Some examples of Chow rings and applications to intersection theory

Alvaro Gonzalez Hernandez

Study group on intersection theory, May 21, 2024

These notes are a creative adaptation (by which I mean, they are borderline plagiarism) of the second section of the book **3264 and All That** [EH16] by David Eisenbud and Joe Harris.

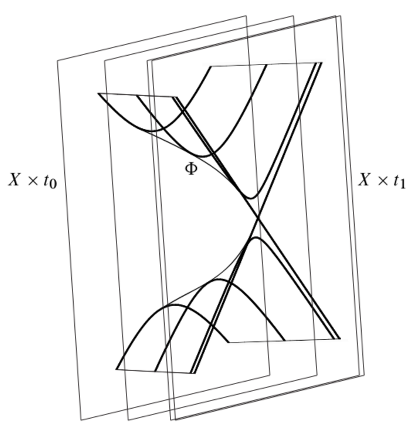
1 Introduction and recap

One of the main goals of this study group is to study the machinery used to solve combinatorial problems involving the intersections of varieties in projective space. As we have seen in previous talks, the main object that describes the intersection theory inside of a variety X is the **Chow ring of** X, whose definition I will recall.

1.1 Defining the Chow ring

Definition 1.1. *Let X be an integral separated scheme of finite type over an algebraically closed field of characteristic 0 and let $Z(X)$ be the* ***group of cycles*** *of X, namely, the free abelian group generated by the set of subvarieties of X. Then, the* ***Chow group*** *of X, $\mathrm{CH}(X)$ is the group of cycles of X modulo rational equivalence.*

There is some subtleties about this definition of **rational equivalence**, but, informally, the idea is that two cycles are rationally equivalent if there is a rationally parametrised family of cycles interpolating between them.



We would like to endow the Chow group of X with a product structure in which the product of two classes $[A]$ and $[B]$ in $\mathrm{CH}(X)$ corresponded to the class of their intersection $[A \cap B]$. There are, however, some problems that we encounter when trying to define this product.
The main one is that there is a natural grading in the Chow group

$$\mathrm{CH}(X) = \bigoplus_{i=0}^{\dim(X)} \mathrm{CH}^i(X)$$

coming from the fact that we can grade the cycles in X according to its codimension, and we would ideally want our Chow ring to be a graded ring with respect to it. However, it is not true that the intersection of two irreducible subvarieties A and B of codimensions a and b is always a subvariety of codimension $a + b$.

There is a situation when this is true, which is when the varieties A and B are **generically transverse**, meaning that they meet transversely at a general point of each component C of $A \cap B$. By meeting transversely at a point p, what I mean is that A, B and X are smooth at p and

$$\operatorname{codim}(T_pA \cap T_pB) = \operatorname{codim}(T_pA) + \operatorname{codim}(T_pB).$$

Here is where it is useful that in Chow groups cycles are defined up to rational equivalence, as it turns out, when X is a smooth quasi-projective variety, given any two classes $\alpha, \beta \in \mathrm{CH}(X)$, we can always find two cycles $A, B \in Z(X)$ such that $[A] = \alpha$ and $[B] = \beta$, and the class $[A \cap B]$ is independent of the choice of A and B. This is what is known as the **moving lemma.**

Now, let X again be a smooth quasi-projective variety. For every two classes $\alpha, \beta \in \mathrm{CH}(X)$, with $\alpha = [A]$ and $\beta = [B]$, we can define their product in $\mathrm{CH}(X)$

$$\alpha \cdot \beta = [A][B] = [A \cap B]$$

and this makes $\mathrm{CH}(X)$ an commutative ring, graded by codimension called the **Chow ring.**

1.2 Affine stratifications of a variety

We have seen in previous talks that we have partial knowledge of the Chow groups of varieties that admit a decomposition known as an **affine stratification**, so let's recall this concept.

Definition 1.2. *A scheme X is said to have an* ***affine stratification*** *if there exist a finite collection of irreducible, locally closed subschemes U_i satisfying:*

1. *X is a disjoint union of the U_i.*
2. *If $\bar{U}_i$ meets U_j, then $\bar{U}_i$ contains U_j.*
3. *Every U_i is isomorphic to $\mathbb{A}^k$ for some k.*

The closures of these strata, $\bar{U}_i$ are called the **closed strata.** We then have the following theorem due to Totaro [Tot14]:

Theorem 1.3. *The classes of the strata in an affine stratification of a scheme X form a basis of* $\mathrm{CH}(X)$.

2 A few examples of Chow rings

We have seen in the past some examples of Chow groups of varieties, and it is not difficult to see how the intersection product gives these groups a ring structure.

Proposition 2.1. *The Chow ring of $\mathbb{A}^n$ is*

$$\mathrm{CH}(\mathbb{A}^n) \cong \mathbb{Z} \cdot [\mathbb{A}^n].$$

Proof. Mayo proved in their talk that scalar multiplication could be used to set a rational equivalence between any affine variety not containing the origin and the empty set, showing that the Chow group of $\mathbb{A}^n$ was $\mathbb{Z}$, with the fundamental class $[\mathbb{A}^n]$ as a generator. This proposition therefore follows from the fact that $[\mathbb{A}^n][\mathbb{A}^n] = [\mathbb{A}^n]$. □

Let's now do a slightly harder example

Proposition 2.2. *The Chow ring of $\mathbb{P}^n$ is*

$$\mathrm{CH}(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/(\zeta^{n+1}),$$

where $\zeta \in \mathrm{CH}^1(\mathbb{P}^n)$ is the equivalence class of a hyperplane. More generally, the class of a variety of codimension k and degree d is $d\zeta^k$.

Proof. Consider the flag of subspaces

$$\{p\} \subset \mathbb{P}^1 \subset \mathbb{P}^2 \subset \cdots \subset \mathbb{P}^n.$$

Tommaso proved in his talk that as a group, $\mathrm{CH}(\mathbb{P}^n) \cong \mathbb{Z}^{n+1}$, using the affine stratification whose open sets are $U_i = \mathbb{P}^i \setminus \mathbb{P}^{i-1}$. Furthermore, $\mathrm{CH}^k(\mathbb{P}^n) \cong \mathbb{Z}$, where the generator in each graded piece was an $(n-k)$-plane (a linear subvariety of codimension k).

Now, from basic linear algebra, we now that the intersection of k hyperplanes H in general position, is an $(n-k)$-plane L, and so,

$$[L] = \zeta^k,$$

with $\zeta = [H]$. Finally, let's recall that the definition of degree of a variety X of dimension k is precisely that it intersects a general k-plane transversely in d points, from which we deduce that $[X]\zeta^{n-k} = d\zeta^n$. Since $\deg(\zeta^n) = 1$, we conclude that $[X] = d\zeta^k$. □

2.1 Some consequences of the structure of the Chow ring of $\mathbb{P}^n$

There are a few nice consequences from this:

Corollary 2.3. *A morphism from $\mathbb{P}^n$ to a quasi-projective variety of dimension strictly less than n is constant.*

Corollary 2.4. *If $X \subset \mathbb{P}^n$ is a variety of dimension m and degree d, then,*

$$\mathrm{CH}_k(\mathbb{P}^n \setminus X) \cong \begin{cases} \mathbb{Z}/d\,\mathbb{Z} & \text{if } k = m \\ \mathbb{Z} & \text{if } m < k \leq n \end{cases}$$

In particular, m and d are determined by the isomorphism class.

Proof. We have seen before that for all $k \geq m$, there exist a short exact sequence (of rings)

$$\mathrm{CH}_k(X) \longrightarrow \mathrm{CH}_k(\mathbb{P}^n) \longrightarrow \mathrm{CH}_k(\mathbb{P}^n \setminus X) \longrightarrow 0$$

The first equality comes from the fact $\mathrm{CH}_m(X) \cong \mathbb{Z}$ as X is irreducible of dimension m, $\mathrm{CH}_m(\mathbb{P}^n) \cong \mathbb{Z}$, as explained before, and the image of $[X]$ in $\mathrm{CH}_m(\mathbb{P}^n)$ is $d\zeta^{n-m}$. The second equality is a consequence of $\mathrm{CH}_k(X) = 0$ whenever $k > m$. □

These two results help convey an idea about closed projective varieties: up to rational equivalence, most of their geometry is determined by their dimension and their degree.

This behaviour is really different to other geometric invariants which are not preserved under rational equivalence, such as the arithmetic genus. There is an interesting discussion of this phenomenon in Eisenbud and Harris book. In particular, the example they propose are the following two non-reduced curves in $\mathbb{P}^3$:

$$C_1 = \mathbb{V}((x,y)^2) \qquad\qquad C_2 = \mathbb{V}(x, y^3)$$

Both of them are rationally equivalent to 3 times the class of a line, but C_1 has arithmetic genus 0 whereas C_2 has arithmetic genus 1.

Another classical result that can be deduced from the structure of the Chow ring of $\mathbb{P}^n$ is **Bézout's theorem:**

Corollary 2.5 (Bézout's theorem)**.** *If $X_1, \dots, X_k \in \mathbb{P}^n$ are subvarieties of codimensions $c_1, \dots, c_k$ with $\sum_{i=1}^k c_i \leq n$ and the X_i intersect generically transversely, then*

$$\deg(X_1 \cap \dots \cap X_k) = \prod_{i=1}^k X_i$$

In particular, two subvarieties $X, Y \in \mathbb{P}^n$ having complementary dimension and intersecting transversely will intersect in exactly $\deg(X) \cdot \deg(Y)$ points.

This theorem can be generalised to the case where the subvarieties $X_1, \dots, X_k$ are Cohen-Macaulay[1] and do not intersect generically transversely, but rather, **dimensionally transversely**, meaning that for every irreducible component Z of $X_i \cap X_j$,

$$\operatorname{codim}(X_i) + \operatorname{codim}(X_j) = \operatorname{codim}(Z)$$

2.2 The Chow ring of products of projective spaces

Finally the last example of Chow ring that I will present is the following:

Proposition 2.6. *The Chow ring of $\mathbb{P}^r \times \mathbb{P}^s$ is*

$$\mathrm{CH}(\mathbb{P}^r \times \mathbb{P}^s) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1}).$$

where $\alpha, \beta \in \mathrm{CH}^1(\mathbb{P}^r \times \mathbb{P}^s)$ denote the pullbacks via the projection maps of the hyperplane classes on $\mathbb{P}^r$ and $\mathbb{P}^s$ respectively. Moreover, the class of they hypersurface defined by a bihomogeneous form of bidegree (d, e) on $\mathbb{P}^r \times \mathbb{P}^s$ is $d\alpha + e\beta$.

Proof. The idea is very similar to the proof for the Chow ring of projective space, which consists in finding an affine stratification of $\mathbb{P}^r \times \mathbb{P}^s$. The way we do this is to consider two flags

$$\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{r-1} \subset \Lambda_r = \mathbb{P}^r$$
$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_{s-1} \subset \Gamma_s = \mathbb{P}^s$$

where $\dim(\Lambda_i) = \dim(\Gamma_i) = i$. Take the closed strata to be

$$\Xi_{a,b} = \Lambda_{r-a} \times \Gamma_{s-b} \subset \mathbb{P}^r \times \mathbb{P}^s,$$

and the open strata

$$\tilde{\Xi}_{a,b} = \Xi_{a,b} \setminus (\Xi_{a-1,b} \cup \Xi_{a,b-1}) \cong \mathbb{A}^{a+b}.$$

Then, from theorem 1.3, we deduce that the Chow group of $\mathbb{P}^r \times \mathbb{P}^s$ is generated by the classes $[\Xi_{a,b}] \in \mathrm{CH}^{a+b}(\mathbb{P}^r \times \mathbb{P}^s)$. Since $\Xi_{a,b}$ is the transverse intersection of a hyperplanes in $\mathbb{P}^r$ and b hyperplanes in $\mathbb{P}^s$, we have that $[\Xi_{a,b}] = \alpha^a \beta^b$ and, in particular, $\alpha^{r+1} = \beta^{s+1} = 0$. Therefore, $\mathrm{CH}(\mathbb{P}^r \times \mathbb{P}^s)$ is a homomorphic image of $\mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1})$.

[1] A locally Noetherian scheme X is **Cohen–Macaulay** if at each point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is Cohen–Macaulay.
The definition of Cohen-Macauly ring is, in my opinion, not very illustrative, so I suggest having in mind that varieties that are Cohen-Macaulay are: complete intersections, all 0-dimensional varieties, all 1-dimensional reduced varieties and all 2-dimensional normal varieties. Examples of varieties that are not Cohen-Macaulay are, for instance, the union of a line and a point in it, or in general, any union of two varieties of different dimensions.

The last thing that we need to check is that the elements $\alpha^a\beta^b$ are all independent over $\mathbb{Z}$ for $0 \leq a \leq r$ and $0 \leq b \leq s$. To see that, consider the pairing:

$$\begin{aligned}\varphi : \mathrm{CH}^{a+b}(\mathbb{P}^r \times \mathbb{P}^s) \times \mathrm{CH}^{r+s-a-b}(\mathbb{P}^r \times \mathbb{P}^s) &\longrightarrow \mathbb{Z} \\ ([X],[Y]) &\longmapsto \deg([X][Y])\end{aligned}$$

It is easy to see that

$$\varphi(\alpha^a\beta^b, \alpha^m\beta^n) = \begin{cases} 1 & \text{if } a+m=r \text{ and } b+n=s \\ 0 & \text{otherwise} \end{cases}$$

as the first case happens when the intersection is transverse and consist on one point, and in the second the intersection is empty. This easily shows that the $\alpha^a\beta^b$ are independent. Finally, if $f(x_0, \ldots, x_r, y_0, \ldots, y_s)$ is a bihomogeneous polynomial of bidegree (d, e), it is easy to check that $f/(x_0^d y_0^e)$ is a rational function, and therefore the variety $X = \mathbb{V}(f)$ satisfies that $[X] = d\alpha + e\beta$. $\square$

3 Computing the degrees of varieties

The degree of a variety of dimension n is the number of intersection points with a n general hypersurfaces. Therefore, it should not come as a surprise to anybody that this notion of degree will be very relevant when studying problems in enumerative geometry such as the following, which we will soon answer:

Let $S \subset \mathbb{P}^3$ be a smooth cubic surface and $L \subset \mathbb{P}^3$ a general line. How many planes containing L are tangent to S?

The key strategy to solve this problem is to translate this problem into computing the degree of a variety. In order to compute this degree, we can follow the following strategy:

1. Identify our variety as the image of a "nice map" (birational and finite, for instance) $f : X \to Y$.
2. The degree of the image of f will be the number of points of its intersection with n general hyperplanes H in Y, and what we will do is to pull-back these intersection, to relate it to an easier computation in X.

Let us see one example of this technique:

3.1 Degree of the dual of a hypersurface

Given a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d, we can consider its **dual variety**, meaning the set of points $X^* \subset (\mathbb{P}^n)^*$ parametrising hypersurfaces that are tangent to X.

In principle, it is not obvious why this set X^* would even be a variety, but one can easily check that it is the image of X under the **Gauss map** that sends a point $p \in X$ to its tangent hyperplane.

In coordinates, if X is the zero locus of the homogeneous polynomial $f(x_0, \ldots, x_n)$, then $\mathcal{G}_X$ is given by

$$\begin{aligned} \mathcal{G}_X \colon \mathbb{P}^n &\longrightarrow (\mathbb{P}^n)^* \\ p &\longmapsto \left[\frac{\partial f}{\partial x_0}(p) : \cdots : \frac{\partial f}{\partial x_n}(p)\right] \end{aligned}$$

This is well defined since X is smooth so the partials have no common zeroes. If $d = 1$, $\mathcal{G}_X$ is constant and X^* is a point, but if $d > 1$ $\mathcal{G}_X$ is finite[2] and birational onto its image (this is not easy to see).

We will use this to study the degree of the dual hypersurface.

Proposition 3.1. *If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d > 1$, then the dual of X is a hypersurface of degree $d(d-1)^{n-1}$.*

Proof. The degree of the dual variety $X^* \subset (\mathbb{P}^n)^*$ is the number of points of intersection of X^* and $n-1$ general hyperplanes $H_i \subset (\mathbb{P}^n)^*$. Since the map $\mathcal{G}_X : X \to X^* \subset (\mathbb{P}^n)^*$ is birational, this is the same as the number of points of intersection of the preimages $\mathcal{G}_X^{-1}(H_i)$.

But also, since $\mathcal{G}_X$ is given by the partial derivatives of the defining equation F of X, the preimages of these hyperplanes are the intersections of X with the hypersurfaces $Z_i \subset \mathbb{P}^n$ of degree $d-1$ in $\mathbb{P}^n$ given by general linear combinations of these partial derivatives. As the partials of F have no common zeros, Bertini's theorem tells us that the hypersurfaces given by $n-1$ general linear combinations will intersect transversely with X. By Bézout's theorem the number of these points of intersection is the product of the degrees of the hypersurfaces, that is, $d(d-1)^{n-1}$. □

[2] A **finite morphism** between two affine varieties X, Y is a dense regular map which induces isomorphic inclusion $k[Y] \hookrightarrow k[X]$ between their coordinate rings, such that $k[X]$ is integral over $k[Y]$. The condition extends to quasi-projective varieties by taking covers of affine open sets.

For example, suppose that X is a smooth cubic curve in $\mathbb{P}^2$. By the above formula, the degree of X^* is 6. Since a general line in $\mathbb{P}^{2*}$ corresponds to the set of lines through a general point $p \in \mathbb{P}^2$, there will be exactly six lines in $\mathbb{P}^2$ through p tangent to X.

Let's give the answer to our question, then:

Let $S \subset \mathbb{P}^3$ be a smooth cubic surface and $L \subset \mathbb{P}^3$ a general line. How many planes containing L are tangent to S?

Since the planes containing the line L form a general line in the dual projective space $\mathbb{P}^{3*}$, the number of such planes tangent to a smooth cubic surface $S \subset \mathbb{P}^3$ is $3 \cdot 2^2 = 12$.

References

[EH16] David Eisenbud and Joe Harris. *3264 and All That*. Cambridge University Press, 4 2016.

[Tot14] Burt Totaro. Chow groups, Chow cohomology, and linear varieties. *Forum of Mathematics, Sigma*, 2:e17, 6 2014.