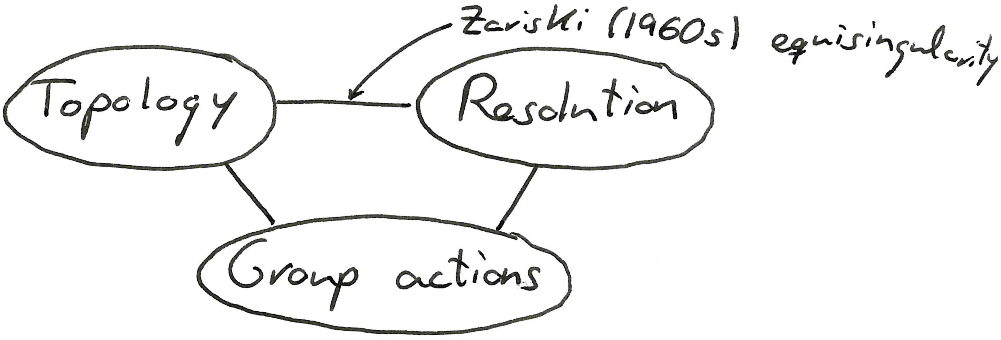
The function fields of varieties defined over positive characteristic fields will generally not be perfect.

Example of a phenomenon that only occurs in positive char.

Abhyankar-Moh (1973) → Let k be an algebraically closed field of characteristic zero. Then, all possible embeddings of the affine line $\mathbb{A}^1_k \longrightarrow \mathbb{A}^2_k$ are the composition of the injection $x \mapsto (x, 0)$ with an automorphism of $\mathbb{A}^2_k$.

Whereas, in positive characteristic, we have embeddings that are not of this sort $\quad x \longmapsto (x + x^{p^2+p}, x^{p^2})$.

Singularities



I. Curves

For simplicity, let's consider plane algebraic curves C over a field k algebraically closed.

Let ξ_0 be the germ of the curve at the origin

Germs of curves $\longleftrightarrow$ Non-zero principal ideals of $\mathcal{O}_{C,0}$ $k[[$

ξ is irreducible $\Leftrightarrow$ cannot be obtained as the sum of 2 non-empty germs.

$\mathcal{O}_{C,0}$ is a local ring, $\mathfrak{m}_0$ is its maximal ideal

$e(\xi) = \{n : f \in \mathfrak{m}_0^n \setminus \mathfrak{m}_0^{n+1}\}$
multiplicity

We will say that the origin is a singular point if $e(\xi) > 1$.

How do we study germs?

Complex case $\Rightarrow$ Choose locall coordinates x, y, then we can assume $\mathcal{O}_{S,0} = \mathbb{C}\{x, y\}$ (ring of convergent power series

In general, we can presume that $\mathcal{O}_{S,0} = K[[x, y]]$ ring of power series. An element $f \in K[[x, y]]$ is what is known as an algebroid curve.

We can study germs from each parametrisations, a.k.a, trying to find $x(t), y(t) \in K[[t]]$ such that $f(x(t), y(t)) = 0$ or, alternatively, $(f) = \mathrm{Ker}(K[[x, y]] \to K[[t]])$.

Assume $\mathrm{char}\,K = 0$. Then, we have the following tool to study singularities:

PUISEUX SERIES

A Puiseux series is a parametrisation of an irreducible algebroid curve of the form

$$\begin{cases} x(t) = t^{m} \rightarrow \text{multiplicity} \\ y(t) = \sum_{n=k}^{\infty} a_n t^n \end{cases}$$

with $a_n \in K$, or, alternative

$$\boxed{y(x) = \sum_{n=k}^{\infty} a_n t^{n/m}}$$

Few facts:

- They are defined over algebraic closures (not necessarily on field
- Can be computed by studying the Newton polygon associat to f.
- If f is reducible over $K[[x, y]]$, we can study f by

$$f = f_1 \cdots f_n$$

<u>Example</u> $\to$ $y^2 = x^3 + x^4 = \left(x^{3/2} \cdot (1+x)^{1/2}\right)^2$

$\Rightarrow$ $y(x) = x^{3/2}$. Taylor expansion of $(1+x)^{1/2}$ around 0

$$= \sum_{n=0}^{\infty} x^{3/2+n} \binom{1/2}{n}$$

Turns out Puiseux expansions determine both the topology and the resolution of a singularity.

From the Puiseux expansions of an irreducible series we can compute their Puiseux pairs (characteristic exponents)

Example above $(3,2)$ or $\{3/2\}$

If we have a reducible germ, we also have to take into account the pairwise intersection multiplicities of the branches.

These things all together determine the equisingularity type

$\Rightarrow$ Tells us how the resolution by blow-ups will be (e.g. $y^2 = x^3$ is $(3,2)$ so, same resolution).

$\Rightarrow$ Has a topological interpretation:

<u>Complex case</u> $\to$ f and g equisingular $\Leftrightarrow$ topologically equivalent. $\exists$ homeomorphism of triples

$$(B_\varepsilon, B_\varepsilon \cap V(f), 0) \cong (B_\varepsilon \cap V(g), 0)$$

with $B_\varepsilon \in \mathbb{C}^2$ small ball around the origin.

<u>Problem</u> $\to$ Puiseux expansions do not behave well when the $p \mid$ multiplicity.

Example $\to$ $y^3 + y^2 = x^4$ in char 2 $\Rightarrow$ easy to see multiplicity is 2 but parametrisation $\begin{cases} x(t) = t^2 \\ y(t) = t^4 + \dots \end{cases}$ does not work.

We can easily consider the opposite parametrisation

Take $y^6 + y = x^4$ 2-problems $= \{x(t) =$

<u>SOLUTION</u> $\to$ Hamburger - Noether expansions (Campillo (1980))

$$\begin{cases} y = a_{01}x + \dots + a_{0h}x^h + x^h z_1 \\ x = a_{12} z_1^2 + \dots + a_{1h_1} z_1^{h_1} + z_1^{h_1} z_2 \\ z_1 = a_{22} z_2^2 + \dots + a_{2h_2} z_2^{h_2} + z_2^{h_2} z_3 \\ \vdots \\ z_{r-1} = a_{r2} z_r^2 + \dots \in k[[z_r]] \end{cases}$$

<u>The solution</u> $\to$ They have characteristic exponents that match with Puiseux

- Also, they are defined over the optimal field extensions.

  Previous example, defined over $\mathbb{F}_2$ and char exponent $(7,2)$

$$y = x z_1$$
$$x = z_1^{m} + . z_1^2 + \dots$$

II. Surfaces

Consider $f \in k[[x_1, \dots, x_n]]$

We then have 2 important notions

- Right equivalence
- Contact equivalence
- Right-simple, contact simple.

Over the complex, Arnold proved that these are all equivalent to be equivalent to an ADE singularity.

<u>Normal forms in char 0</u>	
A_n	$x^{n+1} + y^2 + z^2$
D_n	$x^{n-1} + xy^2 + z^2$
E_6	$x^4 + y^3 + z^2$
E_7	$x^3y + y^3 + z^2$
E_8	$x^5 + y^3 + z^2$

<u>Char 2</u>	
E_6^0	$x^2z + y^3 + z^2$
E_6^1	$x^2z + xyz + y^3 + z^2$
D_{2m}^r	$x^2y + xy^m + xy^{m-r}z + z^2$
$\vdots$	and many more