Masses of central leaves and automorphism groups of abelian varieties

Participants

- Eda Kırımlı
- Enric Florit
- Alvaro Gonzalez Hernandez

Project Leaders

- Valentijn Karemaker
- Soumya Sankar

Motivation

- $K = \overline{\mathbb{F}}_p$
- $(X, \lambda) \in \mathcal{A}_g$ ppav of dimension g
- $\mathcal{S}_g = \{(X, \lambda) \in \mathcal{A}_g :$ X is supersingular $\}$

Fix E_0/K ssEC
and let $X = E_0^g$

* $\mathrm{Aut}(E_0^g) \cong GL_g(\mathrm{End}(E_0))$

but...

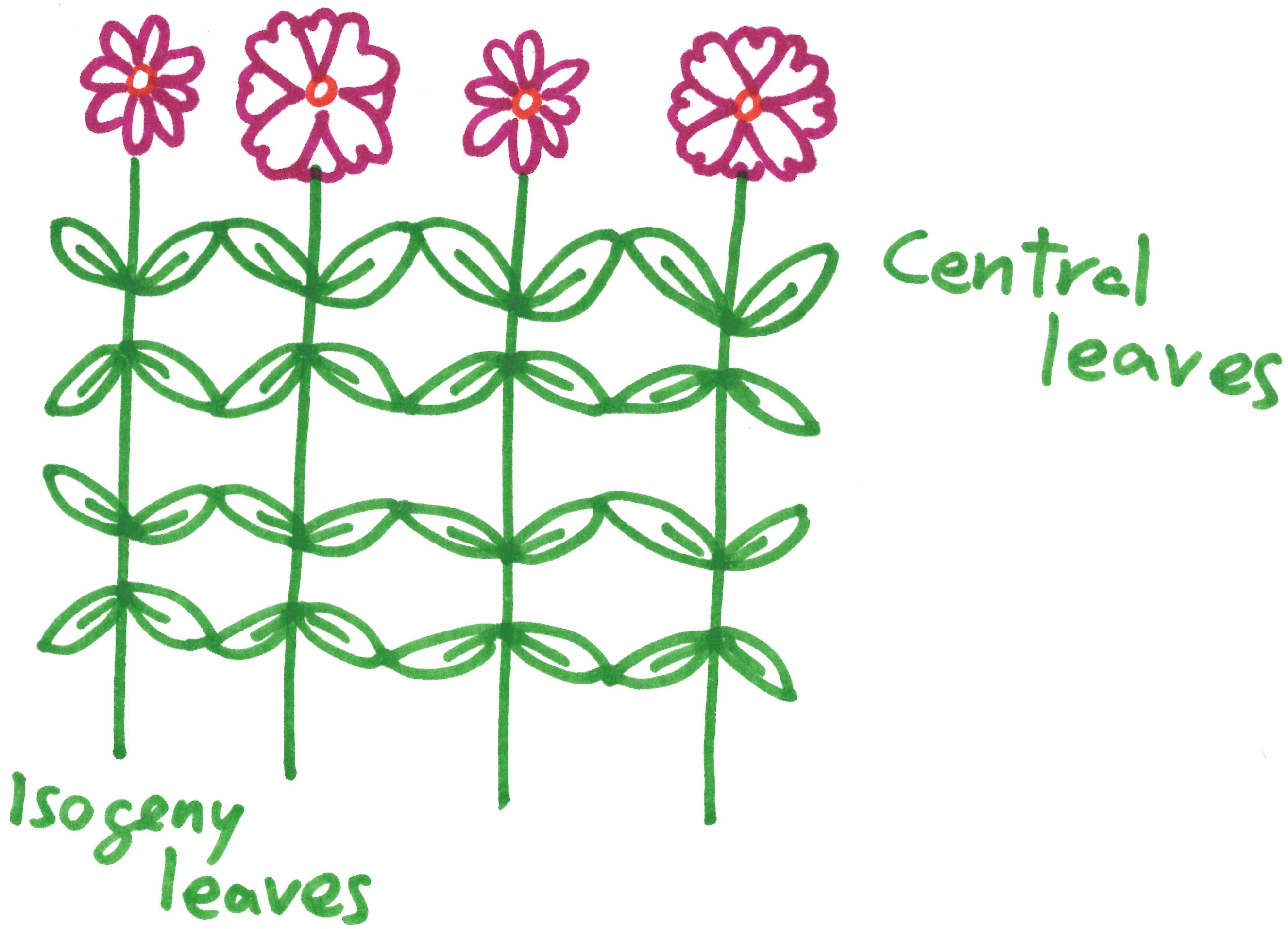
$$\boxed{|\mathrm{Aut}(E_0^g, \mu)| < \infty \; !}$$

polarisation

Let $x = (X, \lambda) \in \mathcal{A}_g$

Central leaf

$$\mathcal{C}(x) := \{ (X', \lambda') \in \mathcal{A}_g : (X', \lambda')[p^\infty] \cong (X, \lambda)[p^\infty] \}$$



$$X \in S_g \Rightarrow \mathcal{C}(X) \text{ is a finite set}$$

$$\mathrm{Mass}(\mathcal{C}(X)) = \sum_{X' \in \mathcal{C}(X)} \frac{1}{|\mathrm{Aut}(X')|}$$

Problem of the week

Fix $g \geq 1$ and $\mathcal{C}(X)$

Given two of the following:

1- $|\mathcal{C}(X)|$

2- $\mathrm{Mass}(\mathcal{C}(X))$

3- List of possible automorphism groups

→ Compute the third

Examples

$\boxed{g=1}$ We know everything!

1- $|S_1| = \left\lfloor \frac{p-1}{2} \right\rfloor = \begin{cases} 0 \\ 1 \\ 2 \end{cases}$ depending on $p \pmod{12}$

$\quad = e(E_0)$

2- $\mathrm{Mass}(S_1) = \frac{p-1}{24}$

3- Possible $\mathrm{Aut}(E_0)$:

$\underline{p=2}$: $\quad SL_2(\mathbb{F}_3)$

$\underline{p=3}$: $\quad C_3 \rtimes C_4$

$\underline{p\geq 5}$: $\quad C_2, C_4, C_6$

$\boxed{g=2}$ Superspecial case studied by Ibukiyama – Katsura – Oort

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$\boxed{g=3}$ Fix a principal polarisation μ on E_0^3

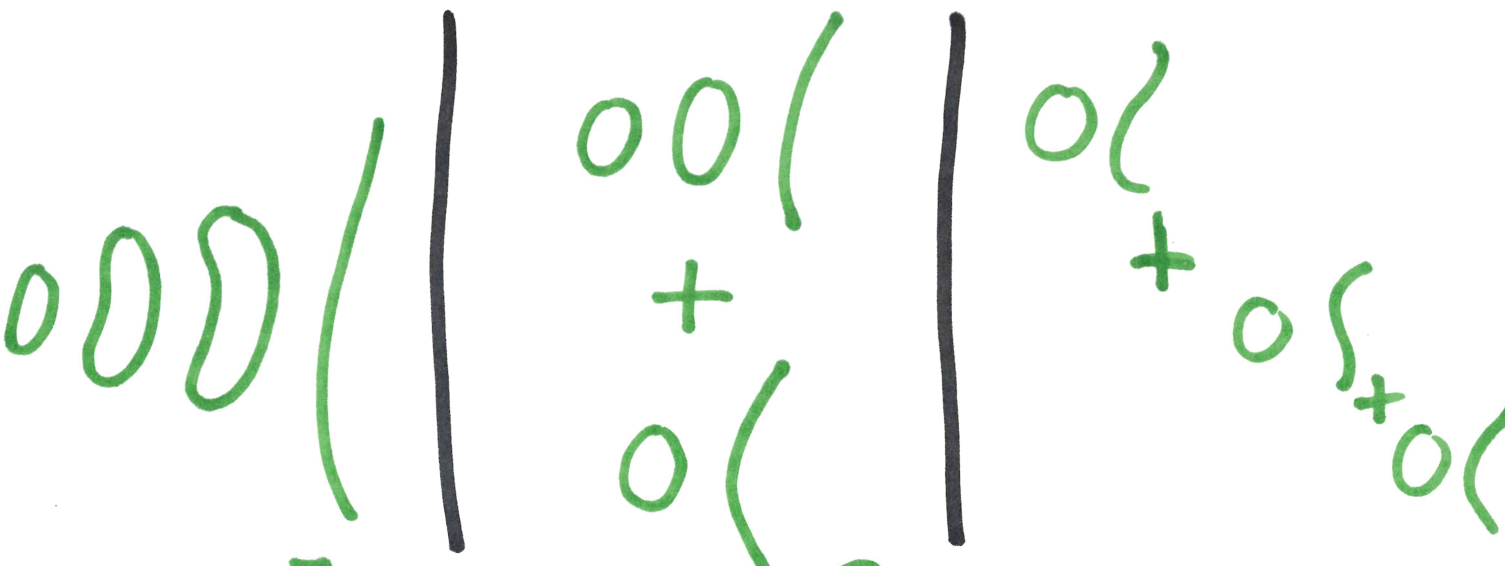
Let $X = (E_0^3, \mu)$

What is $|\mathcal{C}(X)|$?

We know $\mathrm{Mass}(\mathcal{E}(X))$

And the automorphisms?

$$X = \begin{cases} \mathrm{Jac}\left(\begin{smallmatrix}\text{curve of}\\ \text{genus } 3\end{smallmatrix}\right) \\ \mathrm{Jac}\left(\begin{smallmatrix}\text{curve of}\\ \text{genus } 2\end{smallmatrix}\right) \times E \\ E_1 \times E_2 \times E_3 \end{cases}$$



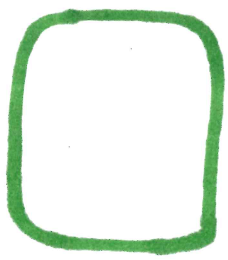
genus 3 | 1x genus 2, 1x genus 1 | 3 x genus 1

Theorem (Torelli, Weil)

Let $G = \mathrm{Aut}(C)$

- C hyperelliptic
  ↳ $\mathrm{Aut}(\mathrm{Jac}(C)) \cong G$
- C non-hyperelliptic
  ↳ $\mathrm{Aut}(\mathrm{Jac}(C)) \cong G \times \{\pm 1\}$

C hyperelliptic



C not hyperelliptic

$$\boxed{p=2}$$

$$\mathrm{Mass}(\mathcal{C}(X)) = \frac{1}{82944}$$

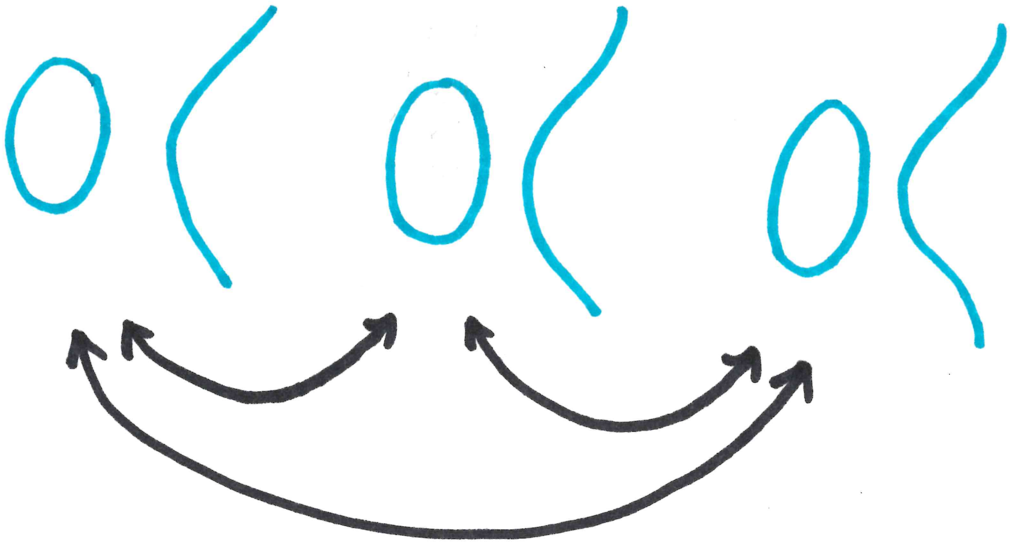
What elements do we know in $\mathcal{C}(X)$?

$$X' = (E_0 \times E_0 \times E_0, \mu_{can})$$

$$E_0 : y^2 + y = x^3 + x$$

$$\mathrm{Aut}(X', \mu_{can}) = SL_2(\mathbb{F}_3)^3 \rtimes S_3$$

$S_3 \circlearrowright$



$$|\mathrm{Aut}(X', \mu_{can})| = 82\,944$$

$$\mathrm{Mass}(\mathcal{C}(X)) = \frac{1}{82\,944}$$

$$\Rightarrow |\mathcal{C}(X)| = 1$$

$\cancel{X}$ $\frac{1}{\text{integer}}$

$$\boxed{p=3}$$

$$\mathrm{Mass}(\mathcal{C}(X)) = \frac{13}{72\,576}$$

$$\Rightarrow |\mathcal{C}(X)| > 1$$

We know:

$$X' = (E_0 \times E_0 \times E_0, \mu_{can})$$

$$E_0: \; y^2 = x^3 - x$$

$$|\mathrm{Aut}(X', \mu_{can})| = 12^3 \times 6$$
$$= 10\,368$$

$$\mathrm{Mass}(\mathcal{C}(X)) - \frac{1}{10\,368} = \frac{1}{12\,096}$$

Can we find something else?

$$F_4 : x_1^4 + x_2^4 + x_3^4 = 0$$

Exercise for the listener

1- Prove $\mathrm{Jac}(F_4)$ is superspecial

2- Show $\mathrm{Aut}(F_4) = PU(3,9)$

$$\Rightarrow \mathrm{Aut}(\mathrm{Jac}(F_4)) = PU(3,9) \times \{\pm 1\}$$

$$|PU(3,9) \times \{\pm 1\}| = 12\,096$$

$$\mathrm{Mass}\left(\mathcal{C}(x)\right) = \frac{1}{12\,096} + \frac{1}{10\,368}$$

$$\Rightarrow \quad |\mathcal{C}(x)| = 2$$

We are done!

What about the rest of the primes?

Small primes:

$\exists$ special curves with many automorphisms

$p \geq 7$

They roughly have the same automorphisms

We can use representation theory to obtain how many elements in $\mathcal{C}(X)$ have each automorphism group...

but it is not easy!

☹

... not even in the superspecial case...

So...

imagine the possibilities!

Further work

- $g=3$ supersingular with $a(X)=2$
- $g=4$ superspecial
- thank the audience ♡ and ask for questions

Superspecial Abelian 3-folds

Prime	Mass	$\mathcal{C}(X)$	$\mathrm{Aut}(X)$
2	$\frac{1}{82944}$	1	$1\times(SL_2(\mathbb{F}_3)^3\rtimes S_3$
3	$\frac{13}{72576}$	2	$1\times((C_4\rtimes C_3)^3\rtimes S_3$ $1\times(PU(3,9)\times\{\pm 1$
5	$\frac{403}{90720}$	3	$1\times(C_6^3\rtimes S_3)$ $1\times(PGL_2(5)\times C_6)$ $1\times(PSL_2(7)\times C_2)$
7	$\frac{95}{2688}$	5	$1\times(C_4^3\rtimes S_3)$ $1\times(S_4\times C_4\times\{\pm 1\}$ $1\times(PGL_2(7)\times C_2)$ $1\times(S_4\times C_2)$ $1\times(\mathrm{Aut}(F_4)\times C_2)$