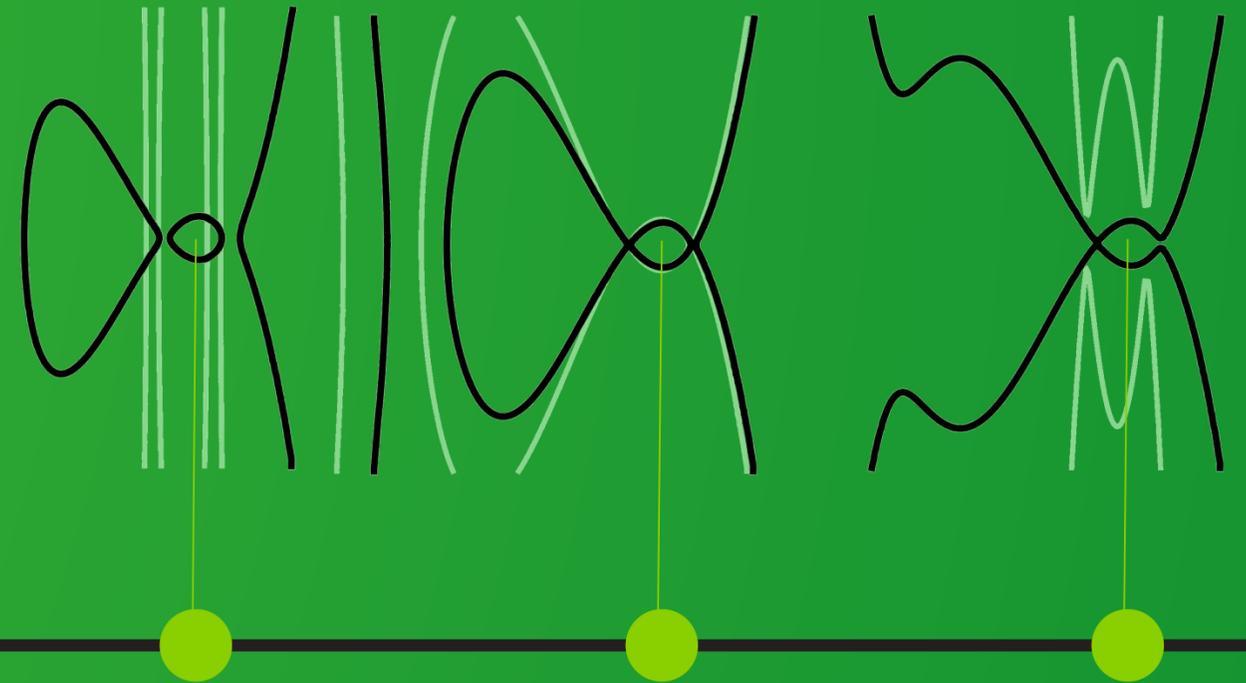


ALVARO GONZALEZ HERNANDEZ

University of Warwick



Intersections of the automorphism and p -rank strata inside M_2

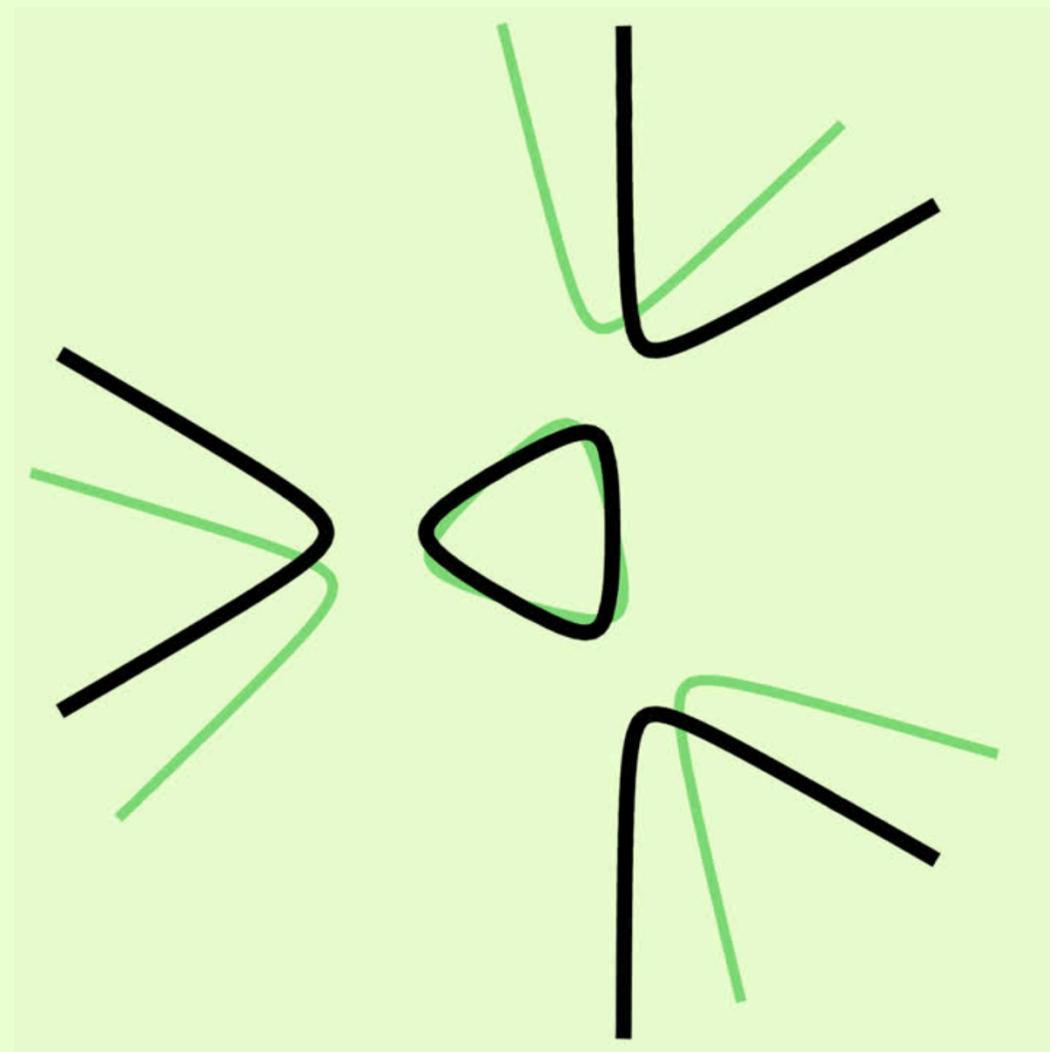
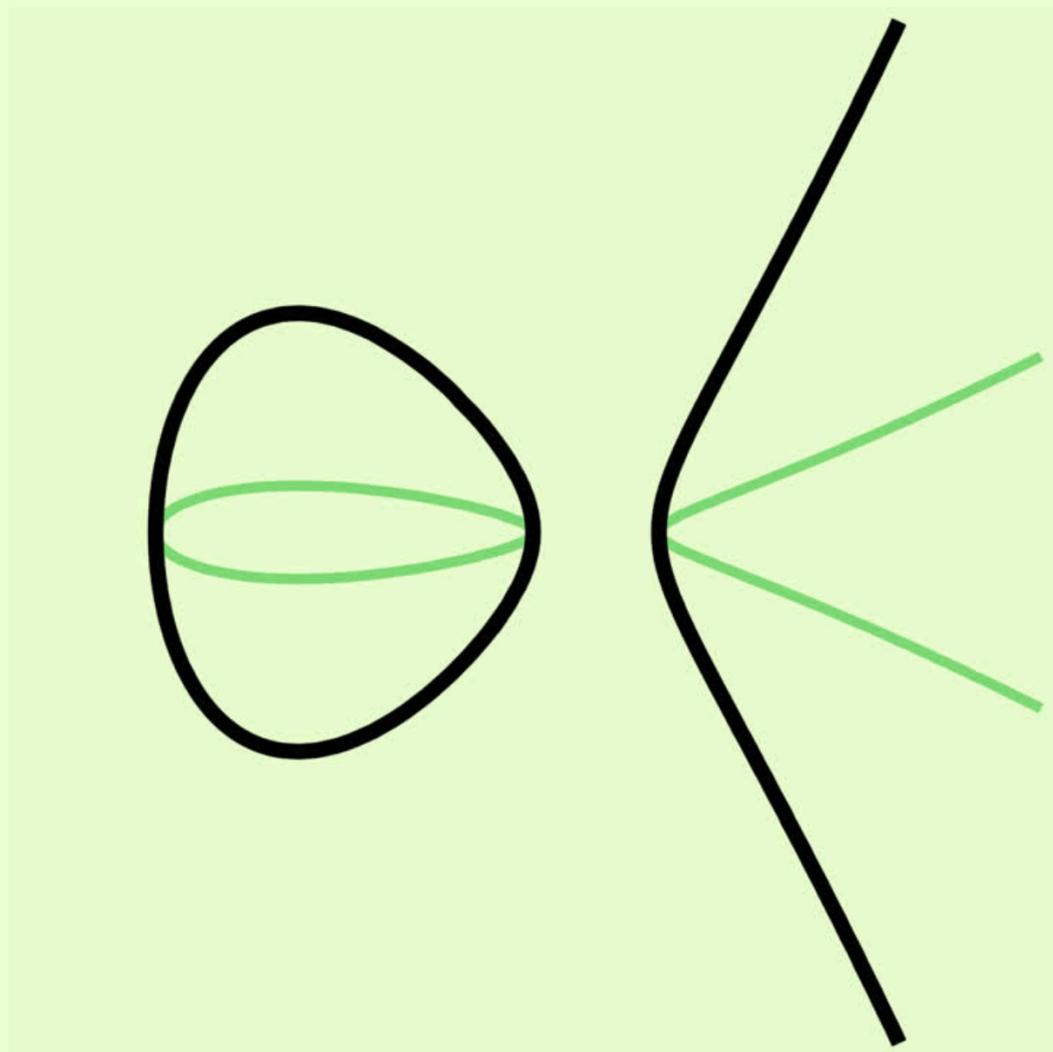
The goal:

**Automorphism
group of an abelian
variety over a field
of characteristic p**

**p -torsion of that
abelian variety**

***in this talk, whenever I say abelian variety I mean principally polarised abelian variety, and the automorphisms that I am considering are the ones preserving the polarisation**

Let's have a look at the case of elliptic curves



The possibilities for the automorphism groups of an elliptic curve are:

$\text{Aut}(E)$	characteristic
-----------------	----------------

C_2	all
-------	-----

C_4	$p \neq 3$
-------	------------

D_6	$p = 3$
-------	---------

C_6	$p \neq 2$
-------	------------

S_4	$p = 2$
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C_4 $p \neq 3$

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C_6 $p \neq 2$

S_4 $p = 2$

$$\left. \begin{array}{l} C_4 \\ D_6 \end{array} \right\} \longrightarrow y^2 = x^3 - x$$

$$\left. \begin{array}{l} C_6 \\ S_4 \end{array} \right\} \longrightarrow y^2 + y = x^3$$

$$E : y^2 = x^3 - x$$

$$j(E) = 1728$$

p	3	5	7	11	13	17	19	23	29	31	37	41	43	47
$\#E(\mathbb{F}_p)$	4	8	8	12	8	16	20	24	40	32	40	32	44	48

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if $p \equiv 3 \pmod{4}$, **then** $\#E(\mathbb{F}_p) = p + 1$

$$E : y^2 + y = x^3$$

$$j(E) = 0$$

p	2	5	7	11	13	17	19	23	29	31	37	41	43	47
$\#E(\mathbb{F}_p)$	3	6	9	12	9	18	27	24	30	36	27	42	36	48
$p + 1$	3	6	8	12	14	18	20	24	30	32	38	42	44	48

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if $p \equiv 2 \pmod{3}$, **then** $\#E(\mathbb{F}_p) = p + 1$

Supersingular elliptic curves

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An elliptic curve E defined over a field of characteristic $p > 3$ is said to be **supersingular** if and only if $\#E(\mathbb{F}_p) = p + 1$.

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p -rank

Let A be an abelian variety of dimension $g \leq 2$ defined over a field k of characteristic p . The p -**rank** f is the integer $0 \leq f \leq g$ such that

$$A[p](\bar{k}) = (\mathbb{Z}/p\mathbb{Z})^f.$$

A is said to be **ordinary** if $f = g$ and **supersingular** if $f = 0$.

$$M_1 \cong \mathbb{A}^1$$

$$p = 29$$

Automorphism group:

C_4
●
-12

C_6
●
0

j-invariant

***p*-rank:**

● ● ●
0 2 6

j-invariant

Moving on to higher dimensions:

We can construct a coarse moduli space A_g parametrisng abelian varieties of dimension g , which will have

$$\dim(A_g) = \frac{1}{2}g(g + 1)$$

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Abelian varieties with a prescribed automorphism group

Abelian varieties with a prescribed p -rank



→ **Strata:**

Unions of finitely many locally closed subsets

Problem:

It is difficult to describe A_g in an explicit way.

Still, we can learn valuable information by focusing our efforts on studying those abelian varieties that are Jacobians of curves.

We can look at the coarse moduli space M_g parametrising curves of genus g , which will have

$$\dim(M_g) = 3g - 3$$

When $g = 2$

$$\dim(M_2) = \dim(A_2) = 3$$

The Torelli map connects both moduli spaces. Pulling back the strata from A_g , we get strata of M_g

The advantage of working with the moduli space of curves is that it has an easier description.

Describing \overline{M}_2

If we assume we are working over an algebraically closed field of characteristic not two, a genus two curve can be always be expressed as:

$$y^2 = \prod_{i=1}^6 (x - \alpha_i)$$

A sextic binary form is determined by a set of 5 invariants $\{J_2, J_4, J_6, J_8, J_{10}\}$ known as the **Igusa invariants, e.g.**

$$J_2 = \frac{1}{8} \sum_{\sigma \in S_6} (\alpha_{\sigma(1)} - \alpha_{\sigma(2)})^2 (\alpha_{\sigma(3)} - \alpha_{\sigma(4)})^2 (\alpha_{\sigma(5)} - \alpha_{\sigma(6)})^2$$

Describing \overline{M}_2

The Igusa invariants fully describe genus 2 curves up to isomorphism. Consider the map:

$$M_2 \longrightarrow \mathbb{P}(1, 2, 3, 4, 5)$$

$$\mathcal{C} \longmapsto [J_2(\mathcal{C}) : J_4(\mathcal{C}) : J_6(\mathcal{C}) : J_8(\mathcal{C}) : J_{10}(\mathcal{C})]$$

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Then, we always have that

$$J_4(\mathcal{C})^2 - J_2(\mathcal{C})J_6(\mathcal{C}) + 4J_8(\mathcal{C}) = 0$$

and \overline{M}_2 is birationally equivalent to the variety

$$\mathbb{V}(J_4^2 - J_2J_6 + 4J_8) \subset \mathbb{P}(1, 2, 3, 4, 5)$$

Strata of automorphism groups and p -ranks of Jacobians of genus 2 curves

$\text{Aut}(\mathcal{C})$	order	characteristic	dim	p -rank	dim
C_2	2	all	3	2	3
$C_2 \times C_2$	4	all	2	1	2
D_4	8	all	1	0	1
C_{10}	10	$p \neq 2, 5$	0		
D_6	12	all	1		
C_2^4	16	$p = 2$	1		
$C_3 \rtimes D_4$	24	$p \neq 2, 3, 5$	0		
$\text{GL}_2(\mathbb{F}_3)$	48	$p \neq 2, 5$	0		
$\text{SL}_2(\mathbb{F}_5)$	120	$p = 5$	0		
$C_2 \wr C_5$	160	$p = 2$	0		

How do they intersect?

My contribution to this topic?

These tables...

$$p \geq 7$$

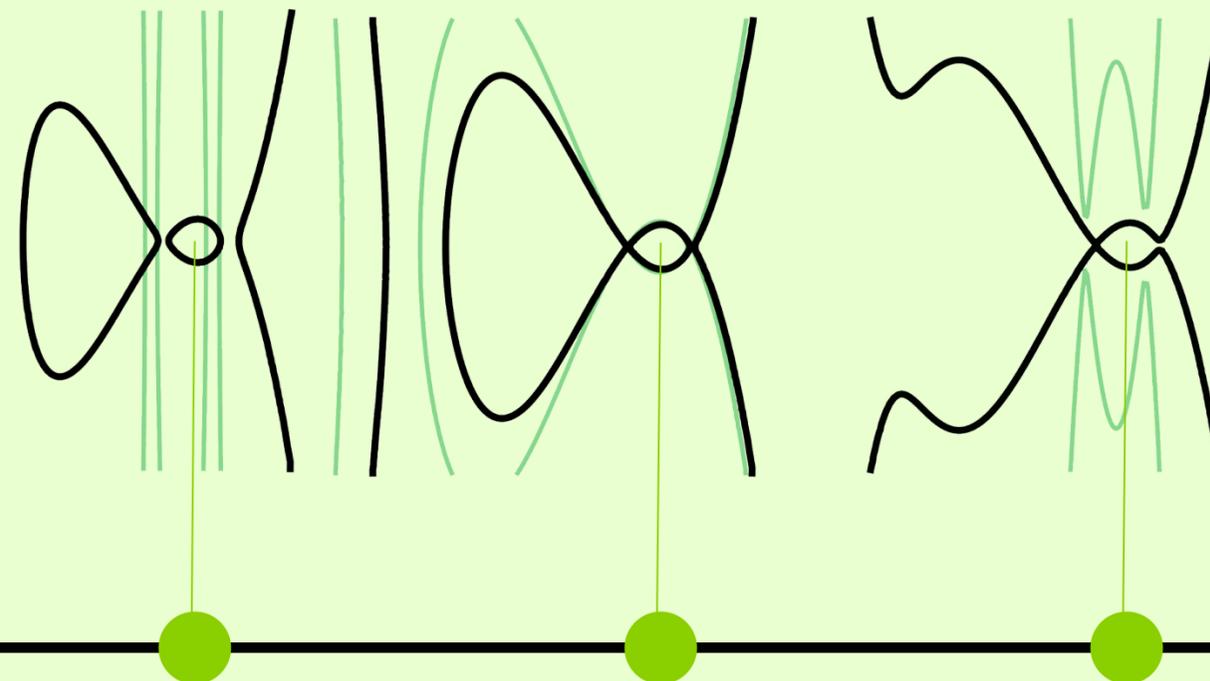
f	a	C_2	$C_2 \times C_2$	D_4	D_6
2	0	3	2	1	1
1	1	2	1	-1	-1
0	1	1	-1	-1	-1
0	2	0 if $p \geq 29$	0 if $p \geq 17$	0 if $p \geq 13$	0 if $p \geq 11$

f	a	$C_3 \rtimes D_4$	$GL_2(\mathbb{F}_3)$	C_{10}
2	0	0 if $p \equiv 1 \pmod{6}$	0 if $p \equiv 1, 3 \pmod{8}$	0 if $p \equiv 1 \pmod{5}$
1	1	-1	-1	-1
0	1	-1	-1	0 if $p \equiv 2, 3 \pmod{5}$
0	2	0 if $p \equiv 5 \pmod{6}$	0 if $p \equiv 5, 7 \pmod{8}$	0 if $p \equiv 4 \pmod{5}$

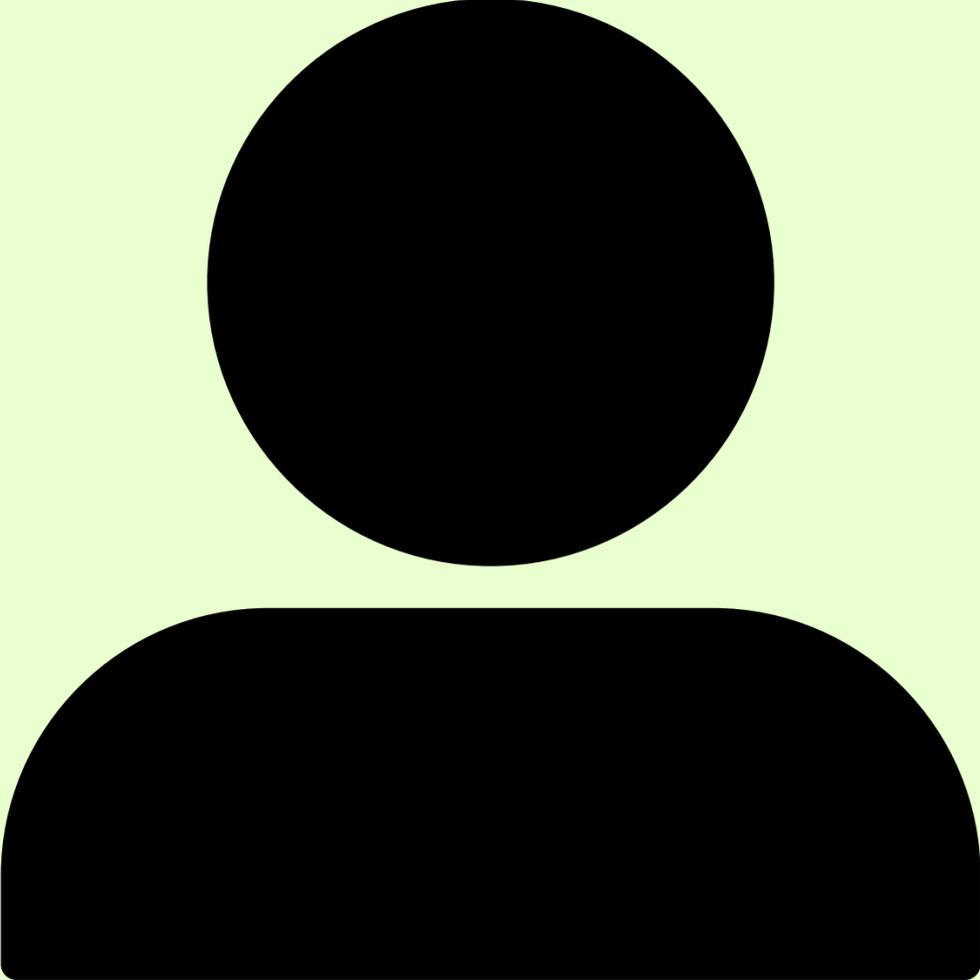
...and a bit more

f	a	C_2	$C_2 \times C_2$	D_4	D_6
2	0	3	2	1	1
1	1	2	1	-1	-1
0	1	1	-1	-1	-1
0	2	0 if $p \geq 29$	0 if $p \geq 17$	0 if $p \geq 13$	0 if $p \geq 11$

$$X_{D_6,2} \subset \mathbb{V}(J_4^2 - J_2 J_6 + 4J_8)$$

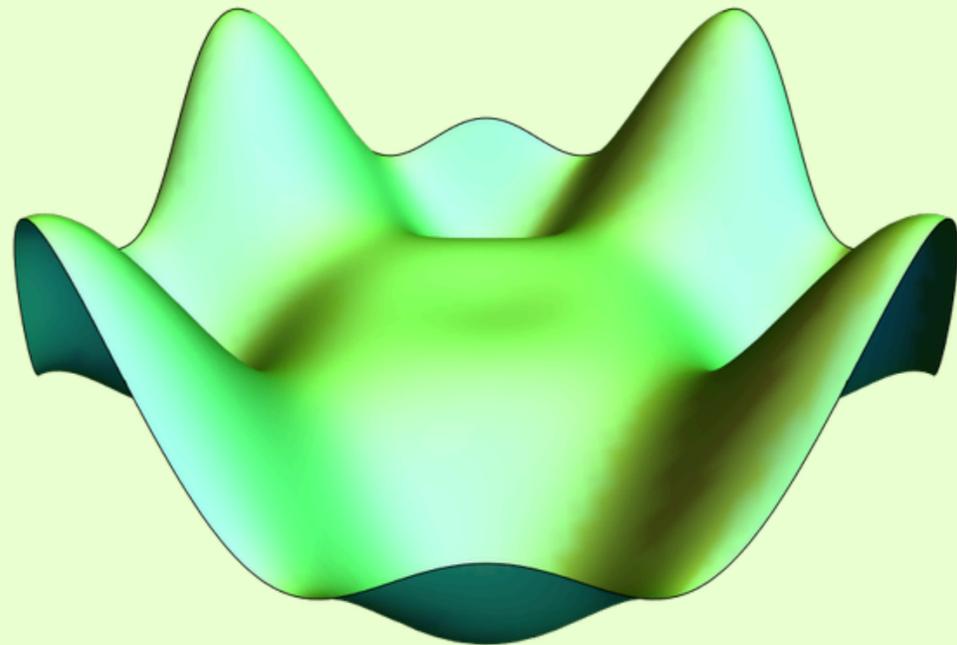


$X_{D_6,2}$



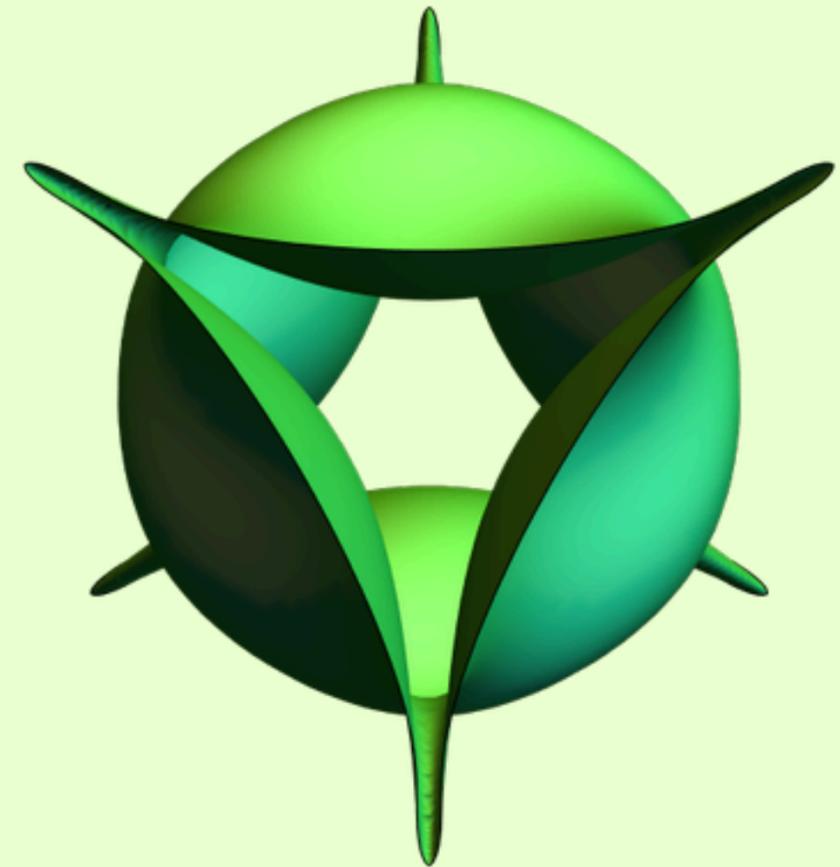
**But Alvaro, last
time we talked, you
only cared about K3
surfaces!**

Abelian surface



→
quotient by a
(symplectic)
automorphism

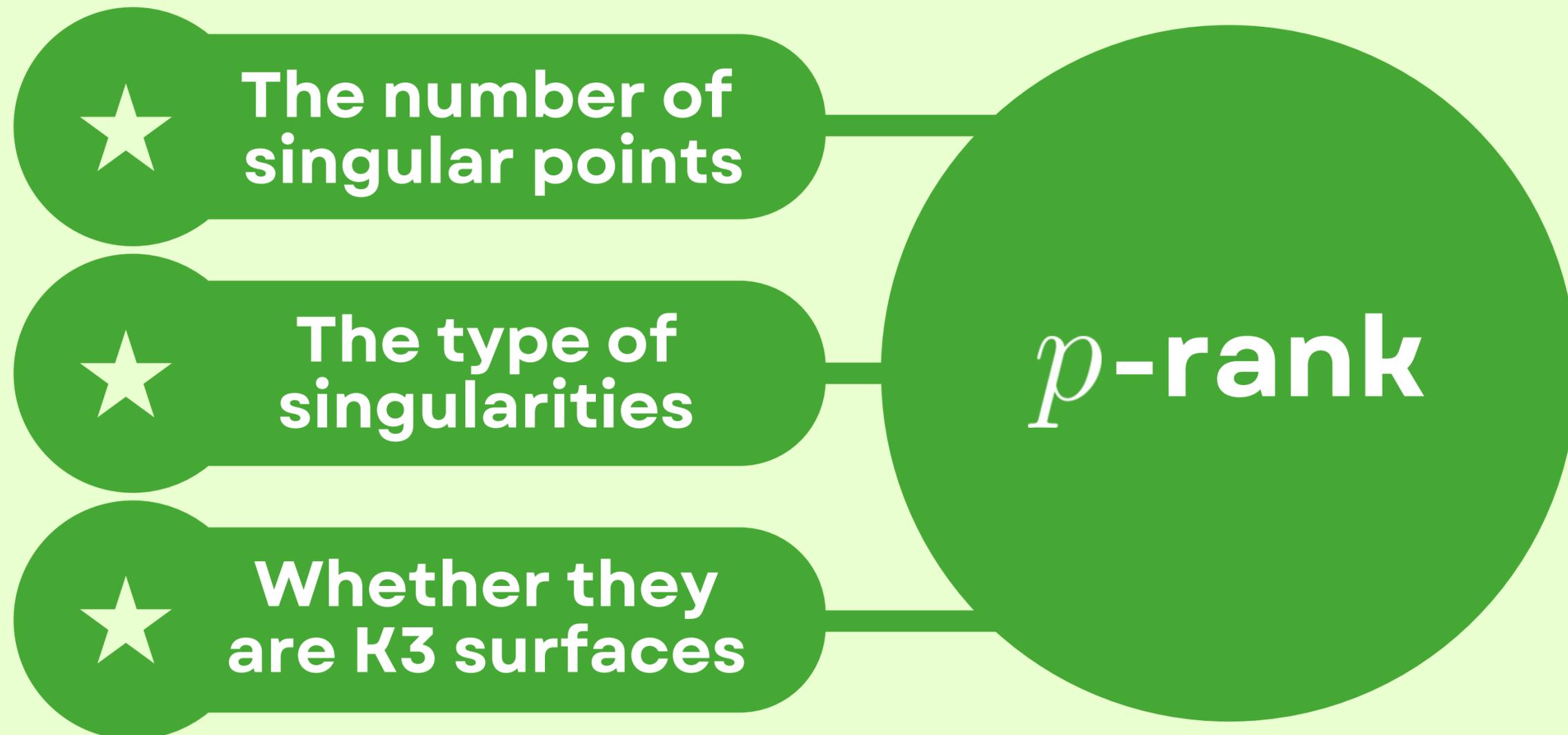
K3 surface



(Generalised
Kummer surface)

Generalised Kummer surfaces

If they are the quotient of an abelian surface in characteristic p by an automorphism of order n



“There are no abelian surfaces with p -rank 1 and an automorphism of order 3”

Great! That means that the K3 quotient of the Jacobian of a genus 2 curve by an automorphism of order 6 can only have 6, 7 or 8 singular points!

***depending on whether the characteristic is 2, 3 or neither**

thank you