Singularities of surfaces in characteristic $p \geq 0$

0. Recap

Last week I gave an introduction to geometry in positive characteristic and the difficulties it entails.

Recall → Inseparable field extensions. K field of characteristic p and not perfect, we have somethimes issues when we consider $X^p - y = 0$ and its splitting field (for y ~~\in~~ Frob$_p$).

I briefly introduced the concept of equisingularity, as a connecti between the topology of a singularity and its resolution process.

For curves

PUISEUX EXPANSIONS
↳ Equisingularity type
$y^2 = x^3 + x^4 \to (3,2)$
$\updownarrow$ e.s.
$y^2 = x^3$

HN EXPANSIONS
Allow the same for positive characteristic

For surfaces → Singularity theory is very difficult over fields of characteristic 0. We cannot hope to achieve complete characterisation.

Solution → Restrict ourselves to a class of nice singularities of surfaces.
(simple elliptic singularities, almost simple critical points...)

I. Rational Double Singularities

Survey article (Alan Durfee) called "Fifteen characterisations of rational double points and simple critical points" (over char 0).

Today, I will present 5 of these characterisations (and in the next lecture I will present at least 2 more)

Let X be a normal projective surface defined over an algebraically closed field, and suppose P is the only singular point of X.

A resolution of the singularity P of X is a birational proper morphism $\pi: X' \longrightarrow X$ where X' is a smooth surface with a reduced divisor $E = \sum^{n} E_i$ (such that every E_i is a complete curve) such that $\pi(E)=\{P\}$ and π is an isomorphism between $X' \setminus E$ and $X \setminus \{P\}$.

We say that π is a minimal resolution if the exceptional locus E contains no (-1)-curves.

→ Let Z be a variety of dim d, recall, Euler-Poincaré $\chi(\mathcal{O}_Z) = \sum_{i=0}^{d} (-1)^i h^i(\mathcal{O}_Z)$

Turns out if $\pi: X' \longrightarrow X$ is a resolution of singularities $\chi(\mathcal{O}_{X'})$ is independent of the choice of resolution.

We say that P is a <u>rational singularity</u> if $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X'})$

Alternatively if the sheaf ~~supp~~ $R^1\pi_* \mathcal{O}_{X'} = 0$ supported at P

Let m denote the maximal ideal of $\mathcal{O}_{X,P}$ and let $k = \mathcal{O}_{X,P}/m$, then, it is well known that for $n \gg 0$

$$\dim_k \mathcal{O}_{X,P}/m^{n+1} = \frac{\mu(\mathcal{O}_{X,P})}{\boxed{d}!} n^d + \text{terms of degree} < d.$$

d ↳ $\dim \mathcal{O}_{X,P}$

$\mu(\mathcal{O}_{X,P})$ is the multiplicity of the singular point P.

A. char $p \geq 0$

We say that P is a rational double point if it is a rational singularity of multiplicity 2.

In that case, it can be checked that $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 3$ and this implies that $\widehat{\mathcal{O}_{X,P}} \cong k[[x,y,z]]/(f)$ for some $f \in k[[x,y,z]]$, so X can be locally analytically described as a hypersurface $f=0$.

There are lists of polynomials (better said, there are a finite families of polynomials that describe, up to analytical isomorphism, how do rational points look like). ⇒ NORMAL FORMS.

B. P is a rational double point if it is analytically isomorphic to $k[[x,y,z]]/(f)$ for one of the f in the list
$p \geq 0$

Let us now look at the resolution aspect of singularities.

Consider $\pi^{-1}(P) = E = \sum E_i$

It is always possible to find a resolution where

1. E_i are non-singular
2. E_i intersect transversely
3. No 3 intersect at a point and $E_i \cap E_j = \emptyset$ or $\{pt\}$.

The intersection matrix of the resolution is negative definite.

C. P is a rational double point if there exists a minimal resolution whose exceptional set consists of curves of genus 0 and self-intersection -2.
$p \geq 0$

There are a limited amount of possibilities, described by Dynkin diagrams.

Let's now focus on the complex case.

Let V be a neighbourhood of $0 \in \mathbb{C}^2$ and let G be a finite group of analytic automorphisms fixing 0.

The quotient space has a structure of a complex analytic space with an isolated singularity.

If $V \cong \mathbb{C}^2/G$ we say that it is a quotient singularity.

An example would be, take G finite subgroup of $GL_2(\mathbb{C})$.

Then, the quotient is algebraic and $k[V] = k[x,y]^G$

D. non-0

P is a rational double point if it is a quotient singularity of $\mathbb{C}^2$.

There are finite families of finite groups of $GL_2(\mathbb{C})$ fixing 0.

<u>Example</u>

$$G = \mathbb{Z}/2\mathbb{Z} = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} \qquad \begin{matrix} x \mapsto -x \\ y \mapsto -y \end{matrix}$$

Invariants are x^2, y^2, xy.

$$k[x,y]^{\mathbb{Z}/2\mathbb{Z}} = k[\underset{a}{\underset{\shortparallel}{x^2}}, \underset{b}{\underset{\shortparallel}{y^2}}, \underset{c}{\underset{\shortparallel}{xy}}] \implies \mathbb{C}^2/G \cong k[a,b,c]/(ab-c^2)$$

In characteristic 2, this does not necessarily make sense (Artin)

II. How to find all possible normal forms in char p

Let U be the complement of P, $U = X \setminus \{P\}$.

We define the local fundamental group of X at P to be

$$\pi = \pi_1(U)$$

This group classifies finite étale coverings of U or pure 2-dimensional schemes Y finite over X, which are étale except above P.

→ Unramified in codim 1

We call a covering of X any finite surjective map $Y \to X$ that is irreducible and normal, and let us call the covering unramified if it is étale above U.

Why are these coverings important?

Proposition (Artin) → If X admits a smooth covering, then π is finite. If such covering is totally ramified along one curve of X, then $\pi = 0$.

In characteristic zero

π finite $\Rightarrow$ X has a smooth covering.

$\pi = 0$ $\Rightarrow$ X smooth (not true in characteristic $p>0$)

E. P is a rational double point if it ~~is~~ P is singular and π is finite.
$p>0$

In char. 0 $\Rightarrow$ $X = \mathbb{A}^2/\pi$

An unramified covering Y of a RPP is either smooth or a rational double point

Studying the ramified curves under the cover provides information about the singularity.

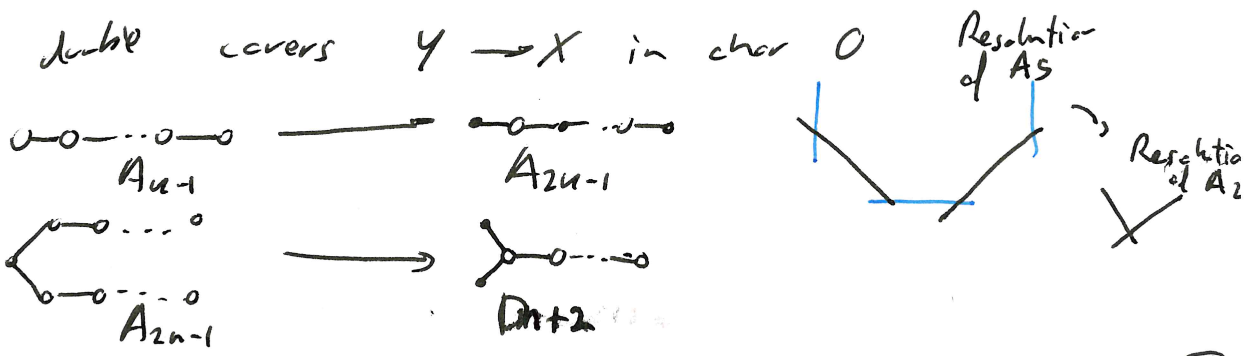
$$\begin{tikzcd} & Y' \arrow[r, "\pi_Y"] & Y \arrow[d] \\ Y_{X'} \arrow[ur, "\text{dominate morphism}"] \arrow[r] & X' \arrow[r, "\pi_X"] & X \end{tikzcd}$$

Y' unramified cover

$Y_{X'}$ normalisation of X' in $k(Y)$

The curves in $Y_{X'}$ which contract in Y' are the ones ramified over X'.

Unramified double covers $Y \to X$ in char 0



Example

A_3 singularity $\qquad z^2 + y^2 + x^4 = 0$

$$z^2 - y^2 + x^4 = 0 \quad \xleftrightarrow{\substack{r = x \\ s = z - y \\ t = z + y}} \quad st + \cancel{x}^4 = 0$$

Consider

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ix \\ -iy \end{pmatrix}$$

invariants are

$$\underset{s}{\underset{\shortparallel}{x^4}}, \ \underset{t}{\underset{\shortparallel}{y^4}}, \ \underset{r}{\underset{\shortparallel}{xy}}$$

A_1 singularity. $\qquad$ Now, consider the cover

$$S^2 = s \qquad S^2 = x^4$$
$$T^2 = t \qquad T^2 = y^4 \Rightarrow \begin{matrix} S = x^2 \\ T = y^2 \end{matrix}$$

$$\to ST + r^2 = 0$$

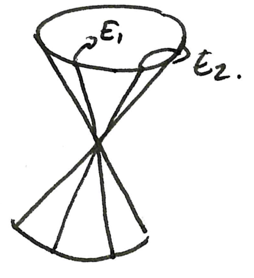
Let's resolve the A_3 singularity.

Consider $\left\{ \begin{matrix} z \mapsto x_1 \end{matrix} \right. \qquad \begin{matrix} x \mapsto x_1 \\ y \mapsto x_1 y \\ z \mapsto x_1 z \end{matrix}$

$$x_1^2 (z^2 - y^2 + x_1^2) = 0$$

The ~~exc~~ proper transform is the surface $z^2 - y^2 + x_1^2 = 0$.

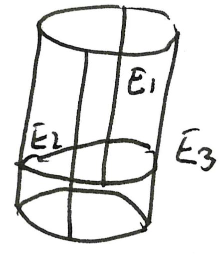
The exceptional divisors are $x_1 = 0 \Rightarrow z^2 - y^2 = 0 \Rightarrow \begin{matrix} E_1 \\ E_2 \end{matrix} \left\{ \begin{matrix} z - y = 0 \\ z + y = 0 \end{matrix} \right.$

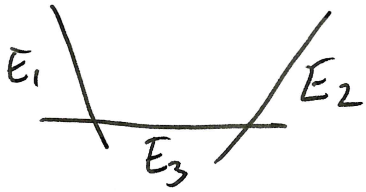


Let us now blow the ~~y~~ chart

$$\left\{ \begin{matrix} x_1 \mapsto y_1 x_1 \\ y_1 \mapsto y_1 \\ z \mapsto y_1 z \end{matrix} \right.$$

$$\Rightarrow \qquad y_1^2 (z^2 - 1 + x_1^2) = 0$$





Let's study which curves they ramified.

$$\begin{cases} S^2 = s \\ T^2 = t \end{cases} \quad \text{ramifies if} \quad \begin{array}{ll} S^2 = 0 = z - y & \Rightarrow E_1 \\ T^2 = 0 = z + y. & \Rightarrow E_2. \end{array}$$

[redacted]

→ Any rational double point in characteristic p can be obtained from one in characteristic zero.

→ Let X be a RDP in char p. The unramified Galois covers of X which lift to characteristic zero are those of order prime to p.

$\Rightarrow$) <u>Idea</u> → The singularities whose ~~…~~ fundamental groups have an order coprime to p will behave the same as in characteristic zero. (they will admit a normal form)

The fundamental group π of X is <u>tame</u> if it has order prime to p.

The A_n singularities are of the form $xy + z^{n+1}$

In char 0.

Corresponds to $\mathbb{Z}/n\mathbb{Z}$: $\begin{pmatrix} \zeta_n^{\,i} & 0 \\ 0 & \zeta_n^{\,n-i} \end{pmatrix}$ ζ_n root of unity. The invariants are

$$\begin{cases} s = x^n \\ t = y^n \\ r = xy \end{cases}$$

These defines a covering $Y \xrightarrow{n} X \sim$

What happens when $p \mid (n+1) \Rightarrow n+1 = p^e m$ with m ⊤

Covering becomes inseparable $\Rightarrow$ Fundamental group π becomes $\mathbb{Z}/m\mathbb{Z}$.

This is because of the fact that $X = \mathbb{A}^2/\mu_{pe}$

To be a bit more concrete, in the case of A_{p-1} in char p, $xy + z^p = 0$ is given as the quotient

$\mathbb{A}^2/\mu_p$ where μ_p is given by the action of the derivation $\delta = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}$ on $k[X,Y]$.

x is invariant, This satisfies that $\delta^p = \delta$. Invariants $\underset{y}{\underset{\shortparallel}{Y^p}}, \underset{x}{\underset{\shortparallel}{X^p}}, \underset{z}{\underset{\shortparallel}{XY}}$

Let $k[[x]] = k[[x_1, \ldots, x_n]]$ be the formal power series in n-variables. Let $k[[x]]^*$ denote the unit group and let $f, g \in k[[x]]$

1. f is right equivalent to g ($f \overset{r}{\sim} g$) if $\exists \varphi \in \mathrm{Aut}(k[[x]])$ s.t. $f = \varphi(g)$. (f is obtained from g by an analytic change of coordinates)

2. f is contact equivalent to g ($f \overset{c}{\sim} g$) $\Leftrightarrow \exists\, \Phi \in \mathrm{Aut}(k[[x]])$ and $u \in k[[x]]^*$ such that $f = u \cdot \Phi(g)$ i.e. ($k[[x]]/\langle f\rangle \cong k[[x]]/\langle g\rangle$ as analytic k-algebras)

Let $J^{(k)} = k[[x]]/m^{k+1}$ be the space of k-jets.

If $f \in k[[x]]$, we write $f^{(k)} \in J^{(k)}$ the power series of f up to terms of order k.

$f \in k[[x]]$ is called contact k-determined if each $g \in k[[x]]$ with $f^{(k)} = g^{(k)}$ is contact equivalent to f.

$\mathfrak{m}$ maximal ideal of $\mathcal{O}_{X,p}$

Theorem (Boubakri, Greuel, Markwig) Let $j(f) = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle$ and let $f \in \mathfrak{m}^2 \setminus \{0\}$ and $\ell \in \mathbb{N}$. If $\mathfrak{m}^{\ell+2} \subset \mathfrak{m} \cdot \langle f \rangle + \mathfrak{m}^2 \cdot j(f)$, then f is contact ~~ℓ~~ k-determined with

$$k = (2\ell - \mathrm{ord}(f) + 2)$$

Better! f is finitely determined if it is k-determined for some k.
Let $\mathrm{ord}(f)$ be the maximum k with $f \in \mathfrak{m}^k$ and $\tau(f)$ the Tjurina number of f:

$$\tau(f) = \dim_k k[[x]] / \langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle.$$

Then, turns out

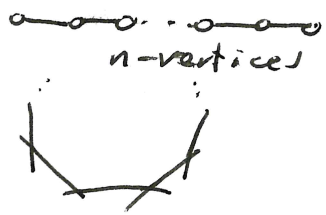
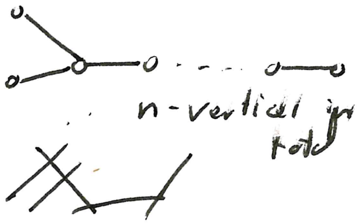
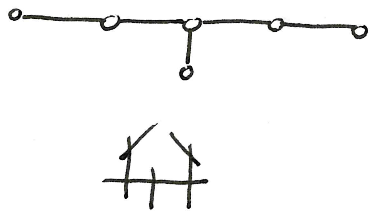
<u>Theorem</u> (Boubakri, Greuel, Markwig, Pham)

f is finitely contact determined $\Leftrightarrow \tau(f) < \infty$

and if $\tau(f) < \infty$, f is contact k-determined where

$$k = 2\tau(f) - \mathrm{ord}(f) + 2$$

So how can we determine all normal forms of points?
We consider space of 2-jets and analyse all possible cases.
We analyse each case individually. If we get that we get that the equation is contact equivalent to an ADE singularity which is contact 2-determined, or that is contact equivalent to something that is not a RDP we move on to a different branch.
If not, we consider the space of 3-jets and so on.
Because $\tau(f)$ is finite for RDP in surfaces the algorithm eventually stops

Type	Normal Form	Local fundamental group		Dual Graph	Milhor - Tjurina numbers
A_n $(n \geq 1)$	$x^{n+1}+y^2+z^2$ $z^2+y^2+x^{n+1}$ $zy+x^{n+1}$	$\mathbb{Z}/(n+1)\mathbb{Z}$	Order $n+1$		n
D_n $(n \geq 4)$	$z^2+x^2y+y^{n-1}$	$BD_{4(n-2)}$ Binary Dihedral	Order $4(n-2)$		n
E_6	$z^2+x^3+y^4$	Binary Tetrahedral	Order 24		6
E_7	$z^2+x^3+xy^3$	Binary Octahedral	Order 48		7
E_8	$z^2+x^3+y^5$	Binary Icosahedral	Order 120		8