Masses of central leaves and automorphism groups of abelian varieties

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Motivation

- $K = \overline{\mathbb{F}_p}$
- $(X, \lambda) \in A_g$ ppar of dimension $g$
- $S_g = \{(X, \lambda) \in A_g : X \text{ is supersingular}\}$
Fix $E_0/K$ as $EC$ and let $X = E_0^g$

\[ \text{but...} \]

\[ |\text{Aut}(E_0^g, \mu)| < \infty \]

\[ \uparrow \text{polarisation} \]
Let $x = (x, x) \in A_g$.

Central leaf $A_g$.

$\exists \delta \exists \gamma (x, x) = [\delta, \gamma]$. Central leaves.
$X \in S_g \implies \mathcal{C}(X)$ is a finite set

\[
\text{Mass}(\mathcal{C}(X)) = \sum_{x' \in \mathcal{C}(X)} \frac{1}{|\text{Aut}(X')|}
\]
Problem of the week

Fix $g \geq 1$ and $C(X)$

Given two of the following:

1. $|C(X)|$
2. Mass $(C(X))$
3. List of possible automorphism groups

→ Compute the third
Examples

\[ g = 1 \] We Know everything!

1. \[ |S_1| = \left| \frac{p-1}{2} \right| = \begin{cases} 0 & \text{depending on } p \ (\text{mod} \ 12) \\ 1 & \frac{p}{2} \end{cases} \]

\[ \mathcal{C}(E_0) \]

2. \[ \text{Mass}(S_1) = \frac{p-1}{24} \]

3. Possible \( \text{Aut}(E_0) \):
   - \( p = 2 \): \( SL_2([\mathbb{F}_3]) \)
   - \( p = 3 \): \( C_3 \times C_4 \)
   - \( p \geq 5 \): \( C_2, C_4, C_6 \)
$g=2$ Superspecial case studied by Ibukiyama-Katsura-Oort

$g=3$ Fix a principal polarisation $\mu$ on $E_0^3$

Let $X=(E_0^3, \mu)$

What is $|\mathcal{O}(X)|$?
We know \( \text{Mass}(\mathbb{C}(X)) \).

And the automorphisms?

\[
X = \begin{cases} 
\text{Jac } (\text{curve of genus 3}) \\
\text{Jac } (\text{curve of genus 2}) \times E \\
E_1 \times E_2 \times E_3 
\end{cases}
\]

\[
\begin{aligned}
001 & \quad 01 \quad 01 \\
0 & \quad 0 & \quad 0 \\
001 & \quad 00 & \quad 00
\end{aligned}
\]

- genus 3
- 1x genus 2
- 2x genus 1
- 3x genus 1
- 05 + 0
Theorem (Torelli, Weil)

Let $G = \text{Aut}(C)$

- $C$ hyperelliptic
  $\Rightarrow \text{Aut}(\text{Jac}(C)) \cong G$

- $C$ non-hyperelliptic
  $\Rightarrow \text{Aut}(\text{Jac}(C)) \cong G \times \{\pm 1\}$

Hyperelliptic

Not hyperelliptic
\[ p = 2 \]

\[
\text{Mass}(C(X)) = \frac{1}{82944}
\]

What elements do we know in \( C(X) \)?

\[ X' = (E_0 \times E_0 \times E_0, \text{mean}) \]

\[ E_0: y^2 + y = x^3 + x \]
\[ \text{Aut}(X', \mu_{\text{can}}) = \text{SL}_2(\mathbb{F}_3)^3 \rtimes S_3 \]

\[ S_3 \
\]

\[ \Rightarrow |C(x)| = 1 \]

\[ |\text{Aut}(X', \mu_{\text{can}})| = 82,944 \]

\[ \text{Mass}(C(X)) = \frac{1}{82,944} \]
Mass \( C(X) \) = \frac{13}{72576} \\
\Rightarrow |C(X)| > 1

We Know:

\[ X' = (E_0 \times E_0 \times E_0, \mu_{can}) \]

\[ E_0: \ y^2 = x^3 - x \]
$|\text{Aut}(X', \mu_{\text{can}})| = 12^3 \times 6$

$= 10368$

$\text{Mass}(C(x)) - \frac{1}{10368} = \frac{1}{12096}$

Can we find something else?

$F_4 : x_1^4 + x_2^4 + x_3^4 = 0$
Exercise for the listener

1. Prove $\text{Jac}(F_4)$ is superspecial

2. Show $\text{Aut}(F_4) = \text{PU}(3,9)$

$$\Rightarrow \text{Aut}(\text{Jac}(F_4))$$

$$\text{PU}(3,9) \times \{\pm 1\}$$

$$|\text{PU}(3,9) \times \{\pm 1\}| = 12096$$
\[ \text{Mass } |C(x)| = \frac{1}{12096} + \frac{1}{10368} \]

\[ \Rightarrow |C(x)| = 2 \]

We are done!

What about the rest of the primes?

Small primes:

3 special curves with many automorphisms

\( p \geq 7 \)

They roughly have the same automorphisms
We can use representation theory to obtain how many elements in $C(X)$ have each automorphism group... but it is not easy!

... not even in the superspecial case...
So...

imagine the possibilities!

Further work

- $g=3$ supersingular with $a(X)=2$
- $g=4$ superspecial
- thank the audience and ask for questions
<table>
<thead>
<tr>
<th>Prime</th>
<th>Mass</th>
<th>$C(X)$</th>
<th>$\text{Aut}(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{1}{82944}$</td>
<td>1</td>
<td>$1 \times (\text{SL}_2(\mathbb{F}_3)^3 \times S_3)$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{13}{72576}$</td>
<td>2</td>
<td>$1 \times ((C_4 \times C_3)^3 \times S_3)$, $1 \times (\text{PU}(3,9) \times 14)$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{403}{90720}$</td>
<td>3</td>
<td>$1 \times (C_6^3 \times S_3)$, $1 \times (\text{PGL}_2(5) \times C_6)$, $1 \times (\text{PSL}_2(7) \times C_2)$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{95}{2688}$</td>
<td>5</td>
<td>$1 \times (C_4^3 \times S_3)$, $1 \times (S_4 \times C_4 \times { \pm 1 })$, $1 \times (\text{PGL}_2(7) \times C_2)$, $1 \times (S_4 \times C_2)$, $1 \times (\text{Aut}(F_4) \times C_2)$</td>
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