Three-dimensional Incompressible Convective Brinkman-Forchheimer Equations

by

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Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Mathematics Institute

July 2019

Contents

Acknowledgements

First and foremost, I wish to thank my supervisor, James Robinson, for countless enlightening conversations, his patient guidance, kind encouragement and all the help he has given me over the course of my PhD studies.

A special thanks to my former MSc advisor at Warsaw, Witold Sadowski, who was my first teacher in the fascinating area of partial differential equations and who inspired me to delve into the subject of fluid mechanics despite my initial reluctance.

I also wish to express my gratitude to Charles Fefferman for his invaluable help with the approximation arguments in Chapter 8.

My thanks also to my two examiners, José Rodrigo and Sergey Zelik, whose corrections and useful comments made this thesis both more accurate and readable.

I gratefully acknowledge the doctoral training award provided by the EPSRC that supported my studies at Warwick.

I also wish to thank all the doctors and medical staff who took care of my knee problems over the past few years and also the physiotherapy team who put me back on my feet after my surgery.

Last, but by no means least, I would like to thank my wonderful wife, Dominika, for her love and support, and for making it all worthwhile.

Declarations

Parts of this thesis form the basis for published or submitted papers as follows:

- Parts of Chapters 3 (especially Sections 3.4 and 3.5) and 4 (Sections 4.3 and 4.4) formed Hajduk and Robinson [2017];
- Chapter 5 as well as Section 4.5 formed Hajduk et al. [2019]. This was joint work with James Robinson (Warwick) and Witold Sadowski (Bristol);
- Chapters 7, 8 and 9 formed Fefferman et al. [2019]. This was joint work with Charles Fefferman (Princeton) and James Robinson (Warwick).

I declare that, to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated, cited, or commonly known.

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

Abstract

This thesis presents a mathematical analysis of the incompressible convective Brinkman–Forchheimer equations in three-dimensional space,

$$\partial_t u - \mu\Delta u + (u\cdot\nabla)u + \alpha u + \beta\left|u\right|^{r-1}u + \nabla p = f, \qquad \operatorname{div} u = 0,$$

where $\alpha, \beta \geq 0$, and $r \geq 1$. These equations describe the motion of a fluid in a saturated porous medium. They can be seen also as the incompressible Navier–Stokes equations with the additional linear and nonlinear terms αu and $\beta\left|u\right|^{r-1}u$. For simplicity, we neglect the linear term throughout the thesis, but all the results presented in this thesis hold also for general $\alpha > 0$. In the thesis we study the influence of the nonlinear term on the existence of weak and strong solutions of the CBF equations and some of their properties.

In particular, we establish that all weak solutions of the 'critical' problem ($r = 3$) verify the Energy Equality

$$\frac{1}{2}\left\|u(T)\right\|^2 + \mu\int_0^T \left\|\nabla u(t)\right\|^2\,\mathrm{d}t + \beta\int_0^T \left\|u(t)\right\|_{L^{r+1}}^{r+1}\,\mathrm{d}s = \frac{1}{2}\left\|u(0)\right\|^2,$$

both on the torus $\mathbb{T}^3$ and on bounded domains $\Omega \subset \mathbb{R}^3$ with smooth boundary. From this fact, we infer the existence of a strong global attractor in the phase space $H \hookrightarrow L^2$ using theory of evolutionary systems developed by Cheskidov [2009].

Moreover, we prove the existence of global-in-time strong solutions on the torus $\mathbb{T}^3$, for two cases: $r > 3$, and $r = 3$ provided that the product of viscosity (μ) and porosity (β) coefficients is not too small, $4\mu\beta \geq 1$. We also establish that strong solutions are unique in the larger class of weak solutions ('weak-strong uniqueness'). Additionally, we provide a 'robustness of regularity' condition for strong solutions of the convective Brinkman–Forchheimer equations when $r \in [1, 3]$.

We also give two general methods of simultaneous approximation in Lebesgue and Sobolev spaces using semigroup theory and finite-dimensional eigenspaces of operators. Furthermore, we provide a simple proof of known characterisation of the domains of the fractional powers of the Laplace and Stokes operators, using the theory of real interpolation spaces. This characterisation is needed to apply our approximation method in the proof of the energy equality on bounded domains.

Chapter 1

Introduction

1.1 Motivation of the model

The three-dimensional incompressible Navier–Stokes equations (NSE) constitute a fundamental model of fluid dynamics. They are given by the system of partial differential equations

$$\partial_t u - \mu \Delta u + (u \cdot \nabla) u + \nabla p = f, \qquad \operatorname{div} u = 0,$$

where $u(x,t) = (u_1, u_2, u_3)$ is an unknown velocity vector field and the scalar function $p(x,t)$ is an unknown pressure. The function $f(x,t) = (f_1, f_2, f_3)$ represents given external forces acting on the fluid (e.g. gravitational forces) and the constant $\mu > 0$ denotes the viscosity of the fluid. We assumed here that the fluid is homogenous with constant density equal to 1. These equations are supplemented by an initial condition $u(x,0) = u_0$ and appropriate boundary conditions $u|_{\partial\Omega}$ for a considered spatial domain $\Omega \subset \mathbb{R}^3$ occupied by the fluid. The importance of this model in physics and mathematics is well illustrated by the fact that the global regularity of its solutions constitutes one of the seven Millennium Prize Problems stated by the Clay Mathematics Institute in May 2000 (for an exact statement of the 'regularity problem' see Fefferman [2006]). The problem described there has remained open since the time of the pioneering works of Leray [1934] and Hopf [1951].

In this thesis we consider the Navier–Stokes equations with the additional nonlinear term $\beta |u|^{r-1} u$ for $r \geq 1$ introduced in the momentum equation

$$\partial_t u - \mu \Delta u + (u \cdot \nabla) u + \beta |u|^{r-1} u + \nabla p = f, \qquad \operatorname{div} u = 0, \tag{1.1}$$

where β is some positive constant. This term is usually called the absorption term.

A similar nonlinearity was studied before in the context of different initial and boundary value problems not necessarily in the area of fluid mechanics. The influence of such a term on the qualitative properties of solutions was studied, among others, by Benilan et al. [1975], Díaz and Herrero [1981] and Bernis [1986]. They studied in particular solutions with compact support, with finite speed of propagation or solutions which become extinct in a finite time.

In the settings described above, the introduction of the absorption term is purely of mathematical nature. This term behaves like a sink inside the domain occupied by the fluid which causes additional dissipation of energy and slows down the fluid flow. There are possible physical justifications for introducing the absorption term in the momentum equation as part of the external force field $h(u) = f - \beta |u|^{r-1} u$ (see de Oliveira [2010] for references). There is also a precise theory of the absorption of forced plane infinitesimal waves according to the Navier–Stokes equations by Truesdell [1953]. The influence of the damping term was studied extensively over the years for various other models in mathematical physics, like the Schrödinger equation (see e.g. Carles and Gallo [2011]), the wave equation (see e.g. Zhou [2005]) or the Euler equation (see e.g. Pan and Zhao [2009]).

One can look at the equations (1.1) from another point of view, more grounded in applications, which comes from the theory of flows in a porous medium. Most mathematical models of porous media are based on Darcy's law so "*Darcy's equation has become the model of choice for the study of the flow of fluids through porous solids due to the pressure gradients, so much that it has now been elevated to the status of a law in physics*" (Rajagopal [2007]). Darcy's empirical flow model assumes a linear relationship between the flow rate and the pressure drop in a porous medium

$$u = -\frac{k}{\mu}\nabla p,$$

where u is the Darcy velocity, k is the permeability of the porous medium, and μ is the dynamic viscosity of the fluid (see Darcy [1856]). Deviations from this scenario are called non-Darcy flows. Compared to the Navier–Stokes equations, this law neglects the acceleration and inertial, and viscous forces. Nature, however, can deviate from Darcy's law, for instance when one deals with high velocity, molecular and ionic effects or in the presence of some non-Newtonian effects in the fluid. In these situations, more adequate models are needed. One such model is the Forchheimer equation which states that the relationship between the flow rate and the pressure gradient is nonlinear at sufficiently high velocities and that this nonlinearity increases with flow rate (see Forchheimer [1901, 1930]). The Darcy–Forchheimer law states that

$$\nabla p = -\frac{\mu}{k} u - \gamma \rho \, |u| \, u,$$

where $\gamma > 0$ is the so-called Forchheimer coefficient, u stands for the Forchheimer velocity, and ρ is the density. The Forchheimer law can be seen as a nonlinear approximation of Darcy's law accounting for the increased pressure drop. See also Giorgi [1997] for the derivation of the Forchheimer law via matched asymptotic expansions. A numerical study supporting the quadratic correction to Darcy's law is given in Firdaouss et al. [1997], for example.

It is natural to generalise the Darcy–Forchheimer law to take into account not only quadratic nonlinearity; indeed, the cubic nonlinearity seems to be the most interesting one mathematically, as we will see later on. Such a generalisation, which also takes into account viscous forces (see Brinkman [1947, 1949]) and acceleration, is called the Brinkman–Forchheimer equations (BF)

$$\partial_t u - \mu \Delta u + \alpha u + \beta \, |u|^{r-1} \, u + \nabla p = f, \qquad \operatorname{div} u = 0. \tag{1.2}$$

This model describes the motion of incompressible fluid flows in a saturated porous medium. The constant μ stands for the positive Brinkman coefficient (effective viscosity). The positive constants α and β follow from the Darcy–Forchheimer law and denote respectively the Darcy (permeability of porous medium) and Forchheimer (proportional to the porosity of the material) coefficients. The BF equations have been used in connection with some real world phenomena, e.g. in the theory of non-Newtonian fluids (see e.g. Shenoy [1994]) or in tidal dynamics (see e.g. Gordeev [1973]; Likhtarnikov [1981]).

By adding to the BF model the inertial term coming from the Navier–Stokes equations [$(u \cdot \nabla)u$, which is called in fluid dynamics the convective or (more generally) advective term] we obtain the incompressible convective Brinkman–Forchheimer equations (CBF)

$$\partial_t u - \mu \Delta u + (u \cdot \nabla) u + \alpha u + \beta \, |u|^{r-1} \, u + \nabla p = f, \qquad \operatorname{div} u = 0, \tag{1.3}$$

where u is the average fluid velocity. This model was originally derived in its classical configuration ($r = 2$) in the framework of thermal dispersion in a porous medium using the method of volume averaging of the velocity and temperature deviations in the pores (see e.g. Hsu and Cheng [1990]). Its applicability is believed to be limited to flows when the velocities are sufficiently high and the porosities are not too small, i.e. when the Darcy law for a porous medium no longer applies. For a discussion of the formulation and limitations of this system see Vafai and Tien [1987] and Nield [1991, 1994, 2000]. The continuum mechanics approach to transport in a saturated porous medium is discussed in Salama and Van Geel [2008a,b]. An extensive study of different models of porous media is collected in the monograph by Nield and Bejan [2017] (5th edition).

Another generalisation of the Darcy–Forchheimer law (additionally taking into account pumping given by similar nonlinearity to the absorption term but with a negative sign) is discussed in Markowich et al. [2016], where an algorithm for continuous data assimilation for the 3D Brinkman–Forchheimer–extended Darcy model for porous media is discussed. The limitations of that extended model are discussed in Vafai and Kim [1995].

In this thesis we adopt the naming convention based on the porous medium approach discussed above. However, in our considerations of the CBF equations the linear term αu poses no additional mathematical difficulties. Therefore, to make our arguments more concise, we disregard this term from our analysis [taking $\alpha = 0$ in (1.3) and effectively considering the equations (1.1), while still calling them the CBF equations]. All the results presented in this thesis hold also for the CBF equations with $\alpha > 0$, and we trust that an interested reader can easily reintroduce the linear term αu.

1.2 Summary of known results

There are numerous mathematical results concerning the BF model (1.2). Most of them consider different values and ranges of parameters μ, α, β and the exponent r. The continuous dependence on the Brinkman and Forchheimer coefficients and the convergence as $\mu \to 0$ of the solutions of the BF equations to the solutions of

$$\partial_t u + \alpha u + \beta |u|^{r-1} u + \nabla p = f, \qquad \operatorname{div} u = 0, \tag{1.4}$$

were studied in Payne and Straughan [1999], Çelebi et al. [2006], Liu and Lin [2007], Louaked et al. [2015] and in the monograph by Straughan [2008]. The long-time behaviour of solutions and the existence of global attractors for the BF equations has been studied in Uğurlu [2008], Wang and Lin [2008], Song and Hou [2012], You et al. [2012], Song [2013] and Zhang et al. [2016] (all with $r = 3$), and in Ouyang and Yang [2009] (with $r \in (2, 7/3]$) [the nonlinearity in all these papers reads: $au + b|u|u + c|u|^{r-1}u;\ a, b, c > 0$]. The existence of a global regular, unique solution and of the global attractor for a version of the BF model (1.2) with fast growing nonlinearities (polynomial growth of order $r \geq 1$) was proved in Kalantarov and Zelik [2012]. Most of the results above are given on bounded domains $\Omega \subset \mathbb{R}^3$ with zero Dirichlet boundary conditions.

There is also an abundance of mathematical results concerning the CBF equations (1.3). Continuous dependence in H^1 (standard Sobolev space defined later on) on the Forchheimer coefficient was established by Çelebi et al. [2005] for weak solutions on bounded domains with $2 < r \leq 3$. Kalantarov and Zelik [2012] also extended their results for the BF equations on bounded domains to the convective case with $r > 3$: existence of global strong solutions, uniqueness of weak solutions (also for $r = 3$ and $\mu = 1$, β large enough) and existence of a global attractor in H^2. Their argument relies on the maximal regularity estimate for the corresponding semi-linear stationary Stokes problem proved using some modification of the nonlinear localisation technique developed in Kalantarov and Zelik [2009]; the nonlinear localisation technique is not necessary if $r \leq 5$ when the standard maximal regularity for the linear Stokes equation can be used, or in the periodic domain where there are no boundary terms coming from integration by parts (see the proof for this case in Section 4.3).

In Cai and Jiu [2008] it was shown that the CBF equations on the whole space $\mathbb{R}^3$ possess global weak solutions for $r \geq 1$, global strong solutions for any $r \geq 7/2$ and that the strong solutions are unique for $7/2 \leq r \leq 5$. Some improvements of these results were given in Zhang et al. [2011] - existence of global-in-time strong solutions for $r > 3$ and their uniqueness for $3 < r \leq 5$. Global existence of strong solutions in $\mathbb{R}^3$ for $r \geq 3$ and $\mu, \beta = 1$ was obtained in Zhou [2012] (note that this is much weaker than our results in Sections 4.3 and 4.4, where we show the same in the periodic case if $r > 3$, $\mu, \beta > 0$ and $r = 3$, $4\mu\beta \geq 1$, respectively). Two local regularity criteria for $1 \leq r < 3$ in terms of Bochner spaces (defined later on) were given there as well: if

$$u \in L^t(0, T; L^s(\mathbb{R}^3)) \qquad \text{with} \qquad \frac{3}{s} + \frac{2}{t} \leq 1,\ 3 < s < \infty,$$

or

$$\nabla u \in L^t(0, T; L^s(\mathbb{R}^3)) \qquad \text{with} \qquad \frac{3}{s} + \frac{2}{t} \leq 2,\ 3/2 < s < \infty,$$

then the local strong solution u remains smooth (in space) on the time interval $(0, T)$. The first of these conditions is the same as the famous *'Serrin condition'* given by Serrin [1962, 1963] for the 3D incompressible Navier–Stokes equations. An improvement of the second of the above regularity criteria for the CBF equations was obtained by Wang and Zhou [2015] assuming that two of the velocity components belong to the weak Lebesgue spaces [$\nabla u_i \in L^t(0, T; L^{s,\infty}(\mathbb{R}^3))$ for $s > 3/2$ and

$t = 2s/(2s-3)$].

Long-time behaviour of solutions for the CBF equations has also been studied. Existence of global and uniform attractors in H^1 and H^2 was established in Song and Hou [2011, 2015] respectively, for $7/2 \le r \le 5$ and for bounded domains $\Omega \subset \mathbb{R}^3$ with smooth boundary. The existence of the trajectory attractor for $r \in (1,3]$ on bounded domains was studied in Zhao et al. [2014], as well as its convergence as $\beta \to 0$ to the trajectory attractor of the Navier–Stokes system. Power-law decay in time of the L^2-norm of weak solutions on the whole space $\mathbb{R}^3$ was shown in Cai and Lei [2010] via a classical Fourier splitting method (as in Schonbek [1985, 1986] for the NSE), for $r > 7/3$. The authors gave also a lower bound for the decay rate for $r \ge 3$. Jia et al. [2011] showed different L^2-decay rates for $r \ge 10/3$ via a self-contained analysis technique based on the auxiliary decay estimates and a rigorous analysis of the heat semigroup $\mathrm{e}^{\Delta t}$. This was further extended by Jiang and Zhu [2012] to $r \ge 3$ using a method established in Zhou [2007]. The upper bound was optimised and an algebraic lower bound for the L^2-decay rate was obtained by Jiang [2012] (an error in the estimates for the lower bound in Cai and Lei [2010] was corrected there) [it is worth mentioning that due to the damping term, the optimal L^2-decay rate is slower for the CBF equations than for the NSE]. These ideas were developed further by Liu and Gao [2017] who proved the L^2-decay of weak solutions for $r > 2$. They also showed the asymptotic stability of strong solutions to the system for $r > 3$ with any $\beta > 0$ and $\beta \ge 1/2$ when $r = 3$.

Long-time properties of solutions were also examined (on bounded domains) by Antontsev and de Oliveira [2010] for the equations (1.1) and by de Oliveira [2010] for a modified version of the Navier–Stokes equations with generalised diffusion

$$-\operatorname{div}\left(|\nabla u|^{q-2}\,\nabla u - u\otimes u\right) \qquad \text{for} \qquad q > 1$$

[$q = 2$ corresponds to the Navier–Stokes case]. In the former paper it was shown that the absorption term, in the absence of body forces ($f \equiv 0$), causes weak solutions of (1.1) to become extinct in a finite time if $0 < r < 1$ and decay exponentially in time if $r = 1$. Provided that the force field vanishes at some time instant and $0 < r < 1$, then the weak solutions also vanish at the same time instant. Additionally, for non-zero body forces decaying at a power-law rate, the solutions decay at analogous power-law rates if $r > 1$. For a general non-zero body force, the solutions exhibit exponential decay in time if $r > 0$. We note that for the NSE the best results that are known in this direction are only in terms of decay in space and time of power-law type (see Antontsev and de Oliveira [2010] for references). In de Oliveira [2010] the extinction in finite time was proved for the CBF model with generalised diffusion. Existence of global-in-time weak solutions in spatial dimensions $n \geq 2$ was given by de Oliveira [2013]. The very technical proof given there is based on the theory of monotone operators, the Lipschitz truncation method (see e.g. Diening et al. [2010]) and the pressure decomposition method discussed by Wolf [2007].

A model similar to the CBF modification of the Navier–Stokes equations, called the tamed NSE, was discussed by Röckner and Zhang [2009]. Instead of the absorption term the authors considered a term $g_n(|u|^2)u$, where g_n was a smooth function that satisfied

$$g_n(r) \equiv 0 \quad \text{if} \quad r \in [0, n] \quad \text{and} \quad g_n(r) = \frac{r - n - 1/2}{\mu} \quad \text{if} \quad r \geq n + 1.$$

From the definition of g_n, it is clear that any bounded strong solution of the NSE satisfies these equations for large enough n. The authors established existence (on the whole space) of a unique, smooth, classical solution for all time starting from smooth initial data. They also showed that the solutions u_n of the tamed NSE converge weakly (as $n \to \infty$) to a 'suitable weak solution' of the NSE, where the notion of suitable weak solutions is that used in the partial regularity results for the NSE (see Scheffer [1977] and Caffarelli et al. [1982]). Thus, the tamed NSE can be viewed as an approximation scheme for the NSE. On the other hand, an approximation of the CBF equations was considered by Zhao and You [2012]; the authors studied convergence of solutions of a family of perturbed compressible CBF problems to the solution of the incompressible CBF equations on a bounded domain.

In this thesis we contribute some new results (listed in Section 1.3) to the list (which is quite long already but probably not exhaustive) given above. We consider the 3D incompressible convective Brinkman–Forchheimer equations either on a smooth bounded domain $\Omega \subset \mathbb{R}^3$ or on the torus $\mathbb{T}^3$.

Throughout the thesis we will call the exponent $r = 3$ 'critical' (not to be confused with the usual notion of critical spaces in which the norm of a solution is invariant under scaling). There are two reasons why we want to call it this way. One being the fact that it lies exactly at the border of exponents for which global regularity of strong solutions is known. The second reason, perhaps more interesting than the first one, is that the critical homogenous CBF equations

$$\partial_t u - \mu \Delta u + (u \cdot \nabla) u + \beta \, |u|^2 \, u + \nabla p = 0, \qquad \operatorname{div} u = 0,$$

are invariant under the same parabolic rescaling as the Navier–Stokes equations. This follows from the following simple proposition.

Proposition 1.1. *Let Ω be the whole space $\mathbb{R}^n$ (or the torus $\mathbb{T}^3$). Let u_λ be the usual parabolic rescaling of the velocity field u:*

$$u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t) \quad \textit{for} \quad \lambda > 0,$$

and let p_λ be the usual rescaling of the pressure function p:

$$p_\lambda(x,t) := \lambda^2 p(\lambda x, \lambda^2 t) \quad \textit{for} \quad \lambda > 0.$$

If u and p solve the homogenous CBF equations

$$\partial_t u - \mu \Delta u + (u \cdot \nabla) u + \beta |u|^{r-1} u + \nabla p = 0, \qquad \operatorname{div} u = 0,$$

then the rescaled functions u_λ, p_λ satisfy

$$\partial_t u_\lambda - \mu \Delta u_\lambda + (u_\lambda \cdot \nabla) u_\lambda + \nabla p_\lambda + \lambda^{3-r} \beta |u_\lambda|^{r-1} u_\lambda = 0, \qquad \operatorname{div} u_\lambda = 0.$$

When $r = 3$ in the above, we obtain the homogenous CBF equations for the rescaled functions u_λ, p_λ.

1.3 Outline of the thesis

In Chapter 2 we introduce some function spaces and notation used throughout the thesis. Chapter 3 discusses existence and properties of weak solutions of the incompressible convective Brinkman–Forchheimer equations with $r \geq 1$. In particular we establish that the *Energy Equality* on the torus $\mathbb{T}^3$ is satisfied by all weak solutions in the critical case $r = 3$. As a consequence we obtain L^2-continuity of all weak solutions, and we show existence of a strong global attractor using the theory of *evolutionary systems* developed by Cheskidov [2009] for dynamical systems without uniqueness.

Existence and properties of strong solutions of the CBF equations are considered in Chapter 4. In particular, we establish global-in-time existence on the torus $\mathbb{T}^3$, for $r > 3$ and in the critical case $r = 3$, provided that the product of the coefficients is not too small, $4\mu\beta \geq 1$. We also prove uniqueness of strong solutions in the larger class of weak solutions (so-called '*weak-strong uniqueness*'). In Chapter 5 we prove a 'robustness of regularity' result, which essentially provides stability of strong solutions in terms of initial data and the forcing function.

In Chapter 6 we introduce real interpolation spaces via the K-method and briefly discuss some of their properties which we need later on. We use these spaces in

Chapter 7 to characterise the domains of fractional powers of the Dirichlet Laplacian and the Stokes operators on bounded domains.

Afterwards, we develop two general simultaneous approximation methods in Lebesgue and Sobolev spaces (Chapter 8). We apply these methods to both the Laplace and Stokes operators. The first scheme (a simpler one) uses the semigroup generated by the operator. However, this method is not sufficient for our application in the next chapter. But, we can apply the second approximation scheme, based on finite-dimensional eigenspaces of the Stokes operator, to prove that the Energy Equality holds also for all weak solutions of the critical CBF equations ($r = 3$) on bounded domains with smooth boundary. This is done in Chapter 9.

In conclusion, we discuss some open problems and possible future work in Chapter 10.

Chapter 2

Preliminaries

We consider the three-dimensional, incompressible convective Brinkman–Forchheimer equations (1.1) with the initial condition

$$u(x,0) = u(0) = u_0(x) \quad \text{on} \quad \Omega,$$

where the initial velocity u_0 is divergence-free and has finite kinetic energy (it belongs to the space $H \subset L^2(\Omega)$, which we will define below). The domain of interest is either the three-dimensional torus $\Omega = \mathbb{T}^3 = [0, 2\pi]^3$ with periodic boundary conditions

$$u(x + 2\pi e_i, t) = u(x,t) \quad \forall\ x \in \mathbb{R}^3, \quad \forall\ t > 0, \quad i = 1, 2, 3$$

(where e_i stand for standard unit vectors forming a basis of the Euclidean space $\mathbb{R}^3$), or an open, bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary and zero Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega, \quad \forall\ t \geq 0.$$

Remark 2.1. *In the analysis of the NSE in the periodic case, it is often convenient to assume a zero mean-value constraint for the functions (i.e. $\int u(x,t)\,\mathrm{d}x = 0$). However, we cannot do this for the CBF equations* (1.1) *because the absorption term $|u|^{r-1} u$ does not preserve this property. Therefore, we cannot use the usual Poincaré inequality $\|u\|_{L^2} \leq c\,\|\nabla u\|_{L^2}$, and we have to control the full H^1-norm instead.*

In what follows, we will often assume for simplicity that the coefficients μ, β are equal to 1 but both of these coefficients can be taken as arbitrary positive constants (note that in the critical case $r = 3$, $\mu = \beta = 1$ implies regularity, see Section 4.4). They affect only the value of the generic constant $c > 0$, which appears in our estimates and whose value can differ from line to line.

2.1 Function spaces

In this section we introduce the basic function spaces and some additional notation used in the thesis.

We recall the standard Lebesgue spaces of vector-valued functions on some arbitrary domain $\Omega \subseteq \mathbb{R}^3$

$$L^p(\Omega) := \left\{ u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3 \ : \ \|u\|_{L^p} < \infty \right\},$$

with the norm

$$\|u\|_{L^p}^p := \int_\Omega |u|^p \, \mathrm{d}x < \infty \quad \text{for} \quad 1 \le p < \infty$$

and

$$\|u\|_{L^\infty} := \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \, .$$

We denote the norm for the Hilbert space L^2 by $\|\cdot\|$ and the scalar product in this space by $\langle u, v\rangle = \int_\Omega uv \, \mathrm{d}x$. We will also use $\langle \cdot, \cdot \rangle$ for other dual pairings. We note that on bounded domains and on the torus $\mathbb{T}^3$ we have nesting of the L^p spaces: $L^p \hookrightarrow L^q$ for $p > q$, where '$\hookrightarrow$' denotes a continuous embedding.

We also recall L^2-based Sobolev spaces $H^k(\Omega) = W^{k,2}(\Omega) \hookrightarrow L^2(\Omega)$ for $k \in \mathbb{N}$, consisting of functions whose distributional derivatives up to order k belong to L^2. We define these spaces on an open, bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. For every $k \in \mathbb{N}$, the space $H^k(\Omega)$, with the scalar product

$$\langle u, v\rangle_{H^k(\Omega)} := \sum_{|\alpha|\le k} \left(\int_\Omega D^\alpha u D^\alpha v \, \mathrm{d}x \right),$$

is a Hilbert space, and the norm in this space is given by

$$\|u\|_{H^k(\Omega)}^2 := \langle u, u\rangle_{H^k(\Omega)} = \sum_{|\alpha|\le kT} \int_\Omega |D^\alpha u|^2 \, \mathrm{d}x.$$

In the periodic case, we can define the Sobolev spaces $H^s(\mathbb{T}^3)$ for $s \ge 0$ by the Fourier expansion

$$H^s(\mathbb{T}^3) := \left\{ u \in L^2(\mathbb{T}^3) \ : \ u(x) = \sum_{k\in\mathbb{Z}^3} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}, \quad \hat{u}_k = \overline{\hat{u}_{-k}}, \quad \|u\|_{H^s(\mathbb{T}^3)} < \infty \right\},$$

where

$$\|u\|^2_{H^s(\mathbb{T}^3)} := \left|\mathbb{T}^3\right| \sum_{k\in\mathbb{Z}^3} (1+|k|^{2s})\,|\hat{u}_k|^2$$

[and $\hat{u}_k := \left|\mathbb{T}^3\right|^{-1} \int_{\mathbb{T}^3} u(x)\mathrm{e}^{-\mathrm{i}k\cdot x}\,\mathrm{d}x \quad \text{for} \quad u \in L^1(\mathbb{T}^3)$].

In this thesis we use the standard notation for the vector-valued function spaces which often appear in the theory of fluid dynamics. For an arbitrary domain $\Omega \subseteq \mathbb{R}^n$ we define:

$$\begin{aligned}
C_0^\infty(\Omega) &:= \{\varphi \in C^\infty(\Omega) \ : \ \operatorname{supp}\varphi \subset\subset \Omega\}, \quad [\text{` } \subset\subset \text{ ' denotes a compact subset}]\\
\mathcal{D}_\sigma(\Omega) &:= \{\varphi \in C_0^\infty(\Omega) \ : \ \operatorname{div}\varphi = 0\},\\
L^q_\sigma(\Omega) &:= \text{closure of } \mathcal{D}_\sigma(\Omega) \text{ in the Lebesgue space } L^q(\Omega),\\
V^s(\Omega) &:= \text{closure of } \mathcal{D}_\sigma(\Omega) \text{ in the Sobolev space } H^s(\Omega) \quad \text{for} \quad s > 0.
\end{aligned}$$

The space of divergence-free test functions in the space-time domain is denoted by

$$\mathcal{D}_\sigma(\Omega_T) := \{\varphi \in C_0^\infty(\Omega_T) \ : \ \operatorname{div}\varphi(\cdot,t) = 0\},$$

where $\Omega_T := \Omega \times [0,T)$ for $T > 0$.

In the periodic case ($\Omega = \mathbb{T}^3$) we define the divergence-free L^2-based spaces by the Fourier expansion. If u is given by $u(x) = \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}$ then a simple computation shows that $\operatorname{div} u(x) = \mathrm{i}(k\cdot\hat{u}_k)\mathrm{e}^{\mathrm{i}k\cdot x}$. This leads to the following definitions:

$$\begin{aligned}
L^2_\sigma(\mathbb{T}^3) &:= \left\{u \in L^2(\mathbb{T}^3) \ : \ k\cdot\hat{u}_k = 0 \quad \text{for all} \quad k \in \mathbb{Z}^3\right\},\\
V^s(\mathbb{T}^3) &:= \left\{u \in H^s(\mathbb{T}^3) \ : \ k\cdot\hat{u}_k = 0 \quad \text{for all} \quad k \in \mathbb{Z}^3\right\} \quad \text{for} \quad s > 0.
\end{aligned}$$

If $\Omega \subseteq \mathbb{R}^3$ is an open, bounded domain with smooth boundary or the torus $\Omega = \mathbb{T}^3$, we denote the Hilbert space $L^2_\sigma(\Omega)$ by $H = H(\Omega)$, and $V^1(\Omega)$ by $V = V(\Omega)$. [Note that $V(\Omega) = H^1_0(\Omega) \cap H(\Omega)$, where the space $H^1_0(\Omega)$ is the subset of $H^1(\Omega)$ that consists of functions vanishing on the boundary (in the sense of trace), and $V(\mathbb{T}^3) = H^1(\mathbb{T}^3) \cap H(\mathbb{T}^3)$.] We use the L^2 inner product on H and H^1 inner product on V. We denote the dual space to a given space X by X', i.e. the dual space to V is denoted by V'.

We also recall the well-known Helmholtz–Weyl decomposition of $L^2(\mathbb{T}^3)$ and $L^2(\Omega)$. Every function $u = (u_1, u_2, u_3)$ from $L^2(\mathbb{T}^3)$[1] can be decomposed into

[1]Note that we use throughout the thesis the X notation instead of X^3 for the spaces of vector-valued functions; it should not cause any confusion since we only consider three-dimensional vector fields.

$$u = v + \nabla\phi,$$

where $v \in H$ (divergence-free) and the scalar function $\phi \in H^1$. We can express this decomposition as

$$L^2(\mathbb{T}^3) = H \oplus G,$$

where G is the orthogonal complement of H ($G \perp H$), which consists of gradients of scalar functions from H^1. A similar decomposition holds also for the general L^p spaces when $p \in (1, \infty)$ (see Fujiwara and Morimoto [1977]).

We will use function spaces with values in a Banach space $(X, \|\cdot\|_X)$. In particular, we will use the space $C([0,T]\,; X)$ consisting of continuous functions $u : [0,T] \to X$ with the norm

$$\|u\|_{C([0,T];X)} := \max_{0\le t\le T} \|u(t)\|_X < \infty.$$

We also recall the Bochner spaces $L^p(0,T;X)$. These consist of *strongly measurable*[2] functions $u : [0,T] \to X$ satisfying

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p \,\mathrm{d}t\right)^{1/p} < \infty, \quad \text{for} \quad 1 \le p < \infty,$$

and for $p = \infty$

$$\|u\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{0\le t\le T} \|u(t)\|_X < \infty.$$

For more details on Bochner spaces see Evans [2010], for example.

2.2 Properties of the absorption term $|u|^{r-1} u$

For notational convenience we will denote the terms connected with the additional nonlinearity in the convective Brinkman–Forchheimer equations by C_r. In order to make the following pairing finite

$$\left|\left\langle |u|^{r-1} u,\, u\right\rangle\right| < \infty$$

we define for $r > 0$ and for all functions $u, v \in L^{r+1}_\sigma$

$$C_r(u,v) := \mathbb{P}\left(|u|^{r-1} v\right),$$

[2]A function $u : [0,T] \to X$ is strongly measurable if it is the limit of a sequence of simple functions that converge in the norm of X for a.e. $t \in [0,T]$.

where $\mathbb{P} : L^p \to L^p_\sigma$ is the 'Leray projection' in L^p (see e.g. Fujiwara and Morimoto [1977] for details); additionally we define

$$C_r(u) := C_r(u,u).$$

We have the following crucial properties of the nonlinearity C_r.

Lemma 2.2. *For every $r \geq 1$ and for all functions $u, v \in L^{r+1}_\sigma$*

$$\langle C_r(u) - C_r(v),\, u - v\rangle = \left\langle |u|^{r-1} u - |v|^{r-1} v,\, u - v \right\rangle \geq c \, \|u - v\|^{r+1}_{r+1}, \tag{2.1}$$

where c is a positive constant depending only on r, and $\langle \cdot, \cdot\rangle$ is the inner product in L^2..

It immediately follows from (2.1) that for $r \geq 1$ the nonlinearity C_r is monotone in the sense that

$$\langle C_r(u) - C_r(v),\, u - v\rangle \geq 0 \tag{2.2}$$

for all $u, v \in L^{r+1}_\sigma$. One can show (2.2) independently even for $r > 0$ by direct computation and using only Young's inequality.

Lemma 2.2 is a consequence of properties of vectors $|u|^{r-1} u$ in $\mathbb{R}^n$ $(n \geq 1)$. The proof of the lower bound (2.1) is taken from DiBenedetto [1993] with some minor changes.

Proof. For all $u, v \in \mathbb{R}^n$ we observe that

$$\left(|u|^{r-1} u - |v|^{r-1} v\right) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \left(|su + (1-s)v|^{r-1} (su + (1-s)v)\right) \mathrm{d}s$$

and hence

$$\begin{aligned}\left(|u|^{r-1} u - |v|^{r-1} v\right) \cdot w = &\int_0^1 |su + (1-s)v|^{r-1} |w|^2 \, \mathrm{d}s \\ &+ (r-1) \int_0^1 |su + (1-s)v|^{r-3} \left([su + (1-s)v] \cdot w\right)^2 \mathrm{d}s,\end{aligned}$$

where $w := u - v$. Therefore for $r \geq 1$, we obtain

$$\left(|u|^{r-1} u - |v|^{r-1} v\right) \cdot w \geq |w|^2 \int_0^1 |su + (1-s)v|^{r-1} \mathrm{d}s.$$

If $|u| \geq |v - u|$ we have

$$|su + (1-s)v| \geq ||u| - (1-s)\,|w|| \geq s\,|w|\,,$$

and we can conclude that $\left(|u|^{r-1} u - |v|^{r-1} v\right) \cdot w \geq \frac{1}{r} |w|^{r+1}$.

On the other hand, if $|u| < |v-u|$, we have

$$
\begin{aligned}
|w|^2 \int_0^1 |su + (1-s)v|^{r-1} \, \mathrm{d}s &\geq |w|^2 \int_0^1 \frac{(|su+(1-s)v|^2)^{(r+1)/2}}{(2-s)^2 |w|^2} \, \mathrm{d}s \\
&\geq \frac{1}{4} \left(\int_0^1 |su + (1-s)v|^2 \, \mathrm{d}s \right)^{(r+1)/2} \\
&= \frac{1}{4 \cdot 3^{(r+1)/2}} \left(|u|^2 + u \cdot v + |v|^2 \right)^{(r+1)/2} \\
&\geq c \, |w|^{r+1}.
\end{aligned}
$$

Finally, we observe an equality for $u, v \in L^{r+1}_\sigma$

$$
\begin{aligned}
\langle C_r(u) - C_r(v), \, w \rangle &= \left\langle \mathbb{P} \left(|u|^{r-1} u - |v|^{r-1} v \right), \, w \right\rangle \\
&= \left\langle |u|^{r-1} u - |v|^{r-1} v, \, w \right\rangle \\
&= \int \left(|u|^{r-1} u - |v|^{r-1} v \right) \cdot w \, \mathrm{d}x,
\end{aligned}
$$

which ends the proof of the lemma due to monotonicity of the integral and the above vector estimates. □

In what follows we will also need to bound the difference

$$
|u|^{r-1} u - |v|^{r-1} v \tag{2.3}
$$

in terms of only u and w, where $w := u - v$.

Lemma 2.3. *Let $u, v \in \mathbb{R}^n$. Then for $r \geq 1$*

$$
\left| |u|^{r-1} u - |v|^{r-1} v \right| \leq (2^{r-2} r) \left(|u|^{r-1} |w| + |w|^r \right).
$$

Proof. First, we consider the following function of one real variable $\varphi : [0,1] \to \mathbb{R}^n$

$$
\varphi(\lambda) := |u - \lambda w|^{r-1} (u - \lambda w),
$$

for $\lambda \in [0,1]$. It is easy to see that

$$
\varphi(1) - \varphi(0) = - \left(|u|^{r-1} u - |v|^{r-1} v \right).
$$

We can easily compute the derivative of φ

$$\varphi'(\lambda) = -r\,|u - \lambda w|^{r-1}\, w.$$

By the Mean Value Theorem we can estimate the difference (2.3)

$$\begin{aligned}
\left| |u|^{r-1} u - |v|^{r-1} v \right| &= |\varphi(1) - \varphi(0)| \le \max_{\lambda \in [0,1]} |\varphi'(\lambda)| \\
&= \max_{\lambda \in [0,1]} \left| -r\,|u - \lambda w|^{r-1} w \right| \le r\,|w| \max_{\lambda \in [0,1]} |u - \lambda w|^{r-1} \\
&\le r\,|w|\,(|u| + |w|)^{r-1} \le r\,|w| \left[2^{r-2} \left(|u|^{r-1} + |w|^{r-1} \right) \right] \\
&\le (2^{r-2} r) \left(|u|^{r-1} |w| + |w|^r \right).
\end{aligned}$$

We used here the simple fact that

$$(a+b)^r \le 2^{r-1}\,(a^r + b^r) \quad \text{for} \quad r \ge 0,\ a, b \ge 0.$$

It comes from observing that the function

$$f(x) := \frac{(1+x)^r}{1+x^r}, \quad x \ge 0,$$

attains its maximum at $x = 1$; so $f(x) \le f(1) = 2^{r-1}$. □

We will also make use of the following lemma, whose proof consists of integration by parts and differentiation of the absolute value function (see Robinson and Sadowski [2014] for the proof in the periodic case or Beirão da Veiga [1987] on the whole space).

Lemma 2.4. *For every $r \ge 1$, if $u \in H^2(\Omega)$, where Ω is either the whole space $\mathbb{R}^3$ or the three-dimensional torus $\mathbb{T}^3$, then*

$$\int_\Omega -\Delta u \cdot |u|^{r-1} u \,\mathrm{d}x \ge \int_\Omega |\nabla u|^2 |u|^{r-1}\,\mathrm{d}x.$$

Explicitly, the left-hand side of the above equals (integrating by parts)

$$\int_\Omega -\Delta u \cdot |u|^{r-1} u \,\mathrm{d}x = \int_\Omega |\nabla u|^2 |u|^{r-1}\,\mathrm{d}x + \frac{(r-1)}{4} \int_\Omega |u|^{r-3} \left| \nabla |u|^2 \right|^2 \mathrm{d}x.$$

In particular, by Lemma 2.4, we can write for the absorption term $|u|^{r-1} u$ with $r \ge 1$

$$\int_\Omega |\nabla u|^2 |u|^{r-1}\,\mathrm{d}x \le \left\langle -\Delta u,\ |u|^{r-1} u \right\rangle \le r \int_\Omega |\nabla u|^2 |u|^{r-1}\,\mathrm{d}x. \tag{2.4}$$

The upper bound in (2.4) follows from the fact that

$$\left|\nabla |u|^2\right|^2 \le 4\,|u|^2\,|\nabla u|^2\,.$$

We recall that the operators $\mathbb{P}$ and Δ commute on the domains $\mathbb{T}^3$ and $\mathbb{R}^3$ but not necessarily on an open, bounded domain $\Omega \subset \mathbb{R}^3$ (see e.g. Robinson et al. [2016] for examples). Therefore, in the periodic case, we can replace $\left\langle -\Delta u,\, |u|^{r-1}\, u\right\rangle$ in (2.4) with $\langle Au,\, C_r(u)\rangle$, provided that u is a divergence-free function, where $A := -\mathbb{P}\Delta$ is the familiar Stokes operator (see Constantin and Foias [1988], for example). Indeed, for $u \in D(A) = V \cap H^2$ we have

$$\begin{aligned}\langle Au,\, C_r(u)\rangle &= \left\langle -\mathbb{P}\Delta u,\, \mathbb{P}\left(|u|^{r-1}\, u\right)\right\rangle = \left\langle -\mathbb{P}\Delta u,\, |u|^{r-1}\, u\right\rangle\\ &= \left\langle -\Delta \mathbb{P} u,\, |u|^{r-1}\, u\right\rangle = \left\langle -\Delta u,\, |u|^{r-1}\, u\right\rangle.\end{aligned}$$

We will also need another lemma from the same paper (Robinson and Sadowski [2014]).

Lemma 2.5. *Take $2 \le p < 3$. Then there exists a constant $c_p > 0$ such that, for every $u \in W^{1,p}(\mathbb{R}^3)$ we have $u \in L^{3(r+1)}(\mathbb{R}^3)$ and*

$$\|u\|^{r+1}_{L^{3(r+1)}(\mathbb{R}^3)} \le c_p \int_{\mathbb{R}^3} |\nabla u|^2\,|u|^{r-1}\,\mathrm{d}x, \tag{2.5}$$

where $r + 1 = p/(3 - p)$. The same is true if Ω is a bounded (perhaps periodic) domain and $u \in W^{1,p}(\Omega)$ with $\int_\Omega u \,\mathrm{d}x = 0$ or $u|_{\partial\Omega} = 0$.

Note that the embedding $W^{1,p} \hookrightarrow L^{3(r+1)}$ is standard. However, the norm on the right-hand side of (2.5) is not the $W^{1,p}$ norm. Nevertheless, it is finite for $u \in W^{1,p}$. We point out that in order to prove the bound (2.5), it is actually not necessary that $u \in W^{1,p}(\Omega)$. It follows from the proof of Lemma 2.5 that whenever the function u belongs to a space in which the space $C_0^\infty(\Omega)$ is dense, and whenever

$$I_r(u) := \int_\Omega |\nabla u|^2\,|u|^{r-1}\,\mathrm{d}x < \infty \quad \text{for} \quad r \ge 1,$$

then we can repeat the argument in the proof of Lemma 2.5 and show that $u \in L^{3(r+1)}(\Omega)$ [for Ω as in Lemma 2.5].

In our application of Lemma 2.5 (see Theorem 4.4) the function u is a strong solution of the CBF equations. In particular, it belongs to the space H^1 in which C_0^∞ is dense. However, on a bounded domain without a zero mean-value assumption (which is the case for solutions of the CBF equations on the torus, see Remark 2.1), the bound (2.5) does not hold, and we have to replace it with

$$\|u\|_{L^{3(r+1)}(\Omega)}^{r+1} \leq c \int_\Omega |\nabla u|^2 \, |u|^{r-1} \, \mathrm{d}x + c \, \|u\|_{L^{r+1}(\Omega)}^{r+1} \, .$$

The L^{r+1} norm on the right-hand side is finite for every strong solution of the CBF equations. Therefore, we can use the bound (2.5) to obtain additional regularity $u \in L^{3(r+1)}$ for a strong solution u.

[Boundedness of the quantity $I_r(u)$ defined above implies as well that the function $u \in W^{1,1}(\Omega)$ belongs also to the certain type of Besov space, namely to the Nikol'skiĭ space[3] $\mathcal{N}^{2/(r+1),\,r+1}$. In particular, we have

$$|u|_{\mathcal{N}^{2/(r+1),\,r+1}(\Omega)}^{r+1} \leq c \, I_r(u), \tag{2.6}$$

where $c > 0$ is a constant depending only on r and Ω (see Lemma 2.1 in Málek et al. [2006] for the details) and the left-hand side is the seminorm of u in $\mathcal{N}^{2/(r+1),\,r+1}$. For $p \in [1, \infty)$ and $s = m + \sigma$, where $m \geq 0$ is an integer and $\sigma \in (0, 1)$, the Nikol'skiĭ spaces $\mathcal{N}^{s,\,p}$ are the subspaces of the L^p functions for which the following norm (this is the norm in the Besov space $B_\infty^{s,\,p}$)

$$\begin{aligned} \|u\|_{\mathcal{N}^{s,\,p}(\Omega)}^p &:= \|u\|_{W^{m,\,p}(\Omega)}^p + |u|_{\mathcal{N}^{s,\,p}(\Omega)}^p \\ &:= \|u\|_{W^{m,\,p}(\Omega)}^p + \sum_{|\alpha|=m} \sup_{0<|h|<\delta} \int_\Omega \frac{|\partial^\alpha u(x+h) - \partial^\alpha u(x)|^p}{|h|^{\sigma p}} \, \mathrm{d}x \end{aligned}$$

is finite. Here $\delta > 0$ is an arbitrary fixed number. For any $\varepsilon \in (0, 1)$ we have the embeddings (see Nikol′skiĭ [1975])

$$\mathcal{N}^{s,\,p} \hookrightarrow W^{s-\varepsilon,\,p} \hookrightarrow \mathcal{N}^{s-\varepsilon,\,p},$$

where the fractional Sobolev spaces $W^{s,\,p}$ are defined as the Besov spaces $B_p^{s,\,p}$ (see e.g. Simon [1990]).

Note that due to lack of a zero mean-value assumption for the velocity field u (see Remark 2.1) we cannot use in our applications the highest order derivative seminorm as an equivalent norm in each of the above spaces. However, in our case $u \in L^{r+1}$ because of the regularity (3.5) of weak solutions of the CBF equations, which, together with (2.6), implies that $u \in \mathcal{N}^{2/(r+1),\,r+1}$. For a general function u (not necessarily a weak solution) on bounded domains, the required regularity

[3] Nikol'skiĭ spaces are a particular case of the Besov spaces when one of the exponents is fixed: $\mathcal{N}^{s,\,p} = B_\infty^{s,\,p}$. See Simon [1990] for more information about these spaces.

follows from Lemma 2.5, since then we have $L^{3(r+1)} \hookrightarrow L^{r+1}$.]

Applying the tools described above we will show in Theorem 4.4 that strong solutions of the convective Brinkman–Forchheimer equations with $r > 3$ possess additional regularity compared to the corresponding solutions of the Navier–Stokes equations.

In the proof of main result of Chapter 5 (Theorem 5.2) it will be crucial to control the L^6-norm of the gradient of a function u by the L^2-norm of Au.

Lemma 2.6. *Let $u \in D(A)$ on the torus $\mathbb{T}^3$. Then there exists a constant $c > 0$ independent of u such that*

$$\|\nabla u\|_{L^6(\mathbb{T}^3)} \leq c\,\|Au\|\,.$$

Proof. First, we apply the Sobolev embedding $H^1 \hookrightarrow L^6$

$$\|\nabla u\|_{L^6} \leq c\,\|\nabla u\|_{H^1} = c\left(\|\nabla u\|^2 + \left\|D^2 u\right\|^2\right)^{1/2}.$$

We can, either by direct computation or by the Poincaré inequality (noting that ∇u has zero mean-value for a periodic function u), verify that

$$\|\nabla u\| \leq c\left\|D^2 u\right\|.$$

Therefore, we have the desired bound

$$\|\nabla u\|_{L^6}^2 \leq c\left\|D^2 u\right\|^2 = c\sum_{m,n=1}^{3}\sum_{k\in\mathbb{Z}^3} k_m^2 k_n^2\,|\hat{u}_k|^2 = c\sum_{k\in\mathbb{Z}^3}|k|^4\,|\hat{u}_k|^2 = c\,\|Au\|^2\,. \quad \square$$

Chapter 3

Weak solutions

In this chapter we first establish the global existence of weak solutions (Section 3.3) of the CBF equations with $r \geq 1$ (we assume for simplicity $f \equiv 0$) on a smooth bounded domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} \partial_t u - \mu \Delta u + (u \cdot \nabla) u + \nabla p + \beta |u|^{r-1} u = 0, \\ \operatorname{div} u = 0, \end{cases} \quad \text{in} \quad \Omega \tag{3.1}$$

with zero Dirichlet boundary condition (which is often called in the literature the '*no-slip*' boundary condition)

$$u = 0 \quad \text{on} \quad \partial\Omega,$$

and initial condition

$$u(x, 0) = u_0 \in H.$$

This result also holds on the torus $\mathbb{T}^3$ (with the same proof).

The main result of this chapter states that all weak solutions of the critical CBF equations ($r = 3$) on the torus (we prove an analogous result on bounded domains in Chapter 9)

$$\begin{cases} \partial_t u - \mu \Delta u + (u \cdot \nabla) u + \nabla p + \beta |u|^2 u = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{3.2}$$

verify the energy equality (see Section 3.4)

$$\frac{1}{2} \|u(T)\|^2 + \mu \int_0^T \|\nabla u(t)\|^2 \, \mathrm{d}t + \beta \int_0^T \|u(t)\|_{L^{r+1}}^{r+1} \, \mathrm{d}t = \frac{1}{2} \|u(0)\|^2 ,$$

for every $T \geq 0$. As a consequence, weak solutions are continuous functions into L^2. Using this fact and the theory of evolutionary systems developed by Cheskidov [2009], we establish existence of a strong global attractor for that case. These two results appeared in Hajduk and Robinson [2017].

3.1 Energy inequality

We will now show an inequality for the solutions of the CBF equations which is analogous to the energy inequality for the Navier–Stokes equations. We treat this as an introduction of and motivation for the definition of weak solutions of this model.

Assuming that u is a smooth, divergence-free function, multiplying both sides of the unforced equation (3.1) by u and then integrating over Ω, we obtain

$$\langle \partial_t u,\, u\rangle + \mu \langle -\Delta u,\, u\rangle + \langle (u\cdot\nabla)u,\, u\rangle + \langle \nabla p,\, u\rangle + \beta \left\langle |u|^{r-1} u,\, u\right\rangle = 0.$$

After integration by parts, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \mu \|\nabla u(s)\|^2 + \beta \|u\|_{L^{r+1}}^{r+1} = 0.$$

The convective term disappears because of the property

$$\langle (u\cdot\nabla)v,\, w\rangle = -\langle (u\cdot\nabla)w,\, v\rangle \qquad \text{for} \qquad u, v, w \in V.$$

The pressure term disappears due to the Helmholtz–Weyl decomposition mentioned earlier. Integrating now over the time interval $[0,t]$ for $t \in [0,T)$, we have

$$\frac{1}{2}\|u(t)\|^2 + \mu \int_0^t \|\nabla u(s)\|^2 \,\mathrm{d}s + \beta \int_0^t \|u(s)\|_{L^{r+1}}^{r+1} \,\mathrm{d}s = \frac{1}{2}\|u(0)\|^2 .$$

Using Galerkin approximations (which we will see in some detail in Section 3.3), we can justify the above computations and write *the first energy inequality* for the CBF equations, in the form

$$\frac{1}{2}\sup_{t\in[0,T]}\|u(t)\|^2 + \mu \int_0^T \|\nabla u(t)\|^2 \,\mathrm{d}t + \beta \int_0^T \|u(t)\|_{L^{r+1}}^{r+1} \,\mathrm{d}t \leq \frac{1}{2}\|u(0)\|^2 . \tag{3.3}$$

Hence, we expect that for $u_0 \in H$, a solution u of the equation (3.1) will have the regularity

$$u \in L^\infty(0,T;\, H) \cap L^2(0,T;\, V) \cap L^{r+1}(0,T; L^{r+1}).$$

From interpolation between the first two spaces, we know that such a function u is actually in the space

$$u \in L^{10/3}\left(0,T; L^{10/3}\right).$$

We can see this using Hölder's inequality with exponents 3/2, 3 and Sobolev's embedding $H^1 \hookrightarrow L^6$

$$\begin{aligned}\int_\Omega |u|^{10/3}\,\mathrm{d}x = \int_\Omega |u|^{4/3}\,|u|^2\,\mathrm{d}x &\le \left(\int_\Omega |u|^2\,\mathrm{d}x\right)^{2/3}\left(\int_\Omega |u|^6\,\mathrm{d}x\right)^{1/3}\\ &\le c\,\|u\|^{4/3}\,\|u\|_{H^1}^2.\end{aligned}$$

Now, integrating over the time interval $[0,T]$, we obtain

$$\begin{aligned}\int_0^T\left(\int_\Omega |u|^{10/3}\,\mathrm{d}x\right)\mathrm{d}t &\le c\int_0^T \|u(t)\|^{4/3}\,\|u(t)\|_{H^1}^2\,\mathrm{d}t\\ &\le c\left(\sup_{0<t<T}\|u(t)\|^{4/3}\right)\left(\int_0^T \|u(t)\|_{H^1}^2\,\mathrm{d}t\right) < \infty.\end{aligned}$$

So, for the absorption exponent r in (3.1) in the range $(0,7/3]$[1], we have no extra information about regularity of u because then

$$L^{10/3} \subset L^{r+1}.$$

On the other hand, for $r > 7/3$, we have more information about the regularity of the function u, since now

$$L^{r+1} \subset L^{10/3}.$$

We also know (see Kalantarov and Zelik [2012] for the proof on bounded domains; we will show in Section 4.3 a simple proof in the periodic case) that for $r > 3$ there exists a global strong solution of the convective Brinkman–Forchheimer equations.

So the range of the parameter r for which we can expect to obtain additional regularity of weak solutions is

$$r \in (7/3, 3]\,.$$

[1]Note, that the 'slightly' singular case $u/|u|^r$, for $r \in (0,1)$, is also included here.

3.2 Weak formulation

In this section, we will present the weak formulation of the unforced problem (3.1).

As for the NSE, we take a test function $\varphi \in \mathcal{D}_\sigma(\Omega_T)$. Multiplying both sides of (3.1) by the function φ, and integrating over the space-time domain Ω_T, we obtain

$$
\begin{aligned}
-\int_0^T \langle u, \partial_t\varphi\rangle \,\mathrm{d}t + \mu \int_0^T \langle \nabla u, \nabla\varphi\rangle \,\mathrm{d}t + \int_0^T \langle (u\cdot\nabla)u, \varphi\rangle \,\mathrm{d}t \\
+ \beta \int_0^T \left\langle |u|^{r-1} u, \varphi \right\rangle \mathrm{d}t = \langle u_0, \varphi(0)\rangle . \qquad (3.4)
\end{aligned}
$$

From (3.4) and from the energy inequality (3.3), we have the following definition of a weak solution for the convective Brinkman–Forchheimer equations without external forces.

Definition 3.1. *We will say that the function u is* a weak solution *on the time interval $[0,T)$ of the convective Brinkman–Forchheimer equations* (3.1) *with the initial condition $u_0 \in H$, if*

$$
u \in L^\infty(0,T;H) \cap L^{r+1}(0,T;L^{r+1}_\sigma) \cap L^2(0,T;V) \qquad (3.5)
$$

and

$$
\begin{aligned}
&-\int_0^t \langle u(s), \partial_t\varphi(s)\rangle \,\mathrm{d}s + \mu \int_0^t \langle \nabla u(s), \nabla\varphi(s)\rangle \,\mathrm{d}s + \int_0^t \langle (u(s)\cdot\nabla)u(s), \varphi(s)\rangle \,\mathrm{d}s \\
&+ \beta \int_0^t \left\langle |u(s)|^{r-1} u(s), \varphi(s) \right\rangle \mathrm{d}s = -\langle u(t), \varphi(t)\rangle + \langle u_0, \varphi(0)\rangle , \qquad (3.6)
\end{aligned}
$$

for almost every $0 < t < T$ and all test functions $\varphi \in \mathcal{D}_\sigma(\Omega_T)$.

A function u is called a *global weak solution* if it is a weak solution for all $T > 0$.

Taking the difference of (3.6) with $t = t_1$ and $t = t_0$, we see that every weak solution u satisfies an equivalent weak formulation

$$
\begin{aligned}
&-\int_{t_0}^{t_1} \langle u(s), \partial_t\varphi(s)\rangle \,\mathrm{d}s + \mu \int_{t_0}^{t_1} \langle \nabla u(s), \nabla\varphi(s)\rangle \,\mathrm{d}s + \int_{t_0}^{t_1} \langle (u(s)\cdot\nabla)u(s), \varphi(s)\rangle \,\mathrm{d}s \\
&+ \beta \int_{t_0}^{t_1} \left\langle |u(s)|^{r-1} u(s), \varphi(s) \right\rangle \mathrm{d}s = -\langle u(t_1), \varphi(t_1)\rangle + \langle u(t_0), \varphi(t_0)\rangle , \qquad (3.7)
\end{aligned}
$$

for all test functions $\varphi \in \mathcal{D}_\sigma(\Omega_T)$, almost all initial times $t_0 \in [0,T)$, including zero, and almost every $t_1 \in (t_0, T)$.

In the setting of the weak formulation (3.7), we have an important class of weak solutions satisfying the energy inequality (3.3).

Definition 3.2. *A* Leray–Hopf weak solution *of the convective Brinkman–Forchheimer equations* (3.1) *with the initial condition* $u_0 \in H$ *is a weak solution satisfying the following* strong energy inequality*:*

$$\|u(t_1)\|^2 + 2\mu \int_{t_0}^{t_1} \|\nabla u(s)\|^2 \, \mathrm{d}s + 2\beta \int_{t_0}^{t_1} \|u(s)\|_{L^{r+1}(\Omega)}^{r+1} \, \mathrm{d}s \leq \|u(t_0)\|^2 , \tag{3.8}$$

for almost every $t_0 \in [0, T)$*, including zero, and every* $t_1 \in (t_0, T)$*.*

It is known that for every $u_0 \in H$ there exists at least one global Leray–Hopf weak solution of (3.1). For the proof of the existence of global Leray–Hopf weak solutions see Antontsev and de Oliveira [2010] (see also Section 3.3 for a sketch of that proof).

Note that Definition 3.1 is silent about the pressure field p. It is well-known that to every weak solution of the NSE we can always associate a corresponding pressure field (see Theorem 2.1 in Galdi [2000]). The same can be shown in a similar way for the CBF equations (but see Chapter 10 for some pressure-related issues).

We note also that the regularity condition (3.5) in Definition 3.1 is not sufficient to explain how a weak solution u satisfies the initial condition $u(0) = u_0$. However, it follows from (3.7) that every weak solution u is L^2-weakly continuous in time (see Lemma 3.4), which allows us to impose the initial condition.

Furthermore, since a weak solution is strictly an equivalence class of functions equal almost everywhere, any such solution can be modified on a set of zero Lebesgue measure without changing it in any essential way. In fact, due to the regularity of the time derivative $\partial_t u$ (see Lemma 3.7 in Robinson et al. [2016] for the NSE case; a similar proof works for the CBF equations), one can modify a weak solution u on a set of measure zero in such a way that (3.6) is satisfied for all $t > 0$. We will assume from now on that every weak solution we consider has been modified on a set of zero Lebesgue measure so that (3.6) and (3.7) are satisfied for all $t > 0$ and all $t_1 \geq t_0 \geq 0$, respectively (cf. also Lemmas 2.1 and 2.2 in Galdi [2000]).

There is a more convenient definition of weak solutions (one of many), in which the test functions depend only on the spatial variables[2].

We consider functions of the form $\psi_h(x, s) := \varphi(x)\theta_h(s)$, where $\varphi \in \mathcal{D}_\sigma(\Omega)$, and $\theta_h(s)$ is a function from the space $C_0^\infty([0, T))$, that equals one for $s \in [0, t]$

[2]A full derivation of this definition can be found for example in Galdi [2000] or Robinson et al. [2016].

and zero for $s \geq t+h$, for a fixed $t \in [0,T)$. Then we see that $\psi_h \in \mathcal{D}_\sigma(\Omega_T)$ and $\psi_h(x,s) = \varphi(x)$ for $s \in [0,t]$. Using $\psi_h(x,s)$ as the test functions in (3.6) we obtain at once the following lemma (see Lemma 2.2 in Galdi [2000] for the details; cf. Lemma 3.6 in Robinson et al. [2016]).

Lemma 3.3. *If u is a weak solution of the convective Brinkman–Forchheimer equations* (3.1) *then*

$$\mu \int_{t_0}^{t_1} \langle \nabla u(s),\, \nabla \varphi \rangle \,\mathrm{d}s + \int_{t_0}^{t_1} \langle (u(s)\cdot\nabla)\, u(s),\, \varphi \rangle \,\mathrm{d}s + \beta \int_{t_0}^{t_1} \left\langle |u(s)|^{r-1} u(s),\, \varphi \right\rangle \mathrm{d}s = -\langle u(t_1),\, \varphi \rangle + \langle u(t_0),\, \varphi \rangle\,, \tag{3.9}$$

for all $\varphi \in \mathcal{D}_\sigma(\Omega)$, for every $t_0 \geq 0$ and $t_1 \geq t_0$.

It turns out that the converse of Lemma 3.3 is also true (see Exercise 3.4 in Robinson et al. [2016] for the proof in the Navier–Stokes case). Namely, if u has the regularity of a weak solution (3.5) and satisfies (3.9), then u is a weak solution of the CBF equations (in the sense of Definition 3.1). Therefore, (3.7) and (3.9) are equivalent definitions of weak solutions of the CBF equations.

The next result tells us the way in which a weak solution of the CBF equations attains the initial condition.

Lemma 3.4. *Every weak solution u of the convective Brinkman–Forchheimer equations is L^2-weakly continuous with respect to time, i.e.*

$$\lim_{t\to t_0} \langle u(t),\, v \rangle = \langle u(t_0),\, v \rangle\,,$$

for every $v \in L^2$ and for every $t_0 \in [0,T)$.

Using the formulation of weak solutions for the CBF equations given in Lemma 3.3 (with test functions depending only on the space variables), we can prove L^2-weak continuity with respect to time. The proof is essentially the same as for the Navier–Stokes equations (cf. Theorem 3.8 in Robinson et al. [2016]).

Proof. We take $v \in L^2(\Omega)$. Using the Helmholtz–Weyl decomposition v can be written as $v = h + \nabla g$, where $h \in H$ and $\nabla g \in G$. By orthogonality of $u(t) \in H$ and ∇g we obtain

$$\langle u(t),\, v \rangle = \langle u(t),\, h + \nabla g \rangle = \langle u(t),\, h \rangle\,.$$

Therefore, we only need to prove that

$$\lim_{t \to t_0} \langle u(t),\, h \rangle = \langle u(t_0),\, h \rangle \quad \text{for all} \quad h \in H.$$

Since $\mathcal{D}_\sigma$ is dense in H we can assume that $h \in \mathcal{D}_\sigma(\Omega)$. For every fixed $t_0 \in [0, T)$, taking the weak formulation from Lemma 3.3 with $t_1 = t$ for the time instant $0 \le t_0 \le t < T$, we can write for such h

$$\begin{aligned}\langle u(t),\, h \rangle - \langle u(t_0),\, h \rangle = &- \mu \int_{t_0}^{t} \langle \nabla u(s),\, \nabla h \rangle \,\mathrm{d}s - \int_{t_0}^{t} \langle (u(s) \cdot \nabla)\, u(s),\, h \rangle \,\mathrm{d}s \\ &- \beta \int_{t_0}^{t} \left\langle |u(s)|^{r-1}\, u(s),\, h \right\rangle \mathrm{d}s.\end{aligned}$$

The functions h and ∇h are bounded and

$$u \in L^2(0, T; V) \cap L^{r+1}(0, T; L^{r+1}_\sigma) \cap L^\infty(0, T; H).$$

Therefore, the right-hand side of the above equality is integrable. Hence, it converges to zero as $t \to t_0^+$. We can follow a similar reasoning for $t < t_0$ and obtain convergence from the left as $t \to t_0^-$, and hence the convergence as $t \to t_0$. $\square$

From Lemma 3.4 it follows that every weak solution of the convective Brinkman–Forchheimer equations satisfy the initial condition in the sense that

$$u(t) \rightharpoonup u_0, \quad \text{as} \quad t \to 0^+.$$

3.2.1 Alternative space of test functions

It is often more convenient to replace the space of test functions $\mathcal{D}_\sigma$ in the weak formulation (3.6) with a different, possibly less restrictive space or a space with different properties. We define here the space $\tilde{\mathcal{D}}_\sigma$ consisting of finite combinations of eigenfunctions of the Stokes operator. Both on a bounded domain in $\mathbb{R}^3$ with a smooth boundary and for the torus $\mathbb{T}^3$, we define

$$\tilde{\mathcal{D}}_\sigma(\Omega_T) := \left\{ \varphi \,:\, \varphi = \sum_{k=1}^{N} \alpha_k(t) a_k(x), \quad \alpha_k \in C_0^1([0, T)),\, T > 0, \quad a_k \in \mathcal{N},\, N \in \mathbb{N} \right\},$$

where $\mathcal{N}$ is the orthonormal basis in H (and orthogonal basis in V) consisting of eigenfunctions of the Stokes operator (see Theorem 2.24 in Robinson et al. [2016] for the proof of existence of the set $\mathcal{N}$), that is $Aa_k = \lambda_k a_k$ for all $k \in \mathbb{N}$, with the eigenfunctions λ_k ordered so that

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots, \qquad \text{and} \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty$$

(this follows from the fact that A is positive, symmetric, self-adjoint operator with compact inverse, see e.g. Temam [1977] or Constantin and Foias [1988] for more details).

The functions in the space $\tilde{\mathcal{D}}_\sigma$ are less regular in time than those in $\mathcal{D}_\sigma$, and they usually do not have compact support in the space domain Ω. However, they have the advantage that their dependence on space and time variables is separated, and that they are directly connected with the Stokes operator.

In Chapter 8 (Section 8.1.3) we will construct a sequence φ_n of approximating functions from the space $\tilde{\mathcal{D}}_\sigma$, with some additional properties. We will then use this sequence in Chapter 9 to prove the energy equality for weak solutions of the critical CBF equations (3.2) on bounded domains (Theorem 9.1). It will be crucial in the proof of this result that the functions φ_n can actually be used as test functions in the weak formulation (3.6). This follows from the following lemma (cf. Lemma 3.11 in Robinson et al. [2016] or Lemma 2.3 in Galdi [2000]).

Lemma 3.5. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ or the torus $\mathbb{T}^3$, and let $r \in (0, 5]$. If $u \in L^\infty(0,T;H) \cap L^{r+1}(0,T;L^{r+1}_\sigma) \cap L^2(0,T;V)$ for all $T > 0$, then the following two statements are equivalent:*

(i) u satisfies (3.6) *for all $\varphi \in \mathcal{D}_\sigma(\Omega_\infty)$,*

(ii) u satisfies (3.6) *for all $\varphi \in \tilde{\mathcal{D}}_\sigma(\Omega_\infty)$.*

Proof. We note that the function u in the assumption is defined on $[0,T)$ for every $T > 0$, so globally in time. The space $\tilde{\mathcal{D}}_\sigma(\Omega_\infty) = \mathcal{D}_\sigma(\Omega \times [0,\infty))$ is given by

$$\left\{ \varphi \,:\, \varphi = \sum_{k=1}^{N} \alpha_k(t) a_k(x), \quad \alpha_k \in C_0^1([0,\infty)), \quad a_k \in \mathcal{N},\, N \in \mathbb{N} \right\};$$

the time part of the functions in $\tilde{\mathcal{D}}_\sigma(\Omega_\infty)$ have compact support in $[0,\infty)$, so in fact have compact support in some $[0,T)$, $T < \infty$. In particular, we have $\tilde{\mathcal{D}}_\sigma(\Omega_\infty) = \bigcup_{T>0} \tilde{\mathcal{D}}_\sigma(\Omega_T)$. Similarly, the test functions $\mathcal{D}_\sigma(\Omega_\infty)$ have compact support (in time) in $[0,\infty)$, so any element in $\mathcal{D}_\sigma(\Omega_\infty)$ is an element of $\mathcal{D}_\sigma(\Omega_T)$ for some $T < \infty$. Therefore, the weak form (3.6) is well-defined in both cases considered in the lemma.

(i) $\Rightarrow$ (ii). It is clear that it suffices to show that the equality (3.6) holds for every fixed time $t > 0$ and every $\varphi \in \tilde{\mathcal{D}}_\sigma$ given by

$$\varphi(x, s) = \alpha(s) a(x),$$

where $a \in \mathcal{N}$ and $\alpha \in C_0^1([0, \infty))$.

First, we notice that we can find a sequence of smooth functions $\alpha_n \in C^\infty([0, \infty))$ with compact support in $[0, t+1)$ such that

$$\alpha_n \to \alpha \quad \text{in} \quad C^1([0, t]).$$

Since functions from $\mathcal{D}_\sigma(\Omega)$ are dense in V we can also find a sequence of functions $\varphi_n \in \mathcal{D}_\sigma(\Omega)$ such that

$$\varphi_n \to a \quad \text{in} \quad H^1(\Omega).$$

From the above we also have that

$$\varphi_n \to a \quad \text{in} \quad L^p(\Omega), \quad \text{for } p \in [1, 6].$$

Then for each n the function ψ_n given by

$$\psi_n(x, s) := \alpha_n(s) \varphi_n(x)$$

is an element of $\mathcal{D}_\sigma(\Omega_\infty)$, so from the assumption (i) we know that (3.6),

$$\begin{aligned} &- \int_0^t \langle u, \partial_t \psi_n \rangle \, \mathrm{d}s + \mu \int_0^t \langle \nabla u, \nabla \psi_n \rangle \, \mathrm{d}s + \int_0^t \langle (u \cdot \nabla) u, \psi_n \rangle \, \mathrm{d}s \\ &+ \beta \int_0^t \left\langle |u|^{r-1} u, \psi_n \right\rangle \mathrm{d}s = - \langle u(t), \psi_n(t) \rangle + \langle u_0, \psi_n(0) \rangle , \end{aligned} \tag{3.10}$$

is satisfied for every n. Furthermore,

$$\psi_n \to \varphi \quad \text{in} \quad C([0, t]; V), \tag{3.11}$$

$$\partial_t \psi_n \to \partial_t \varphi \quad \text{in} \quad L^2(0, t; L^2), \quad \text{and} \tag{3.12}$$

$$\psi_n \to \varphi \quad \text{in} \quad L^{r+1}(0, t; L^{r+1}), \quad \text{for } r \in [0, 5], \tag{3.13}$$

as $n \to \infty$. Now we pass to the limit as $n \to \infty$ in (3.10). Using (3.11) and (3.12) we obtain convergence in the linear terms.

To pass to the limit in the convective term it is enough to notice that, due to the Sobolev embedding, we have

$$\psi_n \to \varphi \quad \text{in} \quad C([0,t];L^6), \quad \text{and consequently} \quad \psi_n \to \varphi \quad \text{in} \quad L^4(0,t;L^6).$$

We note that $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ implies that $(u \cdot \nabla)u \in L^{4/3}(0,T;L^{6/5})$ (see Theorem 3.4 in Robinson et al. [2016]). Hence, we obtain

$$\left| \int_0^t \langle (u(s)\cdot\nabla)u(s),\, (\psi_n(s) - \varphi(s)) \rangle \,\mathrm{d}s \right| \leq \|(u\cdot\nabla)u\|_{L^{4/3}(0,t;L^{6/5})} \|\psi_n - \varphi\|_{L^4(0,t;L^6)},$$

which tends to 0 as $n \to \infty$. Thanks to (3.13) we can pass to the limit in the absorption term as well. Indeed, using Hölder's inequality with exponents $(r+1)' = (r+1)/r$ and $r+1$ (for $r > 0$), we have

$$\left| \int_0^t \left\langle |u(s)|^{r-1} u(s),\, (\psi_n(s) - \varphi(s)) \right\rangle \mathrm{d}s \right| \leq \|u\|^r_{L^{r+1}(0,t;L^{r+1})} \|\psi_n - \varphi\|_{L^{r+1}(0,t;L^{r+1})},$$

which tends to 0 as $n \to \infty$.

(ii) $\Rightarrow$ (i). Let $\varphi \in \mathcal{D}_\sigma(\Omega_\infty)$. Then $\varphi(s) \in V \cap H^k$ for all $k = 1,2,3,\dots$ and every $s > 0$, so we can express $\varphi(s)$ in terms of the Stokes eigenfunctions:

$$\varphi(x,s) = \sum_{k=1}^\infty c_k(s) a_k(x).$$

Define $\psi_n(x,s) := \sum_{k=1}^n c_k(s)a_k(x)$. Note that the coefficients $c_k \in C_0^1([0,\infty))$ since they are given by the inner product of φ with a_k

$$c_k(s) = \langle \varphi(s),\, a_k \rangle.$$

Then $\psi_n \in \tilde{\mathcal{D}}_\sigma(\Omega_\infty)$ and

$$\psi_n \to \varphi \quad \text{in} \quad C([0,t];V).$$

Indeed, we have (note that clearly $\varphi(s) \in D(A) = V \cap H^2$)

$$\begin{aligned}
\sup_{0\leq s\leq t} \|\varphi(s) - \psi_n(s)\|_V^2 = \sup_{0\leq s\leq t} \left\| \sum_{k=n+1}^\infty c_k(s)a_k(x) \right\|^2_{H^1} &\leq \sup_{0\leq s\leq t} \sum_{k=n+1}^\infty (1+\lambda_k)c_k^2(s) \\
&\leq \sup_{0\leq s\leq t} \sum_{k=n+1}^\infty 2\lambda_k c_k^2(s) \leq 2 \sup_{0\leq s\leq t} \sum_{k=n+1}^\infty \frac{\lambda_k^2 c_k^2(s)}{\lambda_n} \\
&\leq 2 \sup_{0\leq s\leq t} \sum_{k=1}^\infty \frac{\lambda_k^2 c_k^2(s)}{\lambda_n} = \frac{2}{\lambda_n} \sup_{0\leq s\leq t} \|A\varphi(s)\|^2
\end{aligned}$$

$$\leq \frac{c}{\lambda_n} \sup_{0\leq s\leq t} \|\varphi(s)\|^2_{H^2} \to 0 \quad \text{as } n\to\infty;$$

we used here the facts that $\lambda_k \geq \lambda_n$ for $k > n$ and $\lambda_n \geq 1$ for n big enough[3]. It follows that

$$\psi_n \to \varphi \quad \text{in} \quad L^{r+1}(0,t;L^{r+1}), \quad \text{for } r \leq 5.$$

Furthermore,

$$\partial_t \psi_n \to \partial_t \varphi \quad \text{in } L^2(0,t;L^2),$$

so we can follow the reasoning in the proof of implication '(i) ⇒ (ii)', which was based on the convergence (3.11)-(3.13). □

3.3 Existence of weak solutions

In this section we give a short sketch of the proof of existence of global-in-time weak solutions of the CBF equations with $r \geq 1$. This result is proved using a standard Galerkin approximation method. It appears in the literature in different settings e.g. in Antontsev and de Oliveira [2010]; de Oliveira [2013] or Markowich et al. [2016], usually in greater generality than our case. The proof of this result follows closely an analogous result for the NSE which can be found in many places (see e.g. Lions [1969]; Temam [1977]; Galdi [2000]; Robinson et al. [2016]). The only issue lies in establishing an a priori estimate for the time derivative of the approximate solution. This issue comes, of course, from the introduction of the absorption term. There is a nice observation connecting the time derivative and the absorption term given in Antontsev and de Oliveira [2010] and this is the main reason why we include a proof of this theorem in our considerations. Additionally, in Chapter 4 we often use formal calculations, which can be made rigorous using Galerkin approximations along similar lines as presented here. The domain considered here is a smooth, bounded domain $\Omega \subset \mathbb{R}^3$ with Dirichlet boundary conditions $u|_{\partial\Omega} = 0$ (in the sense of trace) or the three-dimensional torus $\mathbb{T}^3$.

Theorem 3.6 (Existence of weak solutions for CBF)**.** *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ or the torus $\mathbb{T}^3$. For every function $u_0 \in H$ there exists at least one weak solution of the three-dimensional convective Brinkman–Forchheimer equations with $r \geq 1$. This solution is weakly continuous in L^2 with respect to time and additionally satisfies the energy inequality*

[3]Note that on the torus $\mathbb{T}^3 = [0, 2\pi]^3$ the first eigenvalue λ_1 is known to be greater than 1. However, we cannot assume the same for an arbitrary bounded domain $\Omega \subset \mathbb{R}^3$.

$$\frac{1}{2}\|u(t)\|^2 + \mu \int_0^t \|\nabla u(s)\|^2 \, \mathrm{d}s + \beta \int_0^t \|u(s)\|_{L^{r+1}}^{r+1} \, \mathrm{d}s \leq \frac{1}{2}\|u_0\|^2, \tag{3.14}$$

for all $t \in [0, T]$. *Consequently,* $u(t) \to u_0$ *strongly in* L^2 *as* $t \to 0^+$.

It turns out that the weak solution constructed in Theorem 3.6 satisfies the strong energy inequality (3.8). This result was proved for the Navier–Stokes equations by Ladyzhenskaya [1969] (see also Theorem 4.6 in Robinson et al. [2016]). A similar proof works for the CBF equations as well.

From the above we can infer how the initial condition u_0 is attained by all Leray–Hopf weak solutions (solutions satisfying the strong energy inequality (3.8), in particular, the solution constructed in Theorem 3.6).

Corollary 3.7. *Every Leray–Hopf weak solution* u *of the CBF equations is right continuous in* L^2

$$\|u(t) - u(t_0)\| \to 0 \quad \text{as} \quad t \to t_0^+,$$

at times t_0 *for which* (3.8) *holds.*

In particular, u *satisfies the energy inequality* (3.14) *and converges strongly in* L^2 *to the initial condition* u_0

$$\|u(t) - u(0)\| \to 0 \quad \text{as} \quad t \to 0^+.$$

The right-convergence in L^2 follows from the fact that

$$\|u(t)\| \to \|u(t_0)\| \quad \text{as} \quad t \to t_0^+,$$

which is easy to justify by the means of the strong energy inequality (3.8) (see Corollary 4.8 in Robinson et al. [2016] for the details). Convergence of norms and weak continuity (see Lemma 3.4) in the Hilbert space L^2 imply the strong convergence.

As we already know, all weak solutions of the convective Brinkman–Forchheimer equations are weakly continuous with respect to time. However, to date, it is not known (as is the case for the Navier–Stokes equations) whether all weak solutions of the CBF equations with the absorption exponent $r \in [1, 3)$ satisfy the energy inequality or not. Strong convergence to the initial condition follows from a weak continuity and the energy inequality (3.14), when the latter is satisfied. So, it is also not known whether all weak solutions converge strongly in L^2 to the initial condition. The uniqueness of weak solutions is also an open problem.

However, as we will see in Section 3.4, all weak solutions for the critical case

$r = 3$ on the torus (and on bounded domains; see Chapter 9) verify the energy equality and hence are continuous into L^2. In turn, this guarantees existence of a strong global attractor in the phase space H via theory of 'evolutionary systems' developed by Cheskidov [2009] (see Section 3.5).

Now, we present a sketch of the proof of Theorem 3.6. The argument works both for smooth bounded domains in $\mathbb{R}^3$ and for the torus $\mathbb{T}^3$.

Sketch of the proof. We now define the notion of Galerkin approximations for the convective Brinkman–Forchheimer equations.

Let $\{a_j\}_{j=1}^{\infty}$ be an orthonormal basis in H made of eigenfunctions of the Stokes operator[4] A. Since the domain of the Stokes operator (see Chapter 7 for more details on the domains of operators) is given by

$$D(A) = V \cap H^2 \hookrightarrow H,$$

the set $\{a_j\}$ is also an orthogonal basis in $D(A)$. The choice of the base space $D(A) \hookrightarrow H^2$ is a key idea, allowing us to handle the absorption term $|u|^{r-1}\, u$ for all exponents $r \geq 1$. Since in 3D space we have $H^2 \hookrightarrow L^p$ for every $p \geq 1$, this choice gives us control over all the L^p norms of the approximate solution.

We call the function

$$u_n(x,t) := \sum_{j=1}^{n} c_j^n(t) a_j(x) \tag{3.15}$$

the *n-th Galerkin approximation* of the solution of the CBF equations, if it satisfies the following system of equations $\forall\ j = 1, \ldots, n$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle u_n(t),\, a_j\rangle = &- \mu \langle \nabla u_n(t),\, \nabla a_j\rangle - \langle (u_n(t)\cdot\nabla)u_n(t),\, a_j\rangle \\ &- \beta \left\langle |u_n(t)|^{r-1}\, u_n(t),\, a_j \right\rangle, \end{aligned} \tag{3.16}$$

with the initial condition

$$u_n(x,0) = P_n u_0 := \sum_{j=1}^{n} \langle u_0,\, a_j\rangle\, a_j.$$

The operators $P_n : L^2 \to H$, given by $P_n u := \sum_{j=1}^{n} \langle u,\, a_j\rangle\, a_j$, are the orthogonal projections onto the n-dimensional subspaces V_n spanned by the first n eigenfunctions of the Stokes operator.

[4] Recall the set $\mathcal{N}$ from Section 3.2.1.

For each n, (3.16) gives the system of ordinary differential equations for $c_j^n(t)$ with initial conditions

$$c_j^n(0) = \langle u_0, a_j \rangle .$$

From the classical theory of ODEs (see Hartman [1973]), since the right-hand side of (3.16) is continuous and locally Lipschitz, these equations have unique solutions c_j^n of class C^1 for some time interval $[0, T_n]$.

Proceeding as in Section 3.1 (taking smooth functions u_n as test functions), we can easily derive the energy estimates for the Galerkin approximations [alternatively, we can multiply (3.16) by c_j^n, add these equations over j from 1 to n and integrate in time from 0 to T_n]

$$\sup_{t\in[0,T_n]} \|u_n(t)\|^2 + 2\mu \int_0^{T_n} \|\nabla u_n(t)\| \, dt + 2\beta \int_0^{T_n} \|u_n(t)\|_{L^{r+1}}^{r+1} \, dt \\ \leq \|u_n(0)\|^2 \leq \|u_0\|^2 < \infty. \tag{3.17}$$

From (3.17) and standard results for ordinary differential equations, it follows that we can take $T_n = T$ for all n and for every $T > 0$. Moreover, we obtain from (3.17) that

$$u_n \quad \text{remains bounded in} \quad L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^{r+1}(0,T;L_\sigma^{r+1}). \tag{3.18}$$

We note that

$$\left\| |u_n|^{r-1} u_n \right\|_{L^{(r+1)'}}^{(r+1)'} = \|u_n\|_{L^{r+1}}^{r+1} ,$$

where $(r+1)' = (r+1)/r$. This yields that

$$|u_n|^{r-1} u_n \quad \text{remains bounded in} \quad L^{(r+1)'}(0,T;L^{(r+1)'}).$$

Using the orthogonal projection $P_n : H \to V_n$ and noting that $P_n\mathbb{P} = P_n$, we obtain from (3.16) that

$$\partial_t u_n = \mu P_n(\Delta u_n) - P_n((u_n \cdot \nabla)u_n) - \beta P_n(|u_n|^{r-1} u_n). \tag{3.19}$$

Using (3.17) and the special choice of the basis of V_n, we deduce, arguing as in Lions [1969, Chapter 1, Paragraph 6.4.3], that the sequences $P_n(\Delta u_n)$ and $P_n((u_n \cdot \nabla)u_n)$ are bounded in $L^2(0,T;D(A)')$, where $D(A)' = D(A^{-1})$. Moreover, we recall that $D(A) \hookrightarrow L^{r+1}$ (for every $r \geq 1$) and we observe that P_n are uniformly bounded in $D(A)$, but not in L^{r+1}. Therefore, we have for $\varphi \in D(A)$

$$\left|\left\langle P_n(|u|^{r-1}u),\,\varphi\right\rangle\right| = \left|\left\langle |u|^{r-1}u,\,P_n\varphi\right\rangle\right| \le \|u\|_{L^{r+1}}^r \|P_n\varphi\|_{L^{r+1}}$$
$$\le c\,\|u\|_{L^{r+1}}^r \|P_n\varphi\|_{D(A)} \le c\,\|u\|_{L^{r+1}}^r \|\varphi\|_{D(A)} < \infty. \tag{3.20}$$

Then, from (3.19) and (3.20), we have that

$$\partial_t u_n \quad \text{remains bounded in} \quad L^2(0,T;D(A)') + L^{(r+1)'}(0,T;D(A)'). \tag{3.21}$$

Noting that $L^2(0,T) \hookrightarrow L^{(r+1)'}(0,T)$ for every $r > 0$, we have from (3.21) that

$$\partial_t u_n \quad \text{remains bounded in} \quad L^{(r+1)'}(0,T;D(A)'). \tag{3.22}$$

From (3.18) and (3.22), there exist functions u and v, and a subsequence of (u_n) (which we relabel), such that

$$u_n \to u \qquad \text{weakly-* in} \quad L^\infty(0,T;H), \tag{3.23}$$
$$u_n \to u \qquad \text{weakly in} \quad L^2(0,T;V), \tag{3.24}$$
$$u_n \to u \qquad \text{weakly in} \quad L^{r+1}(0,T;L^{r+1}_\sigma), \tag{3.25}$$
$$|u_n|^{r-1}u_n \to v \qquad \text{weakly in} \quad L^{(r+1)'}(0,T;L^{(r+1)'}), \tag{3.26}$$
$$\partial_t u_n \to \partial_t u \qquad \text{weakly in} \quad L^{(r+1)'}(0,T;D(A)'), \tag{3.27}$$

as $n \to \infty$.

We recall that $D(A) \hookrightarrow V \hookrightarrow H \cong H' \hookrightarrow V' \hookrightarrow D(A)'$, with compact embeddings on the first two inclusions. This, together with (3.24) and (3.27) allows us to use the Aubin–Lions compactness lemma[5] (see Lions [1969, Theorem I-5.1)]) to get that

$$u_n \to u \quad \text{strongly in} \quad L^2(0,T;H) \quad \text{as} \quad n \to \infty. \tag{3.28}$$

Now, we multiply (3.16) by $\varphi \in C^1([0,T])$, with $\varphi(T) = 0$, and then we integrate these equations from 0 to T. We get

$$-\int_0^T \langle u_n(t),\,a_j\rangle\,\varphi'(t)\,\mathrm{d}t + \mu\int_0^T \langle \nabla u_n(t),\,\nabla a_j\rangle\,\varphi(t)\,\mathrm{d}t$$
$$+\int_0^T \langle (u_n(t)\cdot\nabla)u_n(t),\,a_j\rangle\,\varphi(t)\,\mathrm{d}t + \beta\int_0^T \left\langle |u_n(t)|^{r-1}u_n(t),\,a_j\right\rangle\varphi(t)\,\mathrm{d}t$$

[5]Note that this implies that we can prove existence of weak solutions for the CBF equations with any $r > 0$, since $(r+1)' = 1 + 1/r > 1$.

$$= \langle u_n(0),\, a_j \rangle \, \varphi(0). \tag{3.29}$$

Passing to the limit in the linear terms is standard and follows from the weak convergence (3.24). Weak convergence (3.24) and strong convergence (3.28) allows us to pass to the limit in the convective term. For the convergence in the absorption term, we notice that by taking a new subsequence, we may assume that $u_n \to u$ a.e. in Ω_T. This implies that

$$|u_n|^{r-1} u_n \to |u|^{r-1} u \quad \text{a.e. in} \quad \Omega_T. \tag{3.30}$$

Using Lemma 1.3 in Lions [1969], it follows from (3.26) and (3.30) that $v = |u|^{r-1} u$. Showing that the limit function u satisfies the weak formulation and energy inequality is standard. □

3.4 Energy equality in the periodic case for the critical exponent $r = 3$

We consider here the unforced CBF equations on the torus $\mathbb{T}^3$, with the critical value of the absorption exponent $r = 3$

$$\partial_t u - \mu \Delta u + (u \cdot \nabla) u + \nabla p + \beta |u|^2 u = 0, \qquad \operatorname{div} u = 0. \tag{3.31}$$

We want to recall that for the Navier–Stokes equations ($\beta = 0$) it is well-known that for every $u_0 \in H$ there exists at least one global Leray–Hopf weak solution that satisfies the strong energy inequality:

$$\|u(t_1)\|^2 + 2\mu \int_{t_0}^{t_1} \|\nabla u(s)\|^2 \, \mathrm{d}s \leq \|u(t_0)\|^2 . \tag{3.32}$$

This can be found in many places, e.g. in Galdi [2000] or in Robinson et al. [2016]. However, it is not known if all weak solutions have to verify (3.32). The problem of proving equality in (3.32) for weak solutions is also open; there are only partial results in this direction. The first criterion guaranteeing the energy equality was identified by Prodi [1959] and Lions [1960] to be

$$u \in L^4(0, T; L^4(\Omega)). \tag{3.33}$$

Then, a few years later Serrin [1963] proved energy equality under the assumption

$$u \in L^t(0,T;L^s(\Omega)), \quad \text{where} \quad \frac{3}{s} + \frac{2}{t} = 1 \quad \text{and} \quad s \in [3,\infty], \tag{3.34}$$

which is stronger than (3.33). Actually, for $s = 4$, it furnishes $u \in L^8(0,T;L^4)$ which implies (3.33). The result of Lions is a particular case of that stated in Shinbrot [1974], where the assumption (3.33) is replaced by:

$$u \in L^t(0,T;L^s(\Omega)), \quad \text{where} \quad \frac{2}{s} + \frac{2}{t} \leq 1 \quad \text{and} \quad s \geq 4. \tag{3.35}$$

This result extends the condition for energy conservation to a wide range of exponents. Many years later Kukavica [2006] proved energy conservation under the condition on the pressure being locally square integrable

$$p \in L^2_{loc}(\mathbb{R}^3 \times [0,T)).$$

See also a recent review on the energy equality results for the Navier–Stokes equations in Berselli and Chiodaroli [2019].

We now make the observation that by definition weak solutions of the critical CBF equations (3.31) satisfy the condition (3.33) [see regularity condition (3.5) in Definition 3.1]. This suggests that the energy equality holds for all weak solutions of this problem, and we prove this in the following theorem.

Theorem 3.8. *Every weak solution of the critical CBF equations* (3.31) *on the torus* $\mathbb{T}^3$*, with the initial condition* $u_0 \in H$ *satisfies* the energy equality*:*

$$\|u(t_1)\|^2 + 2\mu \int_{t_0}^{t_1} \|\nabla u(s)\|^2 \, \mathrm{d}s + 2\beta \int_{t_0}^{t_1} \|u(s)\|^4_{L^4(\mathbb{T}^3)} \, \mathrm{d}s = \|u(t_0)\|^2 \tag{3.36}$$

for all $0 \leq t_0 < t_1 < T$*. Hence, all weak solutions are continuous functions into the phase space* L^2*, i.e.* $u \in C([0,T]\,; H)$*.*

To the best of our knowledge, the validity of the energy equality is not to date verified for the convective Brinkman–Forchheimer equations (3.1) for the range of exponent values $r \in [1,3)$. For larger values of the exponent $r > 3$, it was already shown that the CBF equations enjoy the existence of global-in-time strong solutions (see proof for bounded domains in Kalantarov and Zelik [2012] and Section 4.3 for the periodic case) and hence the energy equality is satisfied. Theorem 3.8 extends the energy equality to the critical case $r = 3$.

This proof is reminiscent of that for the conditional NSE result (due to Lions [1960][6]), where the energy equality was proved to hold for weak solutions belonging

[6]See also Theorem 4.1 in Galdi [2000] for a more modern approach.

to the Bochner space $L^4(0,T;L^4(\Omega))$. In our case we have to argue more carefully to handle the additional nonlinear term.

The result of Theorem 3.8 was stated without a proof in Antontsev and de Oliveira [2010] for the Navier–Stokes equations modified by an absorption term. A similar result was given in Cheskidov et al. [2010], where the energy equality was proved to hold for weak solutions of the NSE in the functional space

$$L^3(0,T;D(A^{5/12})).$$

Here $D(A^{5/12})$ is the domain of the fractional power of the Stokes operator $A = -\mathbb{P}\Delta$, where $\mathbb{P} : L^2 \to H$ is the Leray projection (for references see Constantin and Foias [1988], Robinson et al. [2016] or Temam [1995]). This space corresponds to the fractional Sobolev space $H^{5/6}$. The main difference in our work is that we cannot use the usual truncations of the Fourier series as an approximating sequence, since we have regularity of solutions in a Lebesgue space rather than in a Sobolev space. Therefore, we use more carefully truncated Fourier series to obtain our result. We adapt the proof given in Galdi [2000], where a specific mollification in time is used.

The main idea of the proof is to use a weak solution as a test function. We cannot do this directly since u is not sufficiently regular in space or time. Therefore, we regularise in time the finite-dimensional approximations of a weak solution and pass to the limit with both the regularisation and spatial approximation parameters. To this end we recall here some standard facts of the theory of mollification.

Let $\eta(t)$ be an even, non-negative, smooth function with compact support contained in the interval $(-1,1)$, such that

$$\int_{-\infty}^{\infty} \eta(s)\,\mathrm{d}s = 1.$$

We denote by η_h a family of mollifiers connected with the function η, i.e.

$$\eta_h(s) := h^{-1}\eta(s/h) \quad \text{for} \quad h > 0.$$

In particular, we have

$$\int_0^h \eta_h(s)\,\mathrm{d}s = \frac{1}{2}. \tag{3.37}$$

For any function $v \in L^q(0,T;X)$, where X is a Banach space and $q \in [1,\infty)$, we denote its mollification in time by v^h

$$v^h(s) := (v * \eta_h)(s) = \int_0^T v(\tau)\eta_h(s-\tau)\,\mathrm{d}\tau \quad \text{for} \quad h \in (0,T).$$

We have the following properties of this mollification (see Lemma 2.5 in Galdi [2000]).

Lemma 3.9. *Let $w \in L^q(0,T;X)$, $1 \le q < \infty$, for some Banach space X. Then $w^h \in C^k([0,T);X)$ for all $k \ge 0$. Moreover,*

$$\lim_{h \to 0} \left\| w^h - w \right\|_{L^q(0,T;X)} = 0.$$

Finally, if $\{w_n\}_{n=1}^{\infty}$ converges to w in $L^q(0,T;X)$, then

$$\lim_{n \to \infty} \left\| w_n^h - w^h \right\|_{L^q(0,T;X)} = 0.$$

Since our domain is the three-dimensional torus, we can approximate functions in L^p spaces using carefully truncated Fourier expansions. The natural truncation of the Fourier series

$$\tilde{S}_n(u) := \sum_{|k| \le n} \hat{u}_n \mathrm{e}^{\mathrm{i}k\cdot x},$$

behaves well in the L^2-based spaces:

$$\left\| \tilde{S}_n(u) - u \right\|_X \to 0 \qquad \text{and} \qquad \left\| \tilde{S}_n(u) \right\|_X \le \|u\|_X$$

for $X = L^2(\mathbb{T}^3)$ or $H^s(\mathbb{T}^3)$. However, the same does not hold in $L^p(\mathbb{T}^3)$ for $p \neq 2$. There is no constant c_p such that

$$\left\| \tilde{S}_n(u) \right\|_{L^p} \le c_p \|u\|_{L^p} \qquad \text{for every} \qquad u \in L^p(\mathbb{T}^3).$$

This follows from the result of Fefferman [1971] concerning the ball multiplier for the Fourier transform (see also Section 1.5 in Robinson et al. [2016] for a brief discussion of this result).

In the periodic setting (and on the whole space $\mathbb{R}^d$) we can overcome this problem by considering truncations over 'cubes' ($|k_j| \le n$) rather than 'balls' ($|k| \le n$) of the Fourier modes. If we define

$$S_n(u) := \sum_{k \in \mathcal{Q}_n} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x},$$

where $\mathcal{Q}_n := [-n,n]^3 \cap \mathbb{Z}^3$, then it follows from good behaviour of the truncation in one-dimensional space that

$$\|S_n(u) - u\|_{L^p} \to 0 \qquad \text{and} \qquad \|S_n(u)\|_{L^p} \leq c_p \|u\|_{L^p}$$

(see e.g. Muscalu and Schlag [2013] for more details). We state this more precisely in the following theorem (see Theorem 1.6 in Robinson et al. [2016] for more details).

Theorem 3.10. *Let $\mathcal{Q}_n := [-n,n]^3 \cap \mathbb{Z}^3$. For every $w \in L^1(\mathbb{T}^3)$ and every $n \in \mathbb{N}$ define*

$$S_n(w) := \sum_{k \in \mathcal{Q}_n} \hat{w}_k \mathrm{e}^{\mathrm{i}k\cdot x}, \tag{3.38}$$

where the Fourier coefficients $\hat{w}_k$ are given by

$$\hat{w}_k := \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} w(x) \mathrm{e}^{-\mathrm{i}k\cdot x} \, \mathrm{d}x.$$

Then for every $1 < p < \infty$ there is a constant c_p, independent of n, such that

$$\|S_n(w)\|_{L^p(\mathbb{T}^3)} \leq c_p \|w\|_{L^p(\mathbb{T}^3)} \quad \textit{for all} \quad w \in L^p(\mathbb{T}^3)$$

and

$$\|S_n(w) - w\|_{L^p(\mathbb{T}^3)} \to 0 \quad \textit{as} \quad n \to \infty.$$

Now we can prove the following density result which will be used in the proof of Theorem 3.8.

Lemma 3.11. *$\mathcal{D}_\sigma(\mathbb{T}^3 \times [0,T))$ is dense in $L^4(0,T; L^4_\sigma(\mathbb{T}^3)) \cap L^2(0,T;V)$.*

We note that in the periodic case the lack of boundaries would allow us to use a mollification in space to prove Lemma 3.11. While this method is simpler than the truncations of the Fourier series which we use, we prefer to use the truncations S_n, because this method is more in line with what we will be doing later on in Chapters 8 and 9. We also think that the convergence of S_n in the Lebesgue spaces is interesting in its own right and not so widely known.

Proof. Let $w \in L^4(0,T; L^4_\sigma(\mathbb{T}^3)) \cap L^2(0,T;V)$ and define

$$w_n^h(x,t) := S_n(w^h(x,t)) = \sum_{k \in \mathcal{Q}_n} \hat{w}_k^h(t) \mathrm{e}^{\mathrm{i}k\cdot x} \quad \text{for} \quad h \in (0,T),$$

where S_n is the same as in (3.38). Clearly, $w_n^h \in \mathcal{D}_\sigma(\mathbb{T}^3 \times [0,T))$. By Theorem 3.10

we have

$$\lim_{n\to\infty} \left\| w_n^h(t) - w^h(t) \right\|_{L^4(\mathbb{T}^3)}^4 = 0 \tag{3.39}$$

and

$$\lim_{n\to\infty} \left\| w_n^h(t) - w^h(t) \right\|_V^2 = \lim_{n\to\infty} \left\| w_n^h(t) - w^h(t) \right\|^2 + \lim_{n\to\infty} \left\| \nabla w_n^h(t) - \nabla w^h(t) \right\|^2 = 0 \tag{3.40}$$

for all $t \in [0, T)$. By Lemma 3.9, for a given $\varepsilon > 0$, we can choose $h > 0$ so small that

$$\int_0^T \left\| w^h(t) - w(t) \right\|_{L^4(\mathbb{T}^3)}^4 \mathrm{d}t < \varepsilon \quad \text{and} \quad \int_0^T \left\| w^h(t) - w(t) \right\|_V^2 \mathrm{d}t < \varepsilon. \tag{3.41}$$

On the other hand, from (3.39), (3.40), and the Lebesgue Dominated Convergence Theorem, we have that for all fixed $h \in (0, T)$

$$\lim_{n\to\infty} \int_0^T \left\| w_n^h(t) - w^h(t) \right\|_{L^4(\mathbb{T}^3)}^4 \mathrm{d}t = 0, \quad \text{and} \quad \lim_{n\to\infty} \int_0^T \left\| w_n^h(t) - w^h(t) \right\|_V^2 \mathrm{d}t = 0, \tag{3.42}$$

since, respectively, $\left\| w_n^h(t) \right\|_{L^4(\mathbb{T}^3)} \le c \left\| w^h(t) \right\|_{L^4(\mathbb{T}^3)}$, and $\left\| w_n^h(t) \right\|_V \le c \left\| w^h(t) \right\|_V$ for all $n \in \mathbb{N}$ and $t \in [0, T)$, and

$$w^h \in L^4(0, T; L^4_\sigma(\mathbb{T}^3)) \cap L^2(0, T; V).$$

Thus, the lemma follows from the relations (3.41), (3.42) and the triangle inequality.

□

Now, we are in a position to prove Theorem 3.8.

Proof. Let $\{u_n\}_{n=1}^\infty \subset \mathcal{D}_\sigma(\mathbb{T}^3 \times [0, T))$ be a sequence converging to a weak solution u in $L^4(0, T; L^4_\sigma(\mathbb{T}^3))$ and in $L^2(0, T; V)$, see Lemma 3.11. For every fixed time instant $t_1 \in (0, T)$, we choose in the weak formulation (3.6) (with $t = t_1$) a sequence of test functions

$$\begin{aligned} \varphi_n^h(x, s) &:= \left(u_n(x, \cdot) \chi_{[0,t_1]}(\cdot) \right)^h (s) = \left(u_n \chi_{[0,t_1]} * \eta_h \right)(x, s) \\ &= \int_0^T u_n(x, \tau) \chi_{[0,t_1]}(\tau) \eta_h(s - \tau) \, \mathrm{d}\tau = \int_0^{t_1} u_n(x, \tau) \eta_h(s - \tau) \, \mathrm{d}\tau, \end{aligned}$$

for $(x, s) \in \mathbb{T}^3 \times [0, T)$, with the parameter h satisfying the following conditions:

$$0 < h < T - t_1 \quad \text{and} \quad h < t_1.$$

We obtain a sequence of equations

$$\begin{aligned}
-\int_0^{t_1} \left\langle u(s),\, \partial_t (u_n \chi_{[0,t_1]})^h(s) \right\rangle \mathrm{d}s + \mu \int_0^{t_1} \left\langle \nabla u(s),\, \nabla (u_n \chi_{[0,t_1]})^h(s) \right\rangle \mathrm{d}s \\
+ \int_0^{t_1} \left\langle (u(s)\cdot\nabla)u(s),\, (u_n \chi_{[0,t_1]})^h(s) \right\rangle \mathrm{d}s + \beta \int_0^{t_1} \left\langle |u(s)|^2 u(s),\, (u_n \chi_{[0,t_1]})^h(s) \right\rangle \mathrm{d}s \\
= -\left\langle u(t_1),\, (u_n \chi_{[0,t_1]})^h(t_1) \right\rangle + \left\langle u(0),\, (u_n \chi_{[0,t_1]})^h(0) \right\rangle . \qquad (3.43)
\end{aligned}$$

Note that our choice of h ensures that $\varphi_n^h(x,T) = 0$. Additionally, observe that the functions φ_n^h are divergence-free, since $\operatorname{div}\varphi_n^h = (\operatorname{div}\varphi_n)^h = 0$, so indeed $\varphi_n^h \in \mathcal{D}_\sigma(\mathbb{T}^3 \times [0,T))$.

We want to pass to the limit in (3.43) as $n \to \infty$. To this end, using Hölder's inequality and Lemma 3.9, we observe the following estimates for the nonlinear terms:

$$\begin{aligned}
&\left| \int_0^{t_1} \left\langle (u(s)\cdot\nabla)u(s),\, (u_n \chi_{[0,t_1]})^h(s) \right\rangle \mathrm{d}s - \int_0^{t_1} \left\langle (u(s)\cdot\nabla)u(s),\, (u \chi_{[0,t_1]})^h(s) \right\rangle \mathrm{d}s \right| \\
&\le \int_0^{t_1} \|u(s)\|_{L^4(\mathbb{T}^3)} \|\nabla u(s)\| \left\| (u_n \chi_{[0,t_1]})^h(s) - (u \chi_{[0,t_1]})^h(s) \right\|_{L^4(\mathbb{T}^3)} \mathrm{d}s \\
&\le \|u\|_{L^4(0,T;L^4_\sigma(\mathbb{T}^3))} \|u\|_{L^2(0,T;V)} \left\| (u_n \chi_{[0,t_1]})^h - (u \chi_{[0,t_1]})^h \right\|_{L^4(0,T;L^4_\sigma(\mathbb{T}^3))} \to 0 \quad (3.44)
\end{aligned}$$

as $n \to \infty$, and

$$\begin{aligned}
&\left| \int_0^{t_1} \left\langle |u(s)|^2 u(s),\, (u_n \chi_{[0,t_1]})^h(s) \right\rangle \mathrm{d}s - \int_0^{t_1} \left\langle |u(s)|^2 u(s),\, (u \chi_{[0,t_1]})^h(s) \right\rangle \mathrm{d}s \right| \\
&\le \int_0^{t_1} \|u(s)\|^3_{L^4(\mathbb{T}^3)} \left\| (u_n \chi_{[0,t_1]})^h(s) - (u \chi_{[0,t_1]})^h(s) \right\|_{L^4(\mathbb{T}^3)} \mathrm{d}s \\
&\le \|u\|^3_{L^4(0,T;L^4_\sigma(\mathbb{T}^3))} \left\| (u_n \chi_{[0,t_1]})^h - (u \chi_{[0,t_1]})^h \right\|_{L^4(0,T;L^4_\sigma(\mathbb{T}^3))} \to 0 \qquad (3.45)
\end{aligned}$$

as $n \to \infty$. Estimating the linear terms in a standard way and using (3.44), (3.45) we can pass in the weak formulation (3.43) to the limit as $n \to \infty$. We arrive at the identity

$$\begin{aligned}
-\int_0^{t_1} \left\langle u(s),\, \partial_t (u \chi_{[0,t_1]})^h(s) \right\rangle \mathrm{d}s + \mu \int_0^{t_1} \left\langle \nabla u(s),\, \nabla (u \chi_{[0,t_1]})^h(s) \right\rangle \mathrm{d}s \\
+ \int_0^{t_1} \left\langle (u(s)\cdot\nabla)u(s),\, (u \chi_{[0,t_1]})^h(s) \right\rangle \mathrm{d}s + \beta \int_0^{t_1} \left\langle |u(s)|^2 u(s),\, (u \chi_{[0,t_1]})^h(s) \right\rangle \mathrm{d}s
\end{aligned}$$

$$= -\left\langle u(t_1),\, (u\chi_{[0,t_1]})^h(t_1)\right\rangle + \left\langle u(0),\, (u\chi_{[0,t_1]})^h(0)\right\rangle.$$

Since the function η_h is even in $(-h,h)$, we have $\dot{\eta}_h(r) = -\dot{\eta}_h(-r)$ and so

$$\begin{aligned}
-\int_0^{t_1} \left\langle u(s),\, \partial_t(u\chi_{[0,t_1]})^h(s)\right\rangle \mathrm{d}s \quad &= -\int_0^{t_1}\left(\int_0^{t_1} \dot{\eta}_h(s-\tau)\,\langle u(s),\, u(\tau)\rangle\,\mathrm{d}\tau\right)\mathrm{d}s \\
\dot{\eta} \text{ is odd} \quad &= \int_0^{t_1}\left(\int_0^{t_1} \dot{\eta}_h(\tau-s)\,\langle u(s),\, u(\tau)\rangle\,\mathrm{d}\tau\right)\mathrm{d}s \\
\text{symmetry of the scalar product} \quad &= \int_0^{t_1}\left(\int_0^{t_1} \dot{\eta}_h(\tau-s)\,\langle u(\tau),\, u(s)\rangle\,\mathrm{d}\tau\right)\mathrm{d}s \\
\text{changing order of integration} \quad &= \int_0^{t_1}\left(\int_0^{t_1} \dot{\eta}_h(\tau-s)\,\langle u(\tau),\, u(s)\rangle\,\mathrm{d}s\right)\mathrm{d}\tau \\
\text{swapping } s \text{ and } \tau \quad &= \int_0^{t_1}\left(\int_0^{t_1} \dot{\eta}_h(s-\tau)\,\langle u(s),\, u(\tau)\rangle\,\mathrm{d}\tau\right)\mathrm{d}s = 0.
\end{aligned}$$

Next, by repeating the arguments in (3.44), (3.45) with $(u\chi_{[0,t_1]})^h$ in place of $(u_n\chi_{[0,t_1]})^h$ and $u\chi_{[0,t_1]}$ in place of $(u\chi_{[0,t_1]})^h$, we obtain

$$\begin{aligned}
\lim_{h\to 0}\int_0^{t_1} \left\langle (u(s)\cdot\nabla)u(s),\, (u\chi_{[0,t_1]})^h(s)\right\rangle \mathrm{d}s &= \int_0^{t_1} \left\langle (u(s)\cdot\nabla)u(s),\, (u\chi_{[0,t_1]})(s)\right\rangle \mathrm{d}s \\
&= \int_0^{t_1} \langle (u(s)\cdot\nabla)u(s),\, u(s)\rangle\,\mathrm{d}s = 0, \\
\lim_{h\to 0}\int_0^{t_1} \left\langle |u(s)|^2\, u(s),\, (u\chi_{[0,t_1]})^h(s)\right\rangle \mathrm{d}s &= \int_0^{t_1} \left\langle |u(s)|^2\, u(s),\, (u\chi_{[0,t_1]})(s)\right\rangle \mathrm{d}s \\
&= \int_0^{t_1} \|u(s)\|_{L^4(\mathbb{T}^3)}^4\,\mathrm{d}s, \\
\lim_{h\to 0}\int_0^{t_1} \left\langle \nabla u(s),\, \nabla(u\chi_{[0,t_1]})^h(s)\right\rangle \mathrm{d}s &= \int_0^{t_1} \left\langle \nabla u(s),\, \nabla(u\chi_{[0,t_1]})(s)\right\rangle \mathrm{d}s \\
&= \int_0^{t_1} \|\nabla u(s)\|^2\,\mathrm{d}s,
\end{aligned}$$

which give us

$$\begin{aligned}
\mu\int_0^{t_1} \|\nabla u(s)\|^2\,\mathrm{d}s + \beta\int_0^{t_1} \|u(s)\|_{L^4(\mathbb{T}^3)}^4\,\mathrm{d}s = &-\lim_{h\to 0}\left\langle u(t_1),\, (u\chi_{[0,t_1]})^h(t_1)\right\rangle \\
&+ \lim_{h\to 0}\left\langle u(0),\, (u\chi_{[0,t_1]})^h(0)\right\rangle.
\end{aligned}$$

Finally, from the fact that u is L^2-weakly continuous in time and from (3.37), we have

$$\left\langle u(t_1),\, (u\chi_{[0,t_1]})^h(t_1)\right\rangle = \int_0^T \eta_h(s)\chi_{[0,t_1]}(t_1-s)\,\langle u(t_1),\, u(t_1-s)\rangle \,\mathrm{d}s$$
$$= \int_0^{t_1} \eta_h(s)\,\langle u(t_1),\, u(t_1-s)\rangle \,\mathrm{d}s = \int_0^{h} \eta_h(s)\,\langle u(t_1),\, u(t_1-s)\rangle \,\mathrm{d}s$$
$$= \frac{1}{2}\left\|u(t_1)\right\|^2 + \int_0^h \eta_h(s)\,\langle u(t_1),\, u(t_1-s)-u(t_1)\rangle \,\mathrm{d}s \to \frac{1}{2}\left\|u(t_1)\right\|^2$$

as $h \to 0$. In the same manner we show that

$$\left\langle u(0),\, (u\chi_{[0,t_1]})^h(0)\right\rangle \to \frac{1}{2}\left\|u(0)\right\|^2 \quad \text{as} \quad h \to 0.$$

Finally, we obtain the identity

$$\frac{1}{2}\left\|u(t_1)\right\|^2 + \mu\int_0^{t_1} \left\|\nabla u(s)\right\|^2 \,\mathrm{d}s + \beta\int_0^{t_1} \left\|u(s)\right\|^4_{L^4(\mathbb{T}^3)} \,\mathrm{d}s = \frac{1}{2}\left\|u(0)\right\|^2 \tag{3.46}$$

for all $t_1 \in (0,T)$. The energy equality (3.36) follows by replacing t_1 with t_0 in (3.46) and taking the difference of the two expressions.

Now we will prove the last part of the theorem, namely that all weak solutions of the critical CBF equations (3.31) are continuous into L^2 with respect to time, i.e.

$$\left\|u(t)-u(t_0)\right\| \to 0 \quad \text{as} \quad t \to t_0, \tag{3.47}$$

for all $t_0 \in [0,T)$.

First, we recall (see Lemma 3.4) that all weak solutions of (3.31) are L^2-weakly continuous with respect to time

$$u(t) \rightharpoonup u(t_0) \quad \text{as} \quad t \to t_0, \tag{3.48}$$

for all $t_0 \in [0,T)$.

Now, let u be a weak solution of (3.31) and take $t_1 = t$ in the energy equality (3.36). We have

$$\left|\left\|u(t)\right\|^2 - \left\|u(t_0)\right\|^2\right| \le 2\mu\left|\int_t^{t_0} \left\|\nabla u(s)\right\|^2 \,\mathrm{d}s\right| + 2\beta\left|\int_t^{t_0} \left\|u(s)\right\|^4_{L^4(\mathbb{T}^3)} \,\mathrm{d}s\right| \to 0,$$

when $t \to t_0$, because the terms under the integral signs are obviously integrable for the weak solution u. Therefore, it follows from the first part of Theorem 3.8 that for all weak solutions of (3.31) we have

$$u(t) \rightharpoonup u(t_0) \quad \text{and} \quad \|u(t)\| \to \|u(t_0)\| \quad \text{as} \quad t \to t_0.$$

The result (3.47) follows, since in a Hilbert space weak convergence and convergence of norms imply strong convergence. □

3.5 Strong global attractor

For a number of basic evolution equations of mathematical physics (including the Navier–Stokes equations) it has been shown that the long-time behaviour of their solutions can be characterised by the 'global attractor' of the equation. Of particular interest are those equations for which the solution of the corresponding Cauchy problem is not unique or the uniqueness is not proved (equations without uniqueness). To construct global attractors for equations without uniqueness the theory of the trajectory (or multi-valued) attractors has been developed. An overview of the theory for autonomous systems can be found in Sell and You [2002], and in Chepyzhov and Vishik [2002] for the non-autonomous case. A survey on trajectory attractors is also contained in Chapter 6 of Miranville and Zelik [2008].

There are several abstract frameworks for studying infinite-dimensional dynamical systems without uniqueness. One method (see Sell [1973]) is to recover uniqueness of solutions by working in a space of semitrajectories $u : [0, \infty) \to X$ and defining a corresponding semiflow $T(\cdot)$ by $T(t)u := u^t$, for $t \geq 0$, where $u^t(s) := u(t+s)$. An example of the use of this method is the proof by Sell [1996] of the existence of a global attractor for the 3D incompressible Navier–Stokes equations (see further results on a trajectory attractor for the 3D NSE in Chepyzhov and Vishik [2002], Sell and You [2002]). Caraballo et al. [2003] compared two canonical methods in this theory by Melnik and Valero [1998] (see also Mel′nik [1997]) and Ball [2000]. The first approach, used by Babin and Vishik [1985], and which goes all the way back to the work by Barbašin [1948], is to consider a set-valued trajectory $t \mapsto T(t)u_0$ in which $T(t)u_0$ consists of all possible points reached at time t by solutions with initial data u_0 (a trajectory is a function of time with values in the set of all subsets of a phase space). Ball's approach considers the generalised semiflow G, where a trajectory is a function of time with values in the phase space, and there may be more than one trajectory with a given initial data.

In Ball [2000] it was shown for the three-dimensional incompressible Navier–Stokes equations that strong L^2-continuity leads to the existence of a global attractor in the phase space H. We have proved in the previous section that all weak solutions of the convective Brinkman–Forchheimer equations with critical exponent $r = 3$

satisfy the energy equality. As a consequence, we obtained unconditional continuity of all weak solutions into L^2. Therefore, we expect to extend the result of Ball to the CBF equations. However, due to technical difficulties we were not able to apply his method of generalised semiflows to our problem. In particular, we were not able to prove Proposition 7.3 in Ball [2000] for weak solutions of the critical CBF equations. Proposition 7.3 is crucial in showing that all L^2-continuous weak solutions (so all weak solutions in our case) form a generalised semiflow G_{CBF} on H, and that this semiflow is pointwise dissipative and asymptotically compact (so a global attractor in H exists for G_{CBF}, see Theorem 3.3 in Ball [2000]).

Strong trajectory attractors constructed via the energy equality are considered also in Vishik et al. [2010] (for a general dissipative reaction-diffusion system) and Chepyzhov et al. [2011] (for the 2D damped Euler equations), for example. The situation in those papers is very similar to the one considered in this thesis and the method presented there should work in our case. However, we did not pursue this approach, and we cannot say with certainty that it can be applied to the critical convective Brinkman–Forchheimer equations. Instead, we make use of the theory of evolutionary systems due to Cheskidov [2009], to show existence of a strong global attractor for (3.31). The evolutionary system $\mathcal{E}$ considered in this chapter is closer to Ball's approach than to that of Melnik and Valero.

We want to point out that there are some issues with the application of Cheskidov's 'evolutionary systems' to the Navier–Stokes equations. These problems are connected with the so-called *exceptional set* (the set of measure zero consisting of the times for which the energy inequality does not hold). For instance, if we remove the initial point $t = t_0$ from the exceptional set in the definition of Leray–Hopf weak solutions (as we did in (3.8) with the initial time $t_0 = 0$), we lose the translation invariance for the set of trajectories. On the other hand, if we allow the initial time to be in the exceptional set then we lose a dissipative estimate and absorbing ball for Leray–Hopf weak solutions of the NSE. Fortunately, in our case the energy equality (3.36) holds and all these problems disappear.

3.5.1 Evolutionary systems

First, we introduce some notation from Cheskidov [2009]. Let $(X, d_s(\cdot,\cdot))$ be a metric space endowed with a metric d_s, which will be referred to as a strong metric. Let $d_w(\cdot,\cdot)$ be another metric on X satisfying the following conditions:

1. X is d_w-compact.

2. If $d_s(u_n, v_n) \to 0$ as $n \to \infty$ for some $u_n, v_n \in X$, then $d_w(u_n, v_n) \to 0$ as $n \to \infty$.

Due to the property 2, $d_w(\cdot,\cdot)$ will be reffered to as a weak metric on X. Note that any d_s-compact set is d_w-compact and any weakly closed set is strongly closed.

Let $C([a,b];X_\tau)$, where $\tau \in \{s,w\}$, be the space of d_τ-continuous X-valued functions on $[a,b]$ endowed with the metric

$$d_{C([a,b];X_\tau)}(u,v) := \sup_{t\in[a,b]} \{d_\tau(u(t),v(t))\}.$$

Let also $C([a,\infty);X_\tau)$ be the space of d_τ-continuous X-valued functions on $[a,\infty)$ endowed with the metric

$$d_{C([a,\infty);X_\tau)}(u,v) := \sum_{T\in\mathbb{N}} \frac{1}{2^T} \frac{\sup\{d_\tau(u(t),v(t)) : a \le t \le a+T\}}{1+\sup\{d_\tau(u(t),v(t)) : a \le t \le a+T\}}.$$

To define an evolutionary system, first let

$$\mathcal{T} := \{I \ : \ I = [T,\infty) \subset \mathbb{R} \quad \text{for } T \in \mathbb{R}, \quad \text{or} \quad I = (-\infty,\infty)\},$$

and for each $I \subset \mathcal{T}$, let $\mathcal{F}(I)$ denote the set of all X-valued functions on I.

Definition 3.12. *A map $\mathcal{E}$ that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}(I) \subset \mathcal{F}(I)$ will be called an* evolutionary system *if the following conditions are satisfied:*

1. $\mathcal{E}([0,\infty)) \neq \emptyset$.
2. $\mathcal{E}(I+s) = \{u(\cdot) \ : \ u(\cdot - s) \in \mathcal{E}(I)\}$ *for all* $s \in \mathbb{R}$.
3. $\{u(\cdot)|_{I_2} \ : \ u(\cdot) \in \mathcal{E}(I_1)\} \subset \mathcal{E}(I_2)$ *for all pairs* $I_1, I_2 \in \mathcal{T}$, *such that* $I_2 \subset I_1$.
4. $\mathcal{E}((-\infty,\infty)) = \big\{u(\cdot) \ : \ u(\cdot)|_{[T,\infty)} \in \mathcal{E}([T,\infty))$ *for all* $T \in \mathbb{R}\big\}$.

We will refer to $\mathcal{E}(I)$ as the *set of all trajectories* on the time interval I. Trajectories in $\mathcal{E}((-\infty,\infty))$ will be called *complete.* To relate the notion of evolutionary systems with the classical notion of semiflows, let $P(X)$ be the set of all subsets of X. For every $t \geq 0$, define a map $R(t) : P(X) \to P(X)$, such that

$$R(t)A := \{u(t) \ : \ u \in A, \ u \in \mathcal{E}([0,\infty))\} \quad \text{for} \quad A \subset X.$$

Note that the assumptions on $\mathcal{E}$ imply that $R(t)$ enjoys the following property:

$$R(t+s)A \subset R(t)R(s)A, \quad A \subset X, \ t,s \geq 0.$$

One can check that a semiflow defines an evolutionary system (see details in Cheskidov [2009]).

Furthermore, we will consider evolutionary systems $\mathcal{E}$ satisfying the following assumptions:

(A) (*Weak compactness*) $\mathcal{E}([0,\infty))$ is a compact set in $C([0,\infty);X_w)$.

(B) (*Energy inequality*) Assume that X is a bounded set in some uniformly convex Banach space $\mathcal{X}$ with the norm denoted by $\|\cdot\|_{\mathcal{X}}$, such that

$$d_s(x,y) = \|x-y\|_{\mathcal{X}} \quad \text{for} \quad x,y \in X.$$

Assume also that for any $\varepsilon > 0$, there exists $\delta > 0$, such that for every trajectory $u \in \mathcal{E}([0,\infty))$ and $t > 0$,

$$\|u(t)\|_{\mathcal{X}} \leq \|u(t_0)\|_{\mathcal{X}} + \varepsilon,$$

for almost every $t_0 \in (t-\delta, t)$.

(C) (*Strong convergence a.e.*) Let $u, u_n \in \mathcal{E}([0,\infty))$ be such that $u_n \to u$ in $C([0,T];X_w)$ for some $T > 0$. Then $u_n(t) \to u(t)$ strongly for a.e. $t \in [0,T]$.

Consider an arbitrary evolutionary system $\mathcal{E}$. For a set $A \subset X$ and $r > 0$, denote an open ball by

$$B_\tau(A,r) := \{u \in X \ : \ d_\tau(u,A) < r\},$$

where

$$d_\tau(u,A) := \inf_{x\in A} \{d_\tau(u,x)\}.$$

We say that a set $A \subset X$ *uniformly attracts* a set $B \subset X$ in the d_τ-metric if for any $\varepsilon > 0$ there exists t_0, such that

$$R(t)B \subset B_\tau(A,\varepsilon), \quad \forall\, t \geq t_0.$$

Based on the above we define an attracting set.

Definition 3.13. *A set $A \subset X$ is a d_τ-attracting set if it uniformly attracts X in the d_τ-metric.*

Using the above definitions we can now define a global attractor in our setting.

Definition 3.14. *A set $\mathcal{A}_\tau \subset X$ is a d_τ-global attractor if $\mathcal{A}_\tau$ is a minimal d_τ-closed, d_τ-attracting set.*

Note that since X may not be strongly compact, the intersection of two strongly closed, strongly attracting sets may not be strongly attracting. Nevertheless, if $\mathcal{A}_\tau$ exists then it is unique (see Theorem 3.6 in Cheskidov [2009]).

The following result (Theorem 3.9 in Cheskidov [2009]) motivates studying evolutionary systems in the context of existence of global attractors. It recovers a similar result from Cheskidov and Foias [2006].

Theorem 3.15. *Every evolutionary system possesses a weak global attractor $\mathcal{A}_w$. Moreover, if a strong global attractor $\mathcal{A}_s$ exists, then $\overline{\mathcal{A}_s}^w = \mathcal{A}_w$ (note that $\overline{\mathcal{A}_s}^w$ denotes the closure of the set $\mathcal{A}_s$ in the topology generated by d_w).*

We now introduce an important notion of asymptotic compactness (for instance, Ball's generalised semiflows possess a global attractor if and only if they are pointwise dissipative and asymptotically compact; see Theorem 3.3 in Ball [2000]).

Definition 3.16. *The evolutionary system $\mathcal{E}$ is asymptotically compact if for any sequence $t_n \to \infty$ as $n \to \infty$, and any $x_n \in R(t_n)X$, the sequence $\{x_n\}_{n=1}^\infty$ is relatively strongly compact (it has a strongly convergent subsequence $\{y_n\} \subset \{x_n\}$).*

The following theorem generalises corresponding results for Ball's generalised semiflows and for classical semiflows (see Hale et al. [1972]; Hale [1988]; Ladyzhenskaya [1991]).

Theorem 3.17. *If an evolutionary system $\mathcal{E}$ is asymptotically compact, then $\mathcal{A}_w$ is a strong global attractor $\mathcal{A}_s$, compact in the strong topology.*

The next result gives sufficient conditions for $\mathcal{E}$ to be asymptotically compact and hence (in the light of Theorem 3.17), for the existence of a strong global attractor for $\mathcal{E}$.

Theorem 3.18. *Let $\mathcal{E}$ be an evolutionary system satisfying the properties* (A), (B), *and* (C). *If every complete trajectory is strongly continuous, i.e. if*

$$\mathcal{E}((-\infty,\infty)) \subset C((-\infty,\infty); X_s),$$

then $\mathcal{E}$ is asymptotically compact.

3.5.2 Application to the critical CBF equations on the torus

In Cheskidov [2009] it was shown that all Leray–Hopf weak solutions of the space-periodic (with zero mean-value assumption $\int_{\mathbb{T}^3} u = 0$) 3D NSE form an evolutionary system $\mathcal{E}$ satisfying conditions (A), (B), and (C). Note that these solutions are defined differently than our Leray–Hopf weak solutions of the CBF equations. Namely, they allow the initial time to be in the exceptional set, whereas our definition excludes the initial time from the exceptional set (cf. Theorem 8.2 in Cheskidov [2009] and Definition 3.2). We will show in this section that all weak solutions of the critical CBF equations form an evolutionary system $\mathcal{E}$ satisfying (A), (B), and (C) as well (the difference in definitions of Leray–Hopf weak solutions is not important here, since in our case the energy equality holds for all weak solutions). We begin by setting our problem into the framework of evolutionary systems.

We define the strong and weak distances by

$$d_s(u,v) := \|u-v\|, \quad d_w(u,v) := \sum_{k\in\mathbb{Z}^3} \frac{1}{2^{|k|}} \frac{|\hat{u}_k - \hat{v}_k|}{1+|\hat{u}_k - \hat{v}_k|}, \quad u,v \in H,$$

where $\hat{u}_k$ and $\hat{v}_k$ are the Fourier coefficients of u and v respectively, and H is the divergence-free subspace of L^2 (as defined in Chapter 2).

Definition 3.19. *A ball $B_\tau(0,R) \subset H$ is called a d_τ-absorbing ball if for any bounded set $A \subset H$, there exists t_0, such that*

$$R(t)A \subset B_\tau(0,R) \quad \forall\, t \geq t_0.$$

For the 3D NSE it is well-known that there exists a strongly absorbing ball (for the proof see e.g. Proposition 13.1 in Constantin and Foias [1988] or Chapter II, Appendix B in Foias et al. [2001]). The same can be proved in a similar way for the critical CBF equations

$$\partial_t u - \mu\Delta u + (u\cdot\nabla)\,u + \nabla p + \beta\,|u|^2\,u = f, \qquad \operatorname{div} u = 0, \tag{3.49}$$

with the forcing function $f \in L^2(\mathbb{T}^3)$ independent of time.

Proposition 3.20. *The critical CBF equations* (3.49) *possess a d_s-absorbing ball $B_s(0,R) \subset H$, where R is any number greater or equal than $\left(2\,\|f\|\,\left|\mathbb{T}^3\right|/\beta\right)^{1/3}$.*

Below we give a proof of this result. Note that in the Navier–Stokes case one can use the Poincaré inequality to obtain the desired bound. We cannot do this in our case since a weak solution u of the CBF equations does not have zero mean-value[7]. We can circumvent this issue by employing the absorption term.

[7] The Poincaré inequality can be used to obtain the existence of an absorbing ball for the critical CBF equations on bounded domains $\Omega \subset \mathbb{R}^3$. Since we prove in Chapter 9 that the energy equality holds also in that case, it follows that the strong global attractor exists there as well.

Proof. We use the energy equality (with the forcing function f independent of time)

$$\frac{1}{2}\|u(t)\|^2 + \mu\int_{t_0}^{t}\|\nabla u(s)\|^2\,\mathrm{d}s + \beta\int_{t_0}^{t}\|u(s)\|_{L^4(\mathbb{T}^3)}^4\,\mathrm{d}s$$
$$= \frac{1}{2}\|u(t_0)\|^2 + \int_{t_0}^{t}\langle f,\,u(s)\rangle\,\mathrm{d}s.$$

Since

$$\|u\|^2 \le \|u\|_{L^4}^2\left|\mathbb{T}^3\right|^{1/2}, \quad \text{which implies that} \quad \|u\|_{L^4}^4 \ge \frac{\|u\|^4}{|\mathbb{T}^3|},$$

we obtain

$$\frac{1}{2}\|u(t)\|^2 \le \frac{1}{2}\|u(t_0)\|^2 + \int_{t_0}^{t}\left\{\langle f,\,u(s)\rangle - \beta\frac{\|u\|^4}{|\mathbb{T}^3|}\right\}\mathrm{d}s$$
$$\le \frac{1}{2}\|u(t_0)\|^2 + \int_{t_0}^{t}\|u(s)\|\left\{\|f\| - \beta\frac{\|u\|^3}{|\mathbb{T}^3|}\right\}\mathrm{d}s.$$

Suppose that $\|u(s)\|^3 \ge \frac{3}{2}\|f\||\mathbb{T}^3|/\beta$ for all $t_0 \le s \le t$; then for all s in this range

$$\|u(s)\|\left(\|f\| - \beta\frac{\|u(s)\|^3}{|\mathbb{T}^3|}\right) \le -\frac{1}{2}\|f\|\left(\frac{3}{2}\|f\||\mathbb{T}^3|/\beta\right)^{1/3} =: -\frac{c}{2}\|f\|^{4/3},$$

and so

$$\|u(t)\|^2 \le \|u(t_0)\|^2 - c(t-t_0)\|f\|^{4/3}. \tag{3.50}$$

We now show that the set $\left\{u \in H \;:\; \|u\|^3 \le 2\varrho\right\}$, where $\varrho := \|f\||\mathbb{T}^3|/\beta$, is absorbing. First we show that once $\|u(t_0)\|^3 \le 2\varrho$ for some $t_0 \ge 0$ then $\|u(t)\|^3 \le 2\varrho$ for all $t \ge t_0$. Suppose for a contradiction that $\|u(t_0)\|^3 \le 2\varrho$ and $\|u(t_1)\|^3 > 2\varrho$ for some $t_1 > t_0$. Set

$$t_0' := \inf\left\{t \in [t_0, t_1] \;:\; \|u(s)\|^3 \ge 2\varrho \quad \text{for all } s \in [t, t_1]\right\}.$$

Since $s \mapsto \|u(s)\|$ is continuous[8] it follows that $\|u(t_0')\|^3 = 2\varrho$ and $\|u(s)\|^3 \ge 2\varrho$ for all $s \in [t_0', t_1]$ so, using (3.50),

$$\|u(t_1)\|^2 \le \|u(t_0')\|^2 - c(t_1 - t_0')\|f\|^{4/3} < \left\|u(t_0')\right\|^2 = (2\varrho)^{2/3},$$

a contradiction.

[8]Recall that the L^2-continuity follows from the energy equality, see Theorem 3.8 in Section 3.4.

Now suppose that $\|u(0)\|^3 > 2\varrho$ and set

$$T := \sup \left\{ t \geq 0 \ : \ \|u(s)\|^3 \geq \tfrac{3}{2}\varrho \quad \text{for every } s \in [0,t] \right\}.$$

Since $s \mapsto \|u(s)\|$ is continuous it follows that either $T = \infty$ or $T < \infty$. Since $\|u(s)\|^3 \geq \frac{3}{2}\varrho$ for all $s \in [0,T]$ it follows from (3.50) that

$$\|u(t)\|^2 \leq \|u(0)\|^2 - ct\|f\|^{4/3} \quad \text{for every } t \in [0,T],$$

and so we must have

$$T \leq \frac{\|u_0\|^2}{c\|f\|^{4/3}} \quad \text{with} \quad \|u(T)\|^3 = \tfrac{3}{2}\varrho.$$

Consequently, the proposition is proved with $R := (2\varrho)^{1/3}$, i.e. we have

$$\|u(t)\| \leq R \quad \text{for every } t \geq T(\|u_0\|) := \frac{\|u_0\|^2}{c\,\|f\|^{4/3}}$$

as required. □

The first part of the proof of the Proposition 3.20 shows that once a weak solution u is in the ball $B_s(0,R) \subset H$ then it stays in it for all time. The second part shows that if u is not initially in the ball $B_s(0,R)$ then it enters it eventually.

Now we let X be a closed absorbing ball for the critical CBF equations (3.49),

$$X := \{u \in H \ : \ \|u\| \leq R\},$$

which is also weakly compact. Then for any bounded set $A \subset H$ there exists[9] a time t_0, such that

$$u(t) \in X \quad \text{for all} \quad t \geq t_0,$$

for every weak solution $u(t)$ with the initial condition $u(0) = u_0 \in A$.

We have shown that all weak solutions of the critical CBF equations on the torus $\mathbb{T}^3$ satisfy the energy equality. Therefore, we can consider an evolutionary system for which a family of trajectories consists of all weak solutions (instead of all Leray–Hopf weak solutions as in Cheskidov [2009]) of the critical convective Brinkman–Forchheimer equations in X. More precisely, we define

[9] This is not true for Leray–Hopf weak solutions of the Navier–Stokes equations as defined in Cheskidov [2009]; see the discussion at the beginning of Section 3.5.

$$\begin{aligned}
\mathcal{E}([T,\infty)) &:= \{u(\cdot)\ :\ u(\cdot) \text{ is a weak solution on } [T,\infty) \text{ and } u(t)\in X\ \forall\, t\in[T,\infty)\},\\
&\quad \text{for } T\in\mathbb{R}, \text{ and}\\
\mathcal{E}(I_\infty) &:= \{u(\cdot)\ :\ u(\cdot) \text{ is a weak solution on } I_\infty \text{ and } u(t)\in X\ \forall\, t\in I_\infty\},\\
&\quad \text{where } I_\infty := (-\infty,\infty).
\end{aligned}$$

Clearly, the properties 1–4 of an evolutionary system $\mathcal{E}$ hold. Therefore, thanks to Theorem 3.15, the weak global attractor $\mathcal{A}_w$ exists for this evolutionary system. Additionally, we can prove the following theorem.

Theorem 3.21. *The weak global attractor $\mathcal{A}_w$ for the evolutionary system $\mathcal{E}$ of the critical CBF equations is a strong global attractor $\mathcal{A}_s$, compact in the strong topology.*

Proof. Since every complete trajectory of the evolutionary system $\mathcal{E}$ for the critical CBF equations is strongly continuous, due to Theorem 3.17 and Theorem 3.18, it is enough to prove that $\mathcal{E}$ satisfies the assumptions (A), (B), and (C).

First note that $\mathcal{E}([0,\infty)) \subset C([0,\infty);H_w)$ by the definition of weak solutions, see (3.48). Now take any sequence $u_n \in \mathcal{E}([0,\infty))$ for $n = 1,2,\ldots.$ Thanks to classical estimates for Leray–Hopf weak solutions of the NSE (Lemma 8.5 in Cheskidov [2009], for more details see Constantin and Foias [1988], for example), which apply also to the CBF equations, there exists a subsequence, still denoted by u_n, that converges to some $u^1 \in \mathcal{E}([0,\infty))$ in $C([0,1];H_w)$ as $n\to\infty$. Passing to a subsequence and dropping a subindex once more, we obtain that $u_n \to u^2$ in $C([0,2];H_w)$ as $n\to\infty$ for some $u^2 \in \mathcal{E}([0,\infty))$. Note that $u^1(t) = u^2(t)$ on $[0,1]$. Continuing this diagonalisation procedure, we obtain a subsequence $\{u_{n_j}\} \subset \{u_n\}$ that converges to some $u \in \mathcal{E}([0,\infty))$ in $C([0,\infty);H_w)$ as $n_j\to\infty$. Therefore, (A) holds.

The energy inequality (B) follows immediately from the energy equality (3.36) [cf. the proof of Lemma 8.6 in Cheskidov [2009] in the Navier–Stokes case].

Let now $u_n, u \in \mathcal{E}([0,T])$ be such that $u_n \to u$ in $C([0,\infty);H_w)$ as $n\to\infty$, for some $T>0$. Classical estimates for the NSE (see e.g. Constantin and Foias [1988] or Robinson et al. [2016]), which hold as well for the CBF equations, imply that the sequence $\{\partial_t u_n\}$ is bounded in $L^{4/3}(0,T;V')$, where V' is the dual space of $V = H^1\cap H$. Since the sequence $\{u_n\}$ is bounded in $L^2(0,T;V)$, by the Aubin–Lions Lemma, there exists a subsequence $\{u_{n_j}\} \subset \{u_n\}$, such that

$$\int_0^T \left\|u_{n_j}(t) - u(t)\right\|^2 \mathrm{d}t \to 0 \quad \text{as} \quad n_j\to\infty.$$

In particular, $\left\|u_{n_j}(t)\right\| \to \|u(t)\|$ as $n_j \to \infty$ for a.e. $t \in [0, T]$, i.e. (C) holds. $\square$

Finally, we note that all the other results from Cheskidov [2009] apply to the critical three-dimensional convective Brinkman–Forchheimer equations (3.49) as well. For instance, the *trajectory attractor* $\mathcal{U}$ exists for the critical CBF equations, and uniformly attracts $\mathcal{E}([0, \infty))$ in $C_{loc}((0, \infty); H)$.

Chapter 4

Strong solutions

In this chapter we consider strong solutions of the convective Brinkman–Forchheimer equations on the torus $\mathbb{T}^3$. The main result of this chapter is the existence of global-in-time strong solutions. We prove two results of this kind depending on the absorption exponent r. We have global regular solutions either for $r > 3$, or for $r = 3$ provided that $4\mu\beta \geq 1$, i.e. the product of the viscosity and porosity is not too small. As we mentioned in Chapter 1, there is no similar result available for the Navier–Stokes equations. In fact, whether such solutions for the NSE exist (or not), is the content of one of the seven Clay Millennium Prize Problems. The CBF equations are better in this regard, at least when the absorption is strong enough. These two results formed part of Hajduk and Robinson [2017]. Another important result, which we establish in this chapter, is the uniqueness of strong solutions in the class of weak solutions. This is of course a stronger result than just uniqueness of strong solutions, and can be found in Hajduk et al. [2019].

In this chapter we will show only formal calculations, which can be made rigorous by the use of a standard Galerkin approximation argument as in Chapter 3, see also Constantin and Foias [1988], Temam [1995], or Galdi [2000], for examples.

We now give a short motivation for the definition of a strong solution based on the energy method. Formal calculations for the unforced Brinkman–Forchheimer equations (3.1) give (apply the Leray projection $\mathbb{P}$ to the equation, multiply it by Au and integrate over the spatial domain)

$$\langle \partial_t u,\, Au \rangle + \mu \langle Au,\, Au \rangle + \langle B(u),\, Au \rangle + \beta \langle C_r(u),\, Au \rangle = 0,$$

where $B(u,v) := \mathbb{P}(u \cdot \nabla)v$ and $B(u) := B(u,u)$ for $u, v \in V$. Therefore, after an integration by parts, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u\right\|^2 + \mu\left\|Au\right\|^2 + \langle B(u),\, Au\rangle + \beta\,\langle C_r(u),\, Au\rangle = 0.$$

Integration over the time interval $[0,T]$ gives

$$\frac{1}{2}\left(\left\|\nabla u(T)\right\|^2 - \left\|\nabla u(0)\right\|^2\right) + \mu\int_0^T \left\|Au(t)\right\|^2\,\mathrm{d}t + \int_0^T \langle B(u),\, Au\rangle\,\mathrm{d}t$$
$$+\,\beta\int_0^T \langle C_r(u),\, Au\rangle\,\mathrm{d}t = 0.$$

Hence, we get the *second energy inequality* for the CBF equations

$$\frac{1}{2}\sup_{t\in[0,T]}\left\|\nabla u(t)\right\|^2 + \mu\int_0^T \left\|Au(t)\right\|^2\,\mathrm{d}t + \int_0^T \langle B(u),\, Au\rangle\,\mathrm{d}t$$
$$+\,\beta\int_0^T \langle C_r(u),\, Au\rangle\,\mathrm{d}t \le \frac{1}{2}\left\|\nabla u(0)\right\|^2.$$

Then, for an initial condition $u_0 \in V$, it follows that the norms

$$\|u\|_{L^\infty(0,T;H^1)} \quad \text{and} \quad \|u\|_{L^2(0,T;H^2)}$$

are bounded.

We now define strong solutions for the convective Brinkman–Forchheimer equations, based on the above considerations.

Definition 4.1 (Strong solution for CBF)**.** *We say that a vector field u is a* strong solution *for the convective Brinkman–Forchheimer equations* (3.1) *corresponding to an initial condition $u_0 \in V$, if it is a weak solution and additionally it possesses higher regularity, i.e.*

$$u \in L^\infty(0,T;H^1)\cap L^2(0,T;H^2).$$

4.1 Local existence of strong solutions

In this section we will prove the local-in-time existence of strong solutions for the unforced convective Brinkman–Forchheimer equations

$$\partial_t u - \mu\Delta u + (u\cdot\nabla)u + \nabla p + \beta\,|u|^{r-1}\,u = 0, \qquad \operatorname{div} u = 0. \tag{4.1}$$

Applying the Leray projection $\mathbb{P}$ to the equations (4.1) we obtain their functional form

$$\partial_t u + \mu A u + B(u) + \beta C_r(u) = 0. \tag{4.2}$$

Theorem 4.2. *For every initial condition $u_0 \in V(\mathbb{T}^3)$ there exists a time $T > 0$ such that a Leray–Hopf weak solution u starting from u_0 is a strong solution of the convective Brinkman–Forchheimer equations (4.1) on the time interval $[0, T]$. Additionally u satisfies the bound*

$$\int_0^T \left(\int_{\mathbb{T}^3} |\nabla u(t)|^2 |u(t)|^{r-1} \, \mathrm{d}x \right) \mathrm{d}t < \infty. \tag{4.3}$$

It follows that u belongs to the spaces

$$L^{r+1}(0, T; L^{3(r+1)}_\sigma(\mathbb{T}^3)) \quad \text{and} \quad L^{r+1}(0, T; \mathcal{N}^{2/(r+1), r+1}(\mathbb{T}^3)), \tag{4.4}$$

where $\mathcal{N}^{2/(r+1), r+1}$ is the Nikol'skiĭ space (as defined in Section 2.2).

Proof. Let u be a global Leray–Hopf weak solution of (4.1) starting from $u_0 \in V$. Multiplying formally (4.2) by Au and integrating over the domain $\mathbb{T}^3$, we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla u\|^2 + \mu \|Au\|^2 + \beta \langle C_r(u), \, Au \rangle \le |\langle B(u), \, Au \rangle| \, . \tag{4.5}$$

First, we estimate the convective term $\langle B(u), \, Au\rangle$ using the Hölder, Sobolev and Young inequalities and also Lemma 2.6

$$\begin{aligned} |\langle B(u), \, Au \rangle| &\le \int_{\mathbb{T}^3} |u| \, |\nabla u| \, |Au| \, \mathrm{d}x \le \|u\|_{L^6} \|\nabla u\|_{L^3} \|Au\|_{L^2} \\ &\le c \|u\|_{H^1} \|\nabla u\|_{L^2}^{1/2} \|\nabla u\|_{L^6}^{1/2} \|Au\| \\ &\le c \|u\|_{H^1}^{3/2} \|Au\|^{3/2} \le c \|u\|_{H^1}^6 + \frac{\mu}{2} \|Au\|^2 \, , \end{aligned} \tag{4.6}$$

where the constant c depends on μ.

We recall that on the torus the operators $-\Delta$ and $\mathbb{P}$ commute, so, using Lemma 2.4, we have the inequality (2.4)

$$\langle C_r(u), \, Au \rangle = \left\langle |u|^{r-1} u, \, -\Delta u \right\rangle \ge \int_{\mathbb{T}^3} |\nabla u|^2 |u|^{r-1} \, \mathrm{d}x \ge 0 \quad \text{for} \quad r \ge 1.$$

Using this fact and also the estimate (4.6) we obtain from (4.5) a differential inequality

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla u\|^2 + \frac{\mu}{2} \|Au\|^2 + \beta \int_{\mathbb{T}^3} |\nabla u|^2 |u|^{r-1} \, \mathrm{d}x \le c \|u\|_{H^1}^6 \, .$$

Noting that u satisfies also (multiplying (4.2) by u; see Chapter 3)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \mu\|\nabla u\|^2 + \beta\|u\|_{L^{r+1}}^{r+1} \le 0, \tag{4.7}$$

we have (adding the above inequalities and dropping some terms on the left-hand side)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^1}^2 + \frac{\mu}{2}\|Au\|^2 + \beta\int_{\mathbb{T}^3}|\nabla u|^2|u|^{r-1}\,\mathrm{d}x \le c\|u\|_{H^1}^6.$$

By setting $X(t) := \|u(t)\|_{H^1}^2$, we rewrite the above in the form

$$X' + \mu\|Au\|^2 + 2\beta\int_{\mathbb{T}^3}|\nabla u|^2|u|^{r-1}\,\mathrm{d}x \le cX^3, \tag{4.8}$$

from which we obtain the differential problem

$$\begin{cases} X' \le cX^3, \\ X(0) = \|u_0\|_{H^1}^2. \end{cases} \tag{4.9}$$

We would obtain the same differential inequality by following the above procedure for the Navier–Stokes equations (details for the NSE case can be found for example in Robinson et al. [2016]). We can conclude that X is no greater than the solution of (4.9) turned into a differential equation instead of a differential inequality. The solution of this ODE blows up in finite time $\tilde{T} = [2cX(0)^2]^{-1}$. Therefore, for $0 \le t \le \tilde{T}/2$

$$\|u(t)\|_{H^1}^2 = X(t) \le \frac{\|u_0\|_{H^1}^2}{\sqrt{1 - 2c\|u_0\|_{H^1}^4 t}} \le c\|u_0\|_{H^1}^2.$$

Using this bound and integrating (4.8) over the time interval $[0, t]$ we obtain

$$\|u(t)\|_{H^1}^2 + \mu\int_0^t\|Au(s)\|^2\,\mathrm{d}s + 2\beta\int_0^t\left(\int_{\mathbb{T}^3}|\nabla u(s)|^2|u(s)|^{r-1}\,\mathrm{d}x\right)\mathrm{d}s < \infty.$$

Hence, we can conclude the proof with $T := \left(4c\|u_0\|_{H^1}^4\right)^{-1}$.

Since we have $\int_0^T\left(\int_{\mathbb{T}^3}|\nabla u(s)|^2|u(s)|^{r-1}\,\mathrm{d}x\right)\mathrm{d}s < \infty$, it follows from the discussion in Section 2.2 [see the bound (2.6)] that $u(s)$ belongs to the Nikol'skiĭ space $\mathcal{N}^{2/(r+1),r+1}(\mathbb{T}^3)$ for a.e. $s \in [0, T]$; the fact that $u \in L^{r+1}(0, T; L_\sigma^{3(r+1)}(\mathbb{T}^3))$ follows from Lemma 2.5. □

Theorem 4.2 tells us that the time of existence of strong solutions of the unforced CBF equations (4.1) can be bounded below in terms of the initial condition

$$T \gtrsim \|u_0\|_{H^1}^{-4}.$$

We recall that we have the same situation for strong solutions of the Navier–Stokes equations. However, for the CBF equations we get the additional bound (4.3). This bound furnishes additional regularity for strong solutions (4.4).

4.2 Global existence for small initial data

In this section we show another result analogous to the classical theorem for the Navier–Stokes equations. Namely, we prove global existence of strong solutions to the CBF equations, subject to small initial data.

Theorem 4.3 (Global existence for small initial data)**.** *There exists a constant* $c > 0$*, such that, if*

$$\|u_0\|_{H^1} \leq c\mu,$$

then a strong solution of the convective Brinkman–Forchheimer equations (4.1) *with* $r \geq 1$*, starting from the initial condition* $u_0 \in V(\mathbb{T}^3)$*, exists for all times* $t \geq 0$.

Again, we present here only the sketch of the proof, which can be made rigorous by a standard approximation argument.

Proof. Taking the inner product of (4.2) with Au, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + \mu\|Au\|^2 + \langle B(u),\, Au\rangle + \beta\langle C_r(u),\, Au\rangle = 0. \tag{4.10}$$

From Lemma 2.4

$$\langle C_r(u),\, Au\rangle \geq 0 \quad \text{for} \quad r \geq 1,$$

so we can drop this term on the left-hand side of (4.10).

Furthermore, we recall the estimate (4.6)

$$|\langle B(u),\, Au\rangle| \leq c\|\nabla u\|^{3/2}\|Au\|^{3/2}.$$

Joining these two facts, we obtain from (4.10)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + \mu\|Au\|^2 \leq c\|\nabla u\|^{3/2}\|Au\|^{3/2} = c\|\nabla u\|\,\|\nabla u\|^{1/2}\|Au\|^{3/2}. \tag{4.11}$$

We observe that $\|\nabla u\| \leq c\|Au\|$. Applying this on the right-hand side of (4.11), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u\right\|^2+\mu\left\|Au\right\|^2\leq c\left\|\nabla u\right\|\left\|Au\right\|^2.$$

Now, we can move the L^2-norm of Au from the left to the right-hand side and get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u\right\|^2\leq 2\left\|Au\right\|^2\left(c\left\|\nabla u\right\|-\mu\right). \tag{4.12}$$

Provided that we start from sufficiently small initial data

$$\left\|\nabla u_0\right\|\leq\frac{\mu}{c},$$

we obtain from (4.12) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u\right\|^2\leq 0,$$

which means that the L^2-norm of ∇u is a non-increasing function of time. Hence, it is bounded for all times $t\geq 0$ and our strong solution u does not blow up (i.e. it is global in time). □

4.3 Global existence for $r>3$

Now we will provide a simple proof of the global-in-time existence of strong solutions for the convective Brinkman–Forchheimer equations in the case $r>3$. This result was given in Kalantarov and Zelik [2012] for a broader class of nonlinearities on bounded domains $\Omega\subset\mathbb{R}^3$ and for more regular initial conditions $u_0\in H^2(\Omega)$, where the proof was based on a nonlinear localisation technique.

Theorem 4.4. *For every initial condition $u_0\in V(\mathbb{T}^3)$ and for every exponent $r>3$, there exists a global-in-time strong solution of the CBF equations* (4.1) *on the torus $\mathbb{T}^3$. Moreover, this solution belongs to the spaces in* (4.4)

$$L^{r+1}(0,T;L^{3(r+1)}_\sigma(\mathbb{T}^3))\quad\textit{and}\quad L^{r+1}(0,T;\mathcal{N}^{2/(r+1),r+1}(\mathbb{T}^3)),$$

for all $T>0$, where $\mathcal{N}^{2/(r+1),r+1}$ is the Nikol'skiĭ space (see Section 2.2*).*

We present here only formal calculations which can be justified rigorously via a Galerkin approximation argument.

Proof. Taking the inner product of (4.2) with Au, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u\right\|^2+\mu\left\|Au\right\|^2+\beta\left\langle C_r(u),\,Au\right\rangle+\left\langle (u\cdot\nabla)u,\,Au\right\rangle=0.$$

Using Lemma 2.4 we note that

$$\langle C_r(u),\, Au\rangle = \left\langle |u|^{r-1}\, u,\, -\Delta u \right\rangle \geq \int_{\mathbb{T}^3} |u|^{r-1}\, |\nabla u|^2 \,\mathrm{d}x.$$

This gives us

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\nabla u\|^2 + \mu\,\|Au\|^2 + \beta \int_{\mathbb{T}^3} |u|^{r-1}\,|\nabla u|^2\,\mathrm{d}x &\leq \int_{\mathbb{T}^3} |u|\,|\nabla u|\,|Au|\,\mathrm{d}x \\ &\leq \frac{1}{2}\left(\frac{1}{\mu}\int_{\mathbb{T}^3} |u|^2\,|\nabla u|^2\,\mathrm{d}x + \mu\int_{\mathbb{T}^3} |Au|^2\,\mathrm{d}x\right)\end{aligned}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\nabla u\|^2 + \mu\,\|Au\|^2 + 2\beta \int_{\mathbb{T}^3} |u|^{r-1}\,|\nabla u|^2\,\mathrm{d}x \leq \frac{1}{\mu}\int_{\mathbb{T}^3} |u|^2\,|\nabla u|^2\,\mathrm{d}x. \tag{4.13}$$

Now we observe the following estimate for $r > 3$:

$$\begin{aligned}\int_{\mathbb{T}^3} |u|^2\,|\nabla u|^2\,\mathrm{d}x &= \int_{\mathbb{T}^3} \left(|u|^2\,|\nabla u|^{4/(r-1)}\right)\left(|\nabla u|^{2(r-3)/(r-1)}\right)\mathrm{d}x \\ &\leq \left(\int_{\mathbb{T}^3} |u|^{r-1}\,|\nabla u|^2\,\mathrm{d}x\right)^{2/(r-1)}\left(\int_{\mathbb{T}^3} |\nabla u|^2\,\mathrm{d}x\right)^{(r-3)/(r-1)} \\ &\leq \beta\mu\left(\int_{\mathbb{T}^3} |u|^{r-1}\,|\nabla u|^2\,\mathrm{d}x\right) + c(\beta,\mu,r)\left(\int_{\mathbb{T}^3} |\nabla u|^2\,\mathrm{d}x\right). \end{aligned}\tag{4.14}$$

In the above we used Hölder's and Young's inequalities with the same exponents $(r-1)/2$ and $(r-1)/(r-3)$. The value of the constant $c(\beta,\mu,r)$ can be computed explicitly

$$c(\beta,\mu,r) = \left(\frac{2}{\beta\mu(r-1)}\right)^{2/(r-3)}\left(\frac{r-3}{r-1}\right).$$

Plugging the estimate (4.14) into (4.13) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\nabla u\|^2 + \mu\,\|Au\|^2 + \beta \int_{\mathbb{T}^3} |u|^{r-1}\,|\nabla u|^2\,\mathrm{d}x \leq \frac{c(\beta,\mu,r)}{\mu}\,\|\nabla u\|^2. \tag{4.15}$$

In particular, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\nabla u\|^2 \leq \frac{c(\beta,\mu,r)}{\mu}\,\|\nabla u\|^2.$$

An application of Gronwall's Lemma yields that $\|\nabla u\|^2$ stays bounded on arbitrarily large time intervals $[0,T]$. We observe that u also satisfies [as in (4.7)]

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \mu\|\nabla u\|^2 + \beta\|u\|_{L^{r+1}}^{r+1} \leq 0.$$

Combining this with (4.15) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^1}^2 + \mu\left(\|\nabla u\|^2 + \|Au\|^2\right) + \beta\left(\int_{\mathbb{T}^3}|u|^{r-1}|\nabla u|^2\,\mathrm{d}x + \|u\|_{L^{r+1}}^{r+1}\right) \leq \frac{c(\beta,\mu,r)}{\mu}\|\nabla u\|^2 < \infty. \tag{4.16}$$

It follows in particular, that $u \in L^\infty(0,T;H^1)$. Then one infers from (4.15) that $\int_0^T \|Au\|^2 < \infty$. Therefore, u is indeed a strong solution on the time interval $[0,T]$ for all $T > 0$.

Additional regularity (4.4) for the function u follows now from the inequality (4.16), Lemma 2.5, and the estimate (2.6). □

4.4 Global existence for coefficients satisfying $4\mu\beta \geq 1$

In this section, we consider the critical case of the convective Brinkman–Forchheimer equations ($r = 3$)

$$\partial_t u - \mu\Delta u + (u\cdot\nabla)u + \beta|u|^2 u + \nabla p = 0, \qquad \operatorname{div} u = 0. \tag{4.17}$$

We prove global-in-time existence of strong solutions of (4.17) for all initial conditions $u_0 \in V$, when the product $\mu\beta$ is not too small. From the point of view of physics this is not a surprising result. It means that when both the viscosity of a fluid and the porosity of a porous medium are large enough, then the corresponding flow stays bounded and regular. What is more interesting is the fact that when the viscosity is small, one can still obtain a regular solution by taking the porosity sufficiently large, and vice versa.

Theorem 4.5. *For every initial condition $u_0 \in V$, there exists a global-in-time strong solution of the critical ($r = 3$) CBF equations* (4.17) *provided that* $4\mu\beta \geq 1$.

We mentioned in Chapter 1 that Zhou [2012] proved global existence of strong solutions in $\mathbb{R}^3$ for $r \geq 3$ and $\mu, \beta = 1$. This result clearly satisfies our condition, since $4\mu\beta = 4$.

Proof. Applying the Leray projection to (4.17) and taking its inner product with Au, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u\right\|^2 + \mu\left\|Au\right\|^2 + \beta\left\langle C_3(u),\, Au\right\rangle \le \left|\left\langle B(u),\, Au\right\rangle\right|.$$

Applying Lemma 2.4, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u\right\|^2 + \mu\left\|Au\right\|^2 + \beta\int_{\mathbb{T}^3}\left|\nabla u\right|^2\left|u\right|^2\mathrm{d}x \le \int_{\mathbb{T}^3}\left|u\right|\left|\nabla u\right|\left|Au\right|\mathrm{d}x. \tag{4.18}$$

We want to estimate the right-hand side in such a way to absorb it with the terms on the left-hand side. Using the Cauchy–Schwarz and Young inequalities we obtain

$$\begin{aligned}\int_{\mathbb{T}^3}\left|u\right|\left|\nabla u\right|\left|Au\right|\mathrm{d}x &\le \left(\int_{\mathbb{T}^3}\left|\nabla u\right|^2\left|u\right|^2\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{T}^3}\left|Au\right|^2\mathrm{d}x\right)^{1/2}\\ &\le \frac{\theta}{2}\int_{\mathbb{T}^3}\left|\nabla u\right|^2\left|u\right|^2\mathrm{d}x + \frac{1}{2\theta}\int_{\mathbb{T}^3}\left|Au\right|^2\mathrm{d}x,\end{aligned}$$

for some positive number $\theta > 0$. We use this estimate in the inequality (4.18) and then move all the terms to the left-hand side to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u\right\|^2 + \left(\mu - \frac{1}{2\theta}\right)\left\|Au\right\|^2 + \left(\beta - \frac{\theta}{2}\right)\int_{\mathbb{T}^3}\left|\nabla u\right|^2\left|u\right|^2\mathrm{d}x \le 0.$$

From the above we see that the norm $\left\|\nabla u(t)\right\|^2$ is not increasing in time, provided that

$$\mu - \frac{1}{2\theta} \ge 0 \quad \text{and} \quad \beta - \frac{\theta}{2} \ge 0 \quad \Longleftrightarrow \quad \mu\beta \ge \frac{1}{4}.$$

Hence, there is no blow-up and the strong solution originating from the initial condition $u_0 \in V$ exists for all times $t > 0$. □

We note that the above argument works only for the critical exponent $r = 3$. For other values of $r \in [1, 3)$ we are not able to balance the exponents in the correct way to absorb the convective term on the left-hand side of (4.18).

4.5 Weak-strong uniqueness

In this section we prove uniqueness of strong solutions of the convective Brinkman–Forchheimer equations for incompressible fluids

$$\partial_t u - \mu\Delta u + (u\cdot\nabla)u + \beta\left|u\right|^{r-1}u + \nabla p = 0, \qquad \operatorname{div} u = 0, \tag{4.19}$$

where $r \in [1,3]$ and the domain is the torus $\mathbb{T}^3$. By *uniqueness*, we mean here uniqueness of strong solutions in the larger class of weak solutions satisfying the energy inequality, which is often called 'weak-strong uniqueness'. Classical uniqueness of strong solutions follows from that result, since every strong solution is by definition a 'more regular' weak solution.

To achieve our goal we need to establish some properties of strong solutions of the CBF equations. We follow here proofs of analogous results for the 3D NSE equations, which can be found in many places, e.g. in Galdi [2000] or Robinson et al. [2016].

4.5.1 Properties of strong solutions

First, we show that due to Definition 4.1 all the terms in the CBF equations are well-defined L^2 functions in the space-time domain.

Lemma 4.6. *Let u be a strong solution of the convective Brinkman–Forchheimer equations with $r \in [1,3]$. Then*

$$\partial_t u, \quad \Delta u, \quad (u \cdot \nabla) u \quad \textit{and} \quad |u|^{r-1} u$$

are all elements of $L^2(0,T;L^2)$.

Proof. We only need to consider the absorption term since the other terms can be dealt with in a similar way as in the analogous result for the Navier–Stokes equations (see Lemma 6.2 in Robinson et al. [2016] for the details). We show that $|u|^{r-1} u$ is square integrable in the space-time domain [equivalently that $u \in L^{2r}(0,T;L^{2r})$] using the nesting of $L^p(\mathbb{T}^3)$ spaces and the Sobolev embedding

$$\begin{aligned} \int_0^T \left\| |u(t)|^{r-1} u(t) \right\|^2 \mathrm{d}t \le \int_0^T \|u(t)\|_{L^{2r}}^{2r} \mathrm{d}t \le c \int_0^T \|u(t)\|_{H^1}^{2r} \mathrm{d}t \\ \le c\, \|u\|_{L^\infty(0,T;H^1)}^{2r} < \infty, \end{aligned} \tag{4.20}$$

for every $r \in [1,3]$. □

Note that we can extend Lemma 4.6 up to $r = 5$. Indeed, by interpolation ($r \ge 3$) and Agmon's inequality in 3D we have

$$\|u\|_{L^{2r}}^{2r} \le \|u\|_{L^6}^{6} \|u\|_{L^\infty}^{2r-6} \le \|u\|_{L^6}^{6} \|u\|_{H^1}^{r-3} \|u\|_{H^2}^{r-3} \le c\, \|u\|_{H^1}^{r+3} \|u\|_{H^2}^{r-3}$$

and therefore we obtain

$$\int_0^T \|u(t)\|_{L^{2r}}^{2r} \,\mathrm{d}t \le c\, \|u\|_{L^\infty(0,T;H^1)}^{r+3} \left(\int_0^T \|u(t)\|_{H^2}^2 \,\mathrm{d}t \right)^{(r-3)/2},$$

which is bounded for any strong solution u. We used Hölder's inequality with exponents $2/(r-3)$ and $2/(5-r)$ in the last inequality. Using this estimate we will in fact be able to prove weak-strong uniqueness for the CBF equations on the 3D torus for any $r \in [1,5]$.

The next result states that for almost all times the Leray projection of the unforced CBF equations (4.19) is equal to zero.

Lemma 4.7. *Let u be a strong solution of the convective Brinkman–Forchheimer equations with $r \in [1,3]$. Then*

$$\int_0^T \left\langle \partial_t u - \mu \Delta u + (u \cdot \nabla) u + \beta \,|u|^{r-1}\, u,\, w \right\rangle \mathrm{d}t = 0 \tag{4.21}$$

for all $w \in L^2(0,T;H)$.

Again, the proof follows the same lines as in the Navier–Stokes case (see Lemma 6.3 in Robinson et al. [2016], for example). We omit it here completely since, due to Lemma 4.6, there are no additional problems caused by the absorption term $|u|^{r-1}\, u$.

The last property which we will need to prove the main result of this section states that a strong solution of the CBF equations (actually any function with the same regularity as a strong solution) can be used as a test function in the weak formulation (3.6).

Lemma 4.8. *Suppose that v is a weak solution of the convective Brinkman–Forchheimer equations with $r \in [1,3]$. If u has the regularity of a strong solution of the CBF equations, that is*

$$u \in L^2(0,T;H^2 \cap V) \cap L^{r+1}(0,T;L_\sigma^{r+1}), \quad \partial_t u \in L^2(0,T;L^2),$$

then for all times $t \in [0,T]$

$$-\int_0^t \langle v,\, \partial_t u \rangle \,\mathrm{d}s + \mu \int_0^t \langle \nabla v,\, \nabla u \rangle \,\mathrm{d}s + \int_0^t \langle (v \cdot \nabla) v,\, u \rangle \,\mathrm{d}s \\ + \beta \int_0^t \left\langle |v|^{r-1}\, v,\, u \right\rangle \mathrm{d}s = \langle v(0),\, u(0) \rangle - \langle v(t),\, u(t) \rangle .$$

In the proof of Lemma 4.8 we need to approximate the function u simultaneously in the Sobolev space H^2 and in the Lebesgue space L^{r+1}. We need an approximation which not only converges in those spaces but which is also uniformly bounded in both of them. We use the truncation of the Fourier series over 'cubes'

$$S_n(u) = \sum_{k \in \mathcal{Q}_n} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x},$$

which was defined in Section 3.4. This kind of approximation was used to show that all weak solutions of the CBF equations with the critical absorption exponent ($r = 3$) satisfy the energy equality in the periodic domain (Theorem 3.8). It follows from Theorem 3.10 that S_n behaves well in the required spaces:

$$\|S_n(u) - u\|_X \to 0 \qquad \text{and} \qquad \|S_n(u)\|_X \le \|u\|_X$$

for $X = H^s(\mathbb{T}^3)$ or $L^p(\mathbb{T}^3)$ for $p \in (1, \infty)$. Approximations with similar properties on bounded domains will be discussed in Chapter 8.

We can now go to the proof of Lemma 4.8.

Proof. For each $t \in [0, T]$ we take

$$u_n(x,t) := S_n(u) = \sum_{|k_j| \le n} \hat{u}_k(t) \mathrm{e}^{\mathrm{i}k\cdot x}.$$

From the preceding discussion we know that the sequence u_n converges in L^2-based spaces and also in $L^{r+1}(\mathbb{T}^3)$ with

$$\|u_n(t)\|_{L^{r+1}} \le c\,\|u(t)\|_{L^{r+1}}$$

for a.e. $t \in [0, T]$, which is the key ingredient in adapting the proof from the Navier–Stokes case (see e.g. Lemma 6.6 in Robinson et al. [2016]). Mollifying u_n in time (see Section 3.4 for details of a similar argument) we obtain a sequence of test functions such that

$$u_n \to u \qquad \text{in} \quad L^2(0,T;H^2), \tag{4.22}$$

$$\partial_t u_n \to \partial_t u \qquad \text{in} \quad L^2(0,T;L^2), \tag{4.23}$$

$$u_n \to u \qquad \text{in} \quad L^{r+1}(0,T;L^{r+1}), \tag{4.24}$$

as $n \to \infty$. We note that $u \in L^2(0,T;H^2)$ and $\partial_t u \in L^2(0,T;L^2)$ implies that $u \in C([0,T];H^1)$ (see e.g. Proposition 1.35 in Robinson et al. [2016]) and hence we also have

$$u_n \to u \quad \text{in} \quad C([0,T]; H^1). \tag{4.25}$$

Since v is a weak solution of the CBF equations, we have

$$\begin{aligned} -\int_0^t \langle v,\, \partial_t u_n \rangle \,\mathrm{d}s + \mu \int_0^t \langle \nabla v,\, \nabla u_n \rangle \,\mathrm{d}s + \int_0^t \langle (v \cdot \nabla) v,\, u_n \rangle \,\mathrm{d}s \\ + \beta \int_0^t \left\langle |v|^{r-1} v,\, u_n \right\rangle \mathrm{d}s = \langle v(0),\, u_n(0) \rangle - \langle v(t),\, u_n(t) \rangle \end{aligned} \tag{4.26}$$

for all $t \in [0,T]$. To prove the lemma it is sufficient to pass to the limit in (4.26).

Passing to the limit in the Navier–Stokes terms is standard and follows from (4.22), (4.23) and (4.25). Therefore, we can focus on the Brinkman–Forchheimer nonlinearity; we note that by standard estimates and (4.24) we have

$$\begin{aligned} \left| \int_0^t \left\langle |v|^{r-1} v,\, u - u_n \right\rangle \mathrm{d}s \right| &\le \int_0^t \int_{\mathbb{T}^3} |v|^r \, |u - u_n| \,\mathrm{d}s \le \int_0^t \|v\|_{L^{r+1}}^r \, \|u - u_n\|_{L^{r+1}} \,\mathrm{d}s \\ &\le \|v\|_{L^{r+1}(0,T;L^{r+1})}^r \, \|u - u_n\|_{L^{r+1}(0,T;L^{r+1})} \to 0 \end{aligned}$$

as $n \to \infty$, which ends the proof. □

4.5.2 Main result

Finally, we can prove the main result of this section; we show that strong solutions are unique in the class of weak solutions satisfying the Energy Inequality (all Leray–Hopf weak solutions, not necessarily constructed via Galerkin approximation method). In the critical case, when $r = 3$ (cubic nonlinearity $|u|^2 u$), since all weak solutions satisfy the Energy Equality (as shown in Section 3.4 for the periodic case; see also Chapter 9 for the proof on bounded domains), this means that strong solutions are unique in the class of all weak solutions.

Theorem 4.9 (Weak-strong uniqueness)**.** *Suppose that u is a strong solution of the convective Brinkman–Forchheimer equations with $r \in [1,3]$ on the time interval $[0,T]$, and that v is any weak solution on $[0,T]$ arising from the same initial condition $v_0 = u_0 \in V$, that satisfies the Energy Inequality*

$$\frac{1}{2} \|v(t)\|^2 + \mu \int_0^t \|\nabla v(s)\|^2 \,\mathrm{d}s + \beta \int_0^t \|v(s)\|_{L^{r+1}}^{r+1} \,\mathrm{d}s \le \frac{1}{2} \|v(0)\|^2$$

for all $t \in [0,T]$. Then $u \equiv v$ on $[0,T]$.

Proof. From Lemma 4.7 and Lemma 4.8 we have for all $t \in [0,T]$

$$\int_0^t \langle \partial_t u,\, v\rangle \,\mathrm{d}s + \mu \int_0^t \langle \nabla u,\, \nabla v\rangle \,\mathrm{d}s + \int_0^t \langle (u\cdot\nabla)u,\, v\rangle \,\mathrm{d}s$$
$$+ \beta \int_0^t \left\langle |u|^{r-1} u,\, v\right\rangle \mathrm{d}s = 0,$$
$$-\int_0^t \langle v,\, \partial_t u\rangle \,\mathrm{d}s + \mu \int_0^t \langle \nabla v,\, \nabla u\rangle \,\mathrm{d}s + \int_0^t \langle (v\cdot\nabla)v,\, u\rangle \,\mathrm{d}s + \beta \int_0^t \left\langle |v|^{r-1} v,\, u\right\rangle \mathrm{d}s$$
$$= \langle v(0),\, u(0)\rangle - \langle v(t),\, u(t)\rangle\,.$$

We add the above equations and obtain

$$2\mu \int_0^t \langle \nabla u,\, \nabla v\rangle \,\mathrm{d}s + \int_0^t \langle (u\cdot\nabla)u,\, v\rangle \,\mathrm{d}s + \int_0^t \langle (v\cdot\nabla)v,\, u\rangle \,\mathrm{d}s$$
$$+\beta \int_0^t \left\langle |u|^{r-1} u,\, v\right\rangle \mathrm{d}s + \beta \int_0^t \left\langle |v|^{r-1} v,\, u\right\rangle \mathrm{d}s = \|u(0)\|^2 - \langle v(t),\, u(t)\rangle\,. \quad (4.27)$$

Our goal now is to obtain an integral inequality for the difference of the solutions $w := v - u$. To this end, we will use the following standard identity

$$\|a-b\|^2 = \|a\|^2 + \|b\|^2 - 2\langle a,\, b\rangle$$

and substitutions to deal with the linear terms. We get

$$2\langle \nabla u,\, \nabla v\rangle = \|\nabla u\|^2 + \|\nabla v\|^2 - \|\nabla w\|^2\,,$$
$$\langle v(t),\, u(t)\rangle = \frac{1}{2}\|u(t)\|^2 + \frac{1}{2}\|v(t)\|^2 - \frac{1}{2}\|w(t)\|^2\,.$$

We use the relation $v = w + u$, and by standard properties of the convective term we obtain

$$\langle (u\cdot\nabla)u,\, v\rangle + \langle (v\cdot\nabla)v,\, u\rangle = \langle (w\cdot\nabla)w,\, u\rangle\,.$$

To deal with the absorption terms we use two different substitutions

$$\left\langle |u|^{r-1} u,\, v\right\rangle + \left\langle |v|^{r-1} v,\, u\right\rangle = \left\langle |u|^{r-1} u,\, w+u\right\rangle + \left\langle |v|^{r-1} v,\, v-w\right\rangle$$
$$= \|u\|_{L^{r+1}}^{r+1} + \|v\|_{L^{r+1}}^{r+1} - \left\langle |u|^{r-1} u - |v|^{r-1} v,\, u-v\right\rangle.$$

Hence, we have from (4.27) the following

$$
\begin{aligned}
&-\mu \int_0^t \|\nabla w\|^2 \,\mathrm{d}s + \mu \int_0^t \|\nabla u\|^2 \,\mathrm{d}s + \mu \int_0^t \|\nabla v\|^2 \,\mathrm{d}s + \int_0^t \langle (w\cdot\nabla)w,\, u\rangle \,\mathrm{d}s \\
&+ \beta \int_0^t \|u\|_{L^{r+1}}^{r+1} \,\mathrm{d}s + \beta \int_0^t \|v\|_{L^{r+1}}^{r+1} \,\mathrm{d}s - \beta \int_0^t \Big\langle |u|^{r-1}\, u - |v|^{r-1}\, v,\, u - v \Big\rangle \,\mathrm{d}s \\
&= \frac{1}{2}\|u(0)\|^2 + \frac{1}{2}\|v(0)\|^2 + \frac{1}{2}\|w(t)\|^2 - \frac{1}{2}\|u(t)\|^2 - \frac{1}{2}\|v(t)\|^2 .
\end{aligned}
$$

After rearranging the terms in the above, we obtain the following equation for the difference w

$$
\frac{1}{2}\|w(t)\|^2 + \mu \int_0^t \|\nabla w\|^2 \,\mathrm{d}s + I_1 = I_2 + I_3 + \int_0^t \langle (w\cdot\nabla)w,\, u\rangle \,\mathrm{d}s, \tag{4.28}
$$

where

$$
\begin{aligned}
I_1 &:= \beta \int_0^t \Big\langle |u(s)|^{r-1}\, u(s) - |v(s)|^{r-1}\, v(s),\, u(s) - v(s) \Big\rangle \,\mathrm{d}s \geq 0, \\
I_2 &:= \frac{1}{2}\|u(t)\|^2 + \mu \int_0^t \|\nabla u(s)\|^2 \,\mathrm{d}s + \beta \int_0^t \|u(s)\|_{L^{r+1}}^{r+1} \,\mathrm{d}s - \frac{1}{2}\|u(0)\|^2 = 0, \\
I_3 &:= \frac{1}{2}\|v(t)\|^2 + \mu \int_0^t \|\nabla v(s)\|^2 \,\mathrm{d}s + \beta \int_0^t \|v(s)\|_{L^{r+1}}^{r+1} \,\mathrm{d}s - \frac{1}{2}\|v(0)\|^2 \leq 0.
\end{aligned}
$$

We employed here the Energy Equality for the strong solution[1] u and the Energy Inequality for the weak solution v, and also the monotonicity of the absorption term (see Lemma 2.2).

Therefore, we have from (4.28)

$$
\frac{1}{2}\|w(t)\|^2 + \mu \int_0^t \|\nabla w(s)\|^2 \,\mathrm{d}s \leq \int_0^t \langle (w(s)\cdot\nabla)w(s),\, u(s)\rangle \,\mathrm{d}s,
$$

which we can estimate in the following way

$$
\begin{aligned}
\frac{1}{2}\|w(t)\|^2 + \mu \int_0^t \|\nabla w\|^2 \,\mathrm{d}s &\leq \left| \int_0^t \langle (w\cdot\nabla)w,\, u\rangle \,\mathrm{d}s \right| \leq \int_0^t \|u\|_{L^\infty} \|w\| \,\|\nabla w\| \,\mathrm{d}s \\
&\leq \frac{1}{2\mu} \int_0^t \|u\|_{L^\infty}^2 \|w\|^2 \,\mathrm{d}s + \frac{\mu}{2} \int_0^t \|\nabla w\|^2 \,\mathrm{d}s \\
&\leq c \int_0^t \|u\|_{H^2}^2 \|w\|^2 \,\mathrm{d}s + \frac{\mu}{2} \int_0^t \|\nabla w\|^2 \,\mathrm{d}s;
\end{aligned}
$$

we used the 3D embedding $H^2 \hookrightarrow L^\infty$ in the last line. Then we have

[1] The fact that strong solutions satisfy the Energy Equality is a simple consequence of Lemma 4.7.

$$\|w(t)\|^2 + \mu \int_0^t \|\nabla w(s)\|^2 \, \mathrm{d}s \leq c \int_0^t \|u(s)\|_{H^2}^2 \|w(s)\|^2 \, \mathrm{d}s$$

and, after dropping the time integral on the left-hand side we obtain

$$\|w(t)\|^2 \leq c \int_0^t \|u(s)\|_{H^2}^2 \|w(s)\|^2 \, \mathrm{d}s.$$

Since u is the strong solution

$$\int_0^t \|u(s)\|_{H^2}^2 \, \mathrm{d}s < \infty \quad \text{for all} \quad t \in [0, T] \,,$$

and so an application of the integral version of the Gronwall Lemma yields that $w(t) = 0$ for all $t \in [0, T]$. □

As a straightforward corollary of Theorem 4.9 we can deduce a weaker result: uniqueness of strong solutions in the class of strong solutions.

Corollary 4.10. *Let u and v be two strong solutions of the convective Brinkman–Forchheimer equations* (4.19) *with $r \geq 0$ on the time interval $[0, T]$, starting from the same initial condition $u_0 \in V$. Then $u \equiv v$ for all times $t \leq T$.*

This result follows from Theorem 4.9 only for the absorption exponents in the range $r \in [1, 3]$; because strong solutions are by definition weak solutions with additional regularity, and they satisfy the energy equality, it suffices to apply Theorem 4.9 to the strong solutions u and v. However, one can prove Corollary 4.10 independently for all exponents $r > 0$, following the proof for the analogous result for the Navier–Stokes equations; the only additional difficulty is in dealing with an extra nonlinear term $C_r(u)$. In this particular case we are able to eliminate the additional nonlinearity from the proof due to its properties. We provide a short sketch of this fact below.

Sketch of the proof. We set $w := u - v$. Then, of course $w(0) = 0$. We subtract the weak formulations of the CBF equations for the functions u and v and obtain an equation for the difference ($\mu, \beta = 1$)

$$-\int_0^t \langle w, \, \partial_t \varphi \rangle \, \mathrm{d}s + \int_0^t \langle \nabla w, \, \nabla \varphi \rangle \, \mathrm{d}s + \int_0^t \langle B(u) - B(v), \, \varphi \rangle \, \mathrm{d}s$$
$$+ \int_0^t \langle C_r(u) - C_r(v), \, \varphi \rangle \, \mathrm{d}s = - \langle w(t), \, \varphi(t) \rangle \,.$$

Using Lemma 4.8, taking as a test function $\varphi := w$, we get

$$\frac{1}{2}\|w(t)\|^2 + \int_0^t \|\nabla w\|^2 \,\mathrm{d}s + \int_0^t \langle C_r(u) - C_r(v),\, w\rangle \,\mathrm{d}s \leq \left| \int_0^t \langle B(u) - B(v),\, w\rangle \,\mathrm{d}s \right|. \tag{4.29}$$

First, we deal with the nonlinearities connected with the operators C_r. We can simply drop this term on the left-hand side of (4.29), since, by monotonicity (Lemma 2.2), we have

$$\langle C_r(u) - C_r(v),\, w\rangle = \left\langle |u|^{r-1} u - |v|^{r-1} v,\, u - v \right\rangle \geq 0 \quad \text{for} \quad r \geq 0.$$

Therefore, we can proceed as in the Navier–Stokes case to finish the proof. □

Chapter 5

Robustness of regularity

In this chapter we deal with the so-called '*robustness of regularity*' result for the solutions of the convective Brinkman–Forchheimer equations on the torus $\mathbb{T}^3$

$$\partial_t u - \mu\Delta u + (u \cdot \nabla)u + \beta\,|u|^{r-1}\,u + \nabla p = 0, \qquad \operatorname{div} u = 0, \tag{5.1}$$

with the absorption exponent $r \in [1,3]$. It generalises the result obtained in Dashti and Robinson [2008] for the Navier–Stokes equations (see also Chernyshenko et al. [2007]). This result formed a significant part of Hajduk et al. [2019]. The local existence of strong solutions and some properties of strong solutions proved in Chapter 4 can be seen as prerequisites for this result.

We take u_0, $v_0 \in V$ and fix $T > 0$. Let u be a strong solution of the CBF equations on the time interval $[0,T]$ with external forces f and initial condition u_0. Similarly, let v be a strong solution of the CBF equations on $[0,T']$ for some $T' < T$, with external forces g and initial condition v_0. We will show that there is an explicit condition, depending only on initial data and on the function u, which allows us to extend the function v to a strong solution on the time interval $[0,T]$.

We consider the following system of equations ($\mu, \beta = 1$)

$$\begin{cases} \partial_t u + Au + B(u) + C_r(u) = f, & u(x,0) = u_0, \\ \partial_t v + Av + B(v) + C_r(v) = g, & v(x,0) = v_0. \end{cases}$$

We denote the difference of solutions by $w := u - v$. Subtracting the above equations we obtain the equation for w

$$\partial_t w + Aw + B(u) - B(v) + C_r(u) - C_r(v) = f - g, \tag{5.2}$$

with the initial condition

$$w(x,0) = u_0 - v_0.$$

In the proof of the robustness of regularity for the above equations, the following simple ODE lemma will be extremely useful. It will allow us to estimate the time of existence for solutions of certain differential inequalities in terms of coefficients of a corresponding differential equation.

Lemma 5.1. *Let $T > 0$, $a > 0$ and $n \in \mathbb{N}$ ($n > 1$). Let $\delta(t)$ be a nonnegative, continuous function on the interval $[0,T]$. Let also y be a nonnegative function, satisfying the following differential inequality*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} y \leq a y^n + \delta(t), \\ y(0) = y_0 \geq 0. \end{cases}$$

We define the quantity

$$\eta := y_0 + \int_0^T \delta(t)\,\mathrm{d}t.$$

If the following condition is satisfied

$$\eta < \frac{1}{[(n-1)aT]^{1/(n-1)}},$$

then

1. *$y(t)$ stays bounded on the interval $[0,T]$, and*
2. *$y(t) \to 0$, as $\eta \to 0$, uniformly on $[0,T]$.*

For the proof of Lemma 5.1 see e.g. Constantin [1986], Dashti and Robinson [2008] or Robinson et al. [2016].

5.1 A priori estimates

Taking into account that u is a strong solution on the time interval $[0,T]$ and that we want to say the same about v, we need to change the form of equation (5.2) to eliminate the unknown function v. From the definition $v = u - w$, so due to the bilinearity of the form B we have the identity

$$B(u) - B(v) = B(u) - B(u-w) = B(u,u) - B(u-w, u-w)$$

$$
\begin{aligned}
&= B(u,u) - B(u,u) - B(w,w) + B(u,w) + B(w,u) \\
&= B(u,w) + B(w,u) - B(w,w).
\end{aligned}
$$

Multiplying now both sides of (5.2) by Aw (we assume here that the function w has sufficient regularity to justify these operations; in our argument in Section 5.2 u and v will be strong solutions on a common time interval, which will allow us to use these estimates) and integrating over the spatial domain, we get

$$
\begin{aligned}
&\langle \partial_t w,\, Aw\rangle + \langle Aw,\, Aw\rangle + \langle B(u) - B(v),\, Aw\rangle + \langle C_r(u) - C_r(v),\, Aw\rangle \\
&\qquad = \langle f - g,\, Aw\rangle .
\end{aligned}
$$

After integration by parts, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2 + \|Aw\|^2 &\le |\langle f - g,\, Aw\rangle| + |\langle B(u,w) + B(w,u) - B(w,w),\, Aw\rangle| \\
&\quad + |\langle C_r(u) - C_r(v),\, Aw\rangle| . \qquad (5.3)
\end{aligned}
$$

We will now estimate all the terms on the right-hand side of the inequality (5.3). Using standard estimates for the bilinear form B (cf. Constantin and Foias [1988] or Dashti and Robinson [2008]) and Lemma 2.6, we can estimate all the terms coming from the Navier–Stokes equations (see also Chapter 9.1 in Robinson et al. [2016]).

We have

$$
\begin{aligned}
|\langle f - g,\, Aw\rangle| \le \langle |f - g|,\, |Aw|\rangle &\le \|f - g\|\, \|Aw\| \\
&\le c\,\|f - g\|^2 + \frac{1}{16}\|Aw\|^2 , \qquad (5.4)
\end{aligned}
$$

$$
\begin{aligned}
|\langle B(u,w),\, Aw\rangle| &\le \langle |u|\,|\nabla w|,\, |Aw|\rangle \le \|u\|_{L^6}\,\|\nabla w\|_{L^3}\,\|Aw\| \\
&\le c\,\|u\|_{H^1}\,\|\nabla w\|^{1/2}\,\|\nabla w\|_{L^6}^{1/2}\,\|Aw\| \\
&\le c\,\|u\|_{H^1}\,\|w\|_{H^1}^{1/2}\,\|Aw\|^{3/2} \\
&\le c\,\|u\|_{H^1}^4\,\|w\|_{H^1}^2 + \frac{1}{16}\|Aw\|^2 , \qquad (5.5)
\end{aligned}
$$

$$
\begin{aligned}
|\langle B(w,u),\, Aw\rangle| &\le \langle |w|\,|\nabla u|,\, |Aw|\rangle \le \|w\|_{L^6}\,\|\nabla u\|_{L^3}\,\|Aw\| \\
&\le c\,\|w\|_{H^1}\,\|\nabla u\|^{1/2}\,\|\nabla u\|_{L^6}^{1/2}\,\|Aw\| \\
&\le c\,\|w\|_{H^1}\,\|\nabla u\|^{1/2}\,\|Au\|^{1/2}\,\|Aw\|
\end{aligned}
$$

$$\leq c\,\|w\|_{H^1}^2\,\|\nabla u\|\,\|Au\| + \frac{1}{16}\,\|Aw\|^2\,, \tag{5.6}$$

$$\begin{aligned}
|\langle -B(w,w),\, Aw\rangle| &\leq \langle |w|\,|\nabla w|,\, |Aw|\rangle \leq \|w\|_{L^6}\,\|\nabla w\|_{L^3}\,\|Aw\| \\
&\leq c\,\|w\|_{H^1}\,\|\nabla w\|^{1/2}\,\|\nabla w\|_{L^6}^{1/2}\,\|Aw\| \\
&\leq c\,\|w\|_{H^1}^{3/2}\,\|Aw\|^{3/2} \\
&\leq c\,\|w\|_{H^1}^{6} + \frac{1}{16}\,\|Aw\|^2\,. \qquad (5.7)
\end{aligned}$$

Summing (5.4)-(5.7) we get

$$\begin{aligned}
|\langle f-g,\, Aw\rangle| + |\langle B(u,w)+B(w,u)-B(w,w),\, Aw\rangle| \leq c\,\|f-g\|^2 \\
+ c\left(\|u\|_{H^1}^4 + \|\nabla u\|\,\|Au\|\right)\|w\|_{H^1}^2 + c\,\|w\|_{H^1}^6 + \frac{1}{4}\,\|Aw\|^2\,. \qquad (5.8)
\end{aligned}$$

We obtain the full H^1-norm of the difference w on the right-hand side of (5.8) and there is only the L^2-norm of the gradient of w on the left-hand side of (5.3). To circumvent that problem we can consider the energy equality for the difference

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|w\|^2 + \|\nabla w\|^2 + \langle B(u)-B(v),\, w\rangle + \langle C_r(u)-C_r(v),\, w\rangle = \langle f-g,\, w\rangle.$$

We note again that $\langle C_r(u)-C_r(v),\, w\rangle \geq 0$ and substituting $v = u - w$ we get

$$\begin{aligned}
\langle B(u)-B(v),\, w\rangle = \langle B(u,u)-B(u-w,u-w),\, w\rangle &= \langle B(u,u),\, w\rangle - \langle B(u,u),\, w\rangle \\
&+ \langle B(u,w),\, w\rangle + \langle B(w,u),\, w\rangle - \langle B(w,w),\, w\rangle \\
&= \langle B(w,u),\, w\rangle\,.
\end{aligned}$$

We used here the fact that $\langle (u\cdot\nabla)v,\, w\rangle = -\,\langle (u\cdot\nabla)w,\, v\rangle$ for $u \in V$ and $v, w \in H^1$.

Therefore, we obtain from the energy equality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|w\|^2 + \|\nabla w\|^2 \leq |\langle f-g,\, w\rangle| + |\langle B(w,u),\, w\rangle|\,.$$

Estimating the nonlinear term gives

$$\begin{aligned}
|\langle B(w,u),\, w\rangle| &\leq \langle |w|\,|\nabla u|,\, |w|\rangle \leq \|w\|_{L^4}^2\,\|\nabla u\| \leq \|w\|^{1/2}\,\|w\|_{L^6}^{3/2}\,\|\nabla u\| \\
&\leq c\,\|w\|_{H^1}^2\,\|\nabla u\|\,,
\end{aligned}$$

from which we conclude

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \|\nabla w\|^2 \le c\|f-g\|^2 + c\|w\|_{H^1}^2(\|\nabla u\|+1). \tag{5.9}$$

To estimate the additional nonlinear terms in (5.3) connected with the operator C_r we use Lemma 2.3

$$|C_r(u)-C_r(v)| \le \left(2^{r-2}r\right)\left(|u|^{r-1}|w|+|w|^r\right) \quad \text{for} \quad r\ge 1,$$

which gives

$$|\langle C_r(u)-C_r(v),\, Aw\rangle| \le \left(2^{r-2}r\right)\left[\left\langle |u|^{r-1}|w|,\, |Aw|\right\rangle + \langle |w|^r,\, |Aw|\rangle\right]. \tag{5.10}$$

We can estimate the first term in (5.10) using Hölder's inequality with three exponents $6/(r-1), 6/(4-r), 2$ and Sobolev's embedding $H^1 \hookrightarrow L^6$

$$\begin{aligned}\left\langle |u|^{r-1}|w|,\, |Aw|\right\rangle &\le \|u\|_{L^6}^{r-1}\|w\|_{L^{6/(4-r)}}\|Aw\| \\ &\le c\|u\|_{H^1}^{r-1}\|w\|_{H^1}\|Aw\| \\ &\le c\|u\|_{H^1}^{2(r-1)}\|w\|_{H^1}^2 + \frac{1}{8}\|Aw\|^2. \end{aligned} \tag{5.11}$$

Using the same bound for L^{2r}-norm as in (4.20) we estimate the second term on the right-hand side of (5.10)

$$\begin{aligned}\langle |w|^r,\, |Aw|\rangle \le \|w\|_{L^{2r}}^r\|Aw\| &\le c\|w\|_{L^{2r}}^{2r} + \frac{1}{8}\|Aw\|^2 \\ &\le c\|w\|_{H^1}^{2r} + \frac{1}{8}\|Aw\|^2. \end{aligned} \tag{5.12}$$

Combining the inequalities (5.11) and (5.12) yields

$$|\langle C_r(u)-C_r(v),\, Aw\rangle| \le c\|u\|_{H^1}^{2(r-1)}\|w\|_{H^1}^2 + c\|w\|_{H^1}^{2r} + \frac{1}{4}\|Aw\|^2. \tag{5.13}$$

We substitute (5.8) and (5.13) in (5.3) and obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2 + \|Aw\|^2 \le c_0\|f-g\|^2 &+ c_1\left(\|u\|_{H^1}^4 + \|\nabla u\|\|Au\| + \|u\|_{H^1}^{2(r-1)}\right)\|w\|_{H^1}^2 \\ &+ c_r\|w\|_{H^1}^{2r} + c_3\|w\|_{H^1}^6. \end{aligned} \tag{5.14}$$

Finally, we add together (5.9) and (5.14)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{H^1}^2 + 2\|\nabla w\|^2 + \|Aw\|^2 \leq c_0\|f-g\|^2 + c_r\|w\|_{H^1}^{2r} + c_3\|w\|_{H^1}^6$$
$$+ c_1\left(\|u\|_{H^1}^4 + \|\nabla u\|\,\|Au\| + \|u\|_{H^1}^{2(r-1)} + \|\nabla u\| + 1\right)\|w\|_{H^1}^2 . \qquad (5.15)$$

This final estimate will be used in the proof of the 'robustness of regularity' result (Theorem 5.2) in the next section.

5.2 Robustness of regularity

We will now prove the following theorem for the convective Brinkman–Forchheimer equations with $r \in [1,3]$ on the periodic domain $\mathbb{T}^3$.

Theorem 5.2. *Assume that $f, g \in L^2(0,T;H)$ and $u_0, v_0 \in V$. Furthermore, let $u \in L^\infty(0,T;V) \cap L^2(0,T;H^2)$ be the strong solution of the convective Brinkman–Forchheimer equations* (5.1) *on the time interval $[0,T]$, with external forces f and initial condition u_0. If*

$$\|u_0 - v_0\|_{H^1}^2 + c_0\int_0^T \|f(t)-g(t)\|^2\,\mathrm{d}t < R(u), \qquad (5.16)$$

where

$$R(u) := c\frac{\exp(-c_2T)}{\sqrt{T}}\exp\left(-c_1\int_0^T\left(\|u\|_{H^1}^4 + \|\nabla u\|\,\|Au\| + \|u\|_{H^1}^{2(r-1)} + \|\nabla u\|\right)\mathrm{d}t\right),$$

for some positive constants c_0, c_1, c_2, c, then the function v solving the CBF equations (5.1)*, with external forces g and initial condition v_0, is also a strong solution on the time interval $[0,T]$ and have the same regularity as the function u.*

The proof of the above theorem is similar to the proof of the analogous result for the Navier–Stokes equations (see Dashti and Robinson [2008] for the details).

Proof. Local existence of strong solutions for the CBF equations (Theorem 4.2) implies that there is $\tilde{T} > 0$ such that $v \in L^\infty(0,T';V) \cap L^2(0,T';H^2)$ for every $T' < \tilde{T}$. We denote by $\tilde{T}$ the maximal time of existence of the strong solution v, i.e.

$$\limsup_{t\to\tilde{T}^-}\|\nabla v(t)\| = \infty.$$

This implies that $\|\nabla w(t)\|$ also blows up at $t = \tilde{T}$, where $w := u - v$. We assume that $\tilde{T} \leq T$, where T is the time of existence of the strong solution u, and obtain a contradiction.

The difference $w = u - v$ satisfies

$$\partial_t w + Aw + B(u,w) + B(w,u) - B(w,w) + C_r(u) - C_r(v) = f - g, \tag{5.17}$$

on the time interval $(0,\tilde{T})$, with the initial condition $w(x,0) = u_0 - v_0$. We know that $\partial_t v \in L^2(0,T';H)$ for every $T' < \tilde{T}$. Furthermore, we have $\tilde{T} \le T$, so obviously also $\partial_t u \in L^2(0,T';H)$ for every $T' < \tilde{T}$. Then, taking the inner product of (5.17) with Aw in L^2 and using our a priori estimate (5.15), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{H^1}^2 + \|Aw\|^2 &\le c_0\|f-g\|^2 + c_r\|w\|_{H^1}^{2r} + c_3\|w\|_{H^1}^6 \\ &\quad + c_1\|w\|_{H^1}^2\left(\|u\|_{H^1}^4 + \|\nabla u\|\,\|Au\| + \|u\|_{H^1}^{2(r-1)} + \|\nabla u\| + 1\right)\end{aligned} \tag{5.18}$$

for appropriate values of the constants c_i, $i \in \{0,1,r,3\}$.

We define the quantities

- $X(t) := \|w(t)\|_{H^1}^2$,
- $\delta(t) := c_0\|f(t) - g(t)\|^2$,
- $\tilde{\gamma}(t) := c_1\left(\|u(t)\|_{H^1}^4 + \|\nabla u(t)\|\,\|Au(t)\| + \|u(t)\|_{H^1}^{2(r-1)} + \|\nabla u(t)\| + 1\right)$.

Inequality (5.18) gives (omitting $\|Aw\|^2$ on the left-hand side)

$$X' \le c_3X^3 + c_rX^r + \tilde{\gamma}(t)X + \delta(t).$$

Using the inequality (valid for $X \ge 0$)

$$X^p \le X^3 + X \quad \text{for} \quad p \in [1,3]$$

and changing the constant c_3, we get

$$X' \le c_3X^3 + \gamma(t)X + \delta(t), \tag{5.19}$$

where $\gamma(t) := \tilde{\gamma}(t) + c_r$.

We now take

$$Y(t) := \exp\left(-\int_0^t \gamma(s)\,\mathrm{d}s\right)X(t)$$

and multiply both sides of (5.19) by $\exp\left(-\int_0^t \gamma(s)\,\mathrm{d}s\right) \le 1$. This way we obtain

$$
\begin{aligned}
Y' &\le c_3 \exp\left(-\int_0^t \gamma(s)\,\mathrm{d}s\right) X^3 + \delta(t)\exp\left(-\int_0^t \gamma(s)\,\mathrm{d}s\right)\\
&\le c_3 \left[\exp\left(2\int_0^t \gamma(s)\,\mathrm{d}s\right)\right] Y^3 + \delta(t)\\
&\le \underbrace{c_3 \left[\exp\left(2\int_0^T \gamma(s)\,\mathrm{d}s\right)\right]}_{=:K} Y^3 + \delta(t).
\end{aligned}
$$

Hence, we have the differential inequality [valid on the time interval $(0, \tilde{T})$]

$$
Y' \le KY^3 + \delta(t),
$$

with the initial condition

$$
Y(0) = \|u_0 - v_0\|_{H^1}^2 .
$$

Therefore, by Lemma 5.1 with $n = 3$, the function $Y(t)$ is uniformly bounded on the time interval $[0, T']$ for every $T' < \tilde{T} \le T$, provided that

$$
Y(0) + \int_0^{T'} \delta(t)\,\mathrm{d}t < \frac{1}{(2KT')^{1/2}},
$$

which clearly holds (since $T' < T$) if we have

$$
Y(0) + \int_0^{T} \delta(t)\,\mathrm{d}t < \frac{1}{(2KT)^{1/2}}.
$$

Substituting all our original variables in the above condition, we obtain

$$
\begin{aligned}
&\|u_0 - v_0\|_{H^1}^2 + c_0 \int_0^T \|f(t) - g(t)\|^2\,\mathrm{d}t\\
&< \frac{\exp(-c_r T)}{\sqrt{2c_3 T}} \exp\left(-c_1 \int_0^T \left(\|u\|_{H^1}^4 + \|\nabla u\|\,\|Au\| + \|u\|_{H^1}^{2(r-1)} + \|\nabla u\| + 1\right)\mathrm{d}t\right),
\end{aligned}
$$

which is (up to a change of constants) the robustness condition (5.16). If this condition is satisfied, it follows that the function $X(t) = \|w(t)\|_{H^1}^2$ is uniformly bounded on the time interval $[0, \tilde{T})$

$$
X(t) = Y(t)\exp\left(\int_0^t \gamma(s)\,\mathrm{d}s\right) \le Y(t)\exp\left(\int_0^T \gamma(s)\,\mathrm{d}s\right) \le C(T) < \infty.
$$

Hence, we finally get that $\|w(t)\|_{H^1} \le C(T)$ for all $t < \tilde{T}$, and consequently

$\|v(t)\|_{H^1} \leq C(T)$ for $t \in [0, \tilde{T})$ as well. It follows that

$$\limsup_{t \to \tilde{T}^-} \|\nabla v(t)\| \leq C(T) < \infty,$$

which contradicts the maximality of the time $\tilde{T}$. Therefore, we have $\tilde{T} > T$ and the function v does not blow up at any time $\tilde{T} \leq T$. Hence, v belongs to the space $L^\infty(0, T; V)$.

Now, directly from the inequality (5.18), it follows that the function v belongs also to the space $L^2(0, T; H^2)$ $\left(\text{since } \int_0^T \|Aw\|^2 = \int_0^T \left\|D^2 w\right\|^2 < \infty\right)$, which proves that it is a strong solution on the time interval $[0, T]$, completing the proof of Theorem 5.2. $\square$

It is worth mentioning that the robustness of regularity result proved in this chapter could also be obtained via the Implicit Function Theorem. Indeed, let us consider the map

$$F : W^{1,2}_{0,\sigma}((0, T) \times \Omega) \to (W^{1,2}_\sigma(\Omega), L^2_\sigma((0, T) \times \Omega))$$

defined by

$$F(u) := (u|_{t=0}, \quad \partial_t u - \mu A u + B(u) + \beta C_r(u) - \mathbb{P} f),$$

where $W^{1,2}_{0,\sigma}((0, T) \times \Omega)$ is a parabolic space of divergence-free functions from L^2 such that first derivative in time and all second derivatives in space belong to L^2. To apply the IFT one has to check that this map is C^1-smooth. Then, if $\bar{u}$ is a strong solution, the condition '$F'(\bar{u})$ is invertible' is equivalent to the unique solvability of the linear problem

$$\partial_t v - \mu \Delta v + (v \cdot \nabla)\bar{u} + (\bar{u} \cdot \nabla)v + \beta r |\bar{u}|^{r-1} v = h(t), \quad \operatorname{div} v = 0, \ v|_{t=0} = v_0.$$

Since the solution $\bar{u}$ is regular enough, this can be shown using energy estimates very similar to the analysis presented above. Thus, the Implicit Function Theorem will give the existence of strong solutions in the neighbourhood of $\bar{u}$.

Chapter 6

Real interpolation spaces

We will use real interpolation spaces based on Hilbert spaces in the next two chapters, and in this chapter we recall some basic theory of real interpolation spaces, generated via the 'K-method'. More theory and applications can be found in the extensive literature in the subject: Lions and Peetre [1964], Bergh and Löfström [1976], Triebel [1978] or Lions and Magenes [1972]. We follow mostly the nice expositions in Lunardi [2009] and Adams and Fournier [2003].

Let $X, Y \hookrightarrow \mathcal{X}$ be two Banach spaces embedded in a common topological Hausdorff vector space $\mathcal{X}$. Then $X \cap Y$ and $X + Y$ are themselves Banach spaces with respect to the norms

$$\begin{aligned} \|u\|_{X\cap Y} &:= \max\{\|u\|_X, \|u\|_Y\}, \\ \|u\|_{X+Y} &:= \inf\{\|u_0\|_X + \|u_1\|_Y \ : \ u = u_0 + u_1, u_0 \in X, u_1 \in Y\}, \end{aligned} \tag{6.1}$$

and $X \cap Y \hookrightarrow X, Y \hookrightarrow X + Y$.

Definition 6.1. *A Banach space X is an* intermediate space *between X and Y if there exist the embeddings*

$$X \cap Y \hookrightarrow X \hookrightarrow X + Y.$$

For fixed $t > 0$ the following functionals define norms on $X \cap Y$ and $X + Y$ respectively, equivalent to those defined above in (6.1)

$$\begin{aligned} J(t, u) &:= \max\{\|u\|_X, t\|u\|_X\}, \\ K(t, u) &:= \inf\{\|u_0\|_X + t\|u_1\|_Y \ : \ u = u_0 + u_1, u_0 \in X, u_1 \in Y\}. \end{aligned}$$

The functional J form the basis for the 'J-method' of real interpolation which is slightly more involved than the K-method (but these two methods are equivalent), and we will not consider it further.

Evidently $K(1,u) = \|u\|_{X+Y}$, and $K(t,u)$ is a continuous and monotonically increasing function of t on the interval $(0,\infty)$. Moreover, we have

$$\min\{1,t\}\,\|u\|_{X+Y} \le K(t,u) \le \max\{1,t\}\,\|u\|_{X+Y}\,.$$

6.1 The K-method

Using the K functional we define the space $(X,Y)_{\theta,q;K}$.

Definition 6.2. *If $0 \le \theta \le 1$ and $1 \le q \le \infty$ we denote by $(X,Y)_{\theta,q;K}$ the space of all $u \in X+Y$ such that the function $t^{-\theta}K(t,u)$ belongs to $L^q_* := L^q(0,\infty;{}^{\mathrm{d}t}/_t)$.*

The next theorem states that $(X,Y)_{\theta,q;K}$ is in fact an intermediate space between X and Y.

Theorem 6.3 (The K-method)**.** *If either $1 \le q < \infty$ and $0 < \theta < 1$ or $q = \infty$ and $0 \le \theta \le 1$, then the space $(X,Y)_{\theta,q;K}$ is a nontrivial Banach space with the norm given by*

$$\|u\|_{\theta,q;K} := \begin{cases} \left(\int_0^\infty t^{-\theta q-1}K(t,u)^q\,\mathrm{d}t\right)^{1/q} & \text{if} \quad 1 \le q < \infty, \\ \operatorname{ess\,sup}_{0<t<\infty}\left\{t^{-\theta}K(t,u)\right\} & \text{if} \quad q = \infty. \end{cases}$$

Furthermore,

$$\|u\|_{X+Y} \le \frac{\|u\|_{\theta,q;K}}{\|t^{-\theta}\min\{1,t\}\|_{L^q_*}} \le \|u\|_{X\cap Y}\,;$$

in particular

$$X \cap Y \hookrightarrow (X,Y)_{\theta,q;K} \hookrightarrow X+Y$$

and $(X,Y)_{\theta,q;K}$ is an intermediate space between X and Y.

Otherwise, if $1 \le q < \infty$ and $\theta \in \{0,1\}$, then $(X,Y)_{\theta,q;K} = \{0\}$.

We also have trivially that

$$X \hookrightarrow (X,Y)_{0,\infty;K} \quad \text{and} \quad Y \hookrightarrow (X,Y)_{1,\infty;K}.$$

There is the following nesting property for these spaces. If $0 < \theta < 1$ and $1 \le p \le q \le \infty$, then

$$(X,Y)_{\theta,p;K} \hookrightarrow (X,Y)_{\theta,q;K}.$$

We define a class of intermediate spaces between X and Y.

Definition 6.4 (Classes of intermediate spaces)**.** *We say that X belongs to the class $\mathcal{K}(\theta, X, Y)$ if for all $u \in X$*

$$K(t,u) \leq C_1 t^{\theta} \left\| u \right\|_X ,$$

where C_1 is a positive constant.

We have the following characterisation of the class $\mathcal{K}$.

Lemma 6.5. *Let $0 \leq \theta \leq 1$ and let X be an intermediate space between X and Y. Then $X \in \mathcal{K}(\theta, X, Y)$ if and only if $X \hookrightarrow (X,Y)_{\theta,\infty,K}$.*

Now we construct intermediate spaces between two intermediate spaces.

Theorem 6.6 (The Reiteration Theorem)**.** *Let $0 \leq \theta_0 < \theta_1 \leq 1$ and let X_{θ_0} and X_{θ_1} be intermediate spaces between X and Y. For $0 \leq \lambda \leq 1$, let*

$$\theta := (1-\lambda)\theta_0 + \lambda\theta_1.$$

If $X_{\theta_i} \in \mathcal{K}(\theta_i, X, Y)$ for $i = 0,1$, and if either $0 < \lambda < 1$ and $1 \leq q < \infty$ or $0 \leq \lambda \leq 1$ and $q = \infty$, then

$$(X_{\theta_0}, X_{\theta_1})_{\lambda,q;K} \hookrightarrow (X,Y)_{\theta,q;K}.$$

6.2 Interpolation spaces

Let $P = \{X_0, X_1\}$ and $Q = \{Y_0, Y_1\}$ be two interpolation pairs of Banach spaces, and let T be a bounded linear operator from $X_0 + X_1$ to $Y_0 + Y_1$ having the property that T is bounded from X_i into Y_i, with norm at most M_i, for $i = 0,1$; that is,

$$\left\| T u_i \right\|_{Y_i} \leq M_i \left\| u_i \right\|_{X_i} \quad \text{for all} \quad u_i \in X_i, \quad i = 0,1.$$

Now we define the notion of an interpolation space.

Definition 6.7 (Interpolation spaces)**.** *If X and Y are intermediate spaces for P and Q, respectively, we call X and Y* interpolation spaces of type θ *for P and Q, where $0 \leq \theta \leq 1$, if every linear operator T as defined above maps X into Y with norm M satisfying*

$$M \leq C M_0^{1-\theta} M_1^{\theta}, \tag{6.2}$$

where the constant $C \geq 1$ is independent of T.

Definition 6.8. *We say that the interpolation spaces X and Y of type θ, are* exact *if the inequality* (6.2) *holds with* $C = 1$.

If $X_i = Y_i$, for $i = 0, 1$, $X = Y$ and $T := Id$, the identity operator on $X_0 + X_1$, then $C = 1$ for all $0 \leq \theta \leq 1$, so no smaller C is possible in (6.2).

The following theorem establishes that K-intermediate spaces defined above (Theorem 6.3) are actually interpolation spaces in the sense of Definition 6.7.

Theorem 6.9 (An Exact Interpolation Theorem)**.** *Let $P = \{X_0, X_1\}$ and $Q = \{Y_0, Y_1\}$ be two interpolation pairs. If either $0 < \theta < 1$ and $1 \leq q < \infty$ or $0 \leq \theta \leq 1$ and $q = \infty$, then the intermediate spaces $(X_0, X_1)_{\theta,q;K}$ and $(Y_0, Y_1)_{\theta,q;K}$ are exact interpolation spaces of type θ for P and Q.*

There are many interesting applications of real interpolation spaces. We will use them to identify the domains of fractional powers of the Laplace and Stokes operators in the next chapter. Then, we will apply the interpolation theory in the context of the Lorentz spaces to obtain some properties of the Stokes operator, required for our approximation schemes in Chapter 8. One can give an alternative construction of the standard Lorentz spaces, using real interpolation (see Theorem 7.26 in Adams and Fournier [2003]).

Theorem 6.10. *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set. If $u \in L^1(\Omega) + L^\infty(\Omega)$, and if $1 < p < \infty$, $1 \leq q \leq \infty$, and $\theta = p' = 1 - 1/p$, then*

$$L^{p,q}(\Omega) = (L^1(\Omega), L^\infty(\Omega))_{\theta,q;K},$$

with equality of norms:

$$\|u\|_{L^{p,q}(\Omega)} = \|u\|_{\theta,q;K}\,.$$

Corollary 6.11. *If $1 \leq p_1 < p < p_2 \leq \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, then by the Reiteration Theorem* 6.6*, up to equivalence of norms, we have*

$$L^{p,q}(\Omega) = (L^{p_1}(\Omega), L^{p_2}(\Omega))_{\theta,p;K}.$$

Chapter 7

Domains of the fractional powers of operators

In this chapter we characterise explicitly the fractional power spaces (i.e. the domains of fractional powers of some linear operator) of the Dirichlet Laplacian on a sufficiently smooth bounded domain Ω, and do the same for the Stokes operator. As we restrict to L^2-based spaces our arguments are largely elementary.

First we show that when dealing with self-adjoint compact-inverse operators on a Hilbert space these fractional power spaces are (real) interpolation spaces; this allows us to give relatively simple arguments to identify the concrete examples of fractional power spaces that will be of interest later. While most of the results of this chapter are not new, we present them in what we hope is a relatively simple and accessible way. One key tool is a simple but powerful observation (Lemma 7.3) that gives sufficient conditions for interpolation to 'preserve intersections', i.e. conditions such that

$$(X \cap Z, Y \cap Z)_\theta = (X, Y)_\theta \cap Z,$$

a result that does not hold in general (in our applications Z will enforce certain 'side conditions'). We combine these results to give two approximation theorems in Chapter 8, and then use the more involved weighted-truncation method to prove the validity of the energy equality for weak solutions of the critical ($r = 3$) CBF equations on bounded domains in Chapter 9.

Results of this chapter are not new, but straightforward proofs are hard to find in the literature. The characterisation of the domains of the Dirichlet Laplacian can be found in the papers by Grisvard [1967], Fujiwara [1967], and Seeley [1972]. Note that Fujiwara's statement is not correct for $\theta = 3/4$, and that Seeley also gives the corresponding characterisation for the operators in L^p-based spaces. For the Stokes operator $\mathcal{A} := -\mathbb{P}\Delta$, Giga [1985] and Fujita and Morimoto [1970] both showed that $D(\mathcal{A}) = D(A) \cap H_\sigma$, where A was the Dirichlet Laplacian and H_σ the divergence-free subspace of L^2; the former in the greater generality of L^p-based spaces. We use a key idea from the proof of Fujita & Morimoto in our argument in Section 7.3.3. The content of this and the next two chapters can be found in Fefferman et al. [2019].

7.1 Abstract setting

In this and in the next chapter we suppose that H is a separable Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and that A is a positive, self-adjoint operator on H with compact inverse. In this case A has a complete set of orthonormal eigenfunctions $\{w_n\}$ with corresponding eigenvalues $\lambda_n > 0$, which we order so that $\lambda_{n+1} \geq \lambda_n$.

Recall that for any $\alpha \geq 0$ we can define $D(A^\alpha)$ as the subspace of H, where

$$D(A^\alpha) := \left\{ u = \sum_{j=1}^{\infty} \hat{u}_j w_j \ : \ \sum_{j=1}^{\infty} \lambda_j^{2\alpha} |\hat{u}_j|^2 < \infty \right\}. \tag{7.1}$$

For $\alpha < 0$ we can take this space to be the dual of $D(A^{-\alpha})$; the expression in (7.1) can then be understood as an element in the completion of the space of finite sums with respect to the $D(A^\alpha)$ norm defined below in (7.2). For all $\alpha \in \mathbb{R}$ the space $D(A^\alpha)$ is a Hilbert space with inner product

$$\langle u, v \rangle_{D(A^\alpha)} := \sum_{j=1}^{\infty} \lambda_j^{2\alpha} \hat{u}_j \hat{v}_j$$

and corresponding norm

$$\|u\|_{D(A^\alpha)}^2 := \sum_{j=1}^{\infty} \lambda_j^{2\alpha} |\hat{u}_j|^2 \tag{7.2}$$

[note that $D(A^0)$ coincides with H]. We can define $A^\alpha \colon D(A^\alpha) \to H$ as the mapping

$$\sum_{j=1}^{\infty} \hat{u}_j w_j \mapsto \sum_{j=1}^{\infty} \lambda_j^{\alpha} \hat{u}_j w_j,$$

and then $\|u\|_{D(A^\alpha)} = \|A^\alpha u\|$. Note that A^α also makes sense as a mapping from $D(A^\beta) \to D(A^{\beta-\alpha})$ for any $\beta \in \mathbb{R}$, and that for $\beta \geq \alpha \geq 0$ we have

$$D(A^\beta) = \{u \in D(A^{\beta-\alpha}) \ : \ A^{\beta-\alpha}u \in D(A^\alpha)\}. \tag{7.3}$$

We can define a semigroup $e^{-\theta A}\colon H \to H$ by setting

$$e^{-\theta A}u := \sum_{j=1}^\infty e^{-\theta\lambda_j}\langle u, w_j\rangle w_j, \qquad \theta \geq 0; \tag{7.4}$$

this extends naturally to $D(A^\alpha)$ for any $\alpha > 0$, and for $\alpha < 0$ we can interpret $\langle u, w_j\rangle$ via the natural pairing between $D(A^\alpha)$ and $D(A^{-\alpha})$ [or, alternatively, as $\hat{u}_j$ in the definition (7.1)]. Then for all $u \in D(A^\alpha)$ we have

$$\|e^{-\theta A}u\|_{D(A^\beta)} \leq \begin{cases} C_{\beta-\alpha}\theta^{-(\beta-\alpha)}\|u\|_{D(A^\alpha)} & \beta \geq \alpha, \\ e^{-\lambda_1\theta}\lambda_1^{\beta-\alpha}\|u\|_{D(A^\alpha)} & \beta < \alpha, \end{cases} \tag{7.5}$$

where we can take $C_\gamma = \sup_{\lambda\geq 0}\lambda^\gamma e^{-\lambda}$ (the exact form of the constant is unimportant, but note that $C_\gamma < \infty$ for every $\gamma \geq 0$), and

$$\left\|e^{-\theta A}u - u\right\|_{D(A^\alpha)} \to 0 \quad \text{as} \quad \theta \to 0^+. \tag{7.6}$$

In particular, (7.6) means that $e^{-\theta A}$ is a strongly continuous semigroup on $D(A^\alpha)$ for every $\alpha \in \mathbb{R}$.

7.2 Domains of fractional powers

We first present a very quick treatment of the fractional powers of unbounded self-adjoint compact-inverse operators on a Hilbert space; in this case it is easy to show that the fractional power spaces are given as real interpolation spaces (cf. Chapter 1 of Lions and Magenes [1972], from which we quote a number of results in what follows).

7.2.1 Real interpolation for Hilbert spaces

We recall the method of 'real interpolation', due to Lions & Peetre (Lions [1959]; Lions and Peetre [1964]) as adopted by Lions & Magenes; their θ-intermediate space corresponds to the $(\theta, 2; K)$ interpolation space in the more general theory covered in the previous chapter.

As in Chapter 6 we suppose that X and Y are Banach spaces, both continuously embedded in some Hausdorff topological vector space $\mathcal{X}$. For any $u \in X + Y$

we define

$$K(t,u) := \inf_{\substack{x,y\,:\,x+y=u,\\ x\in X,\,y\in Y}} \left(\|x\|_X^2 + t^2\|y\|_Y^2\right)^{1/2}; \tag{7.7}$$

we follow Lions and Magenes [1972] in choosing this particular form for K. We define

$$(X,Y)_\theta := \left\{u \in X+Y \;:\; t^{-\theta}K(t,u) \in L^2(0,\infty;\mathrm{d}t/t)\right\}; \tag{7.8}$$

this is a Banach space with norm

$$\|u\|_\theta := \left\|t^{-\theta}K(t,u)\right\|_{L^2(0,\infty;\mathrm{d}t/t)}.$$

[Since $a^2+t^2b^2 \le (a+tb)^2 \le 2(a^2+t^2b^2)$ this is equivalent to the standard definition of the space $(X,Y)_{\theta,2;K}$ as given in Section 6.1. The definition we adopt here (following Lions & Magenes) is more suited to the Hilbert space case.]

7.2.2 Fractional power spaces via real interpolation

We now give a simple proof that the fractional power spaces of A are given by real interpolation spaces when A is a positive, unbounded self-adjoint operator with a compact inverse (cf. Theorem I.15.1 in Lions and Magenes [1972]).

Lemma 7.1. *Suppose that A is a positive, unbounded self-adjoint operator with a compact inverse and domain $D(A)$ in a Hilbert space H (as in Section 7.1). Then*

$$(H, D(A))_\theta = D(A^\theta), \qquad 0 < \theta < 1. \tag{7.9}$$

[A similar result holds for general positive self-adjoint operators on Hilbert spaces. One can obtain (7.9) using complex interpolation provided that the imaginary powers of A are bounded, which they are in this case (Seeley [1971a]); since real and complex interpolation spaces coincide for Hilbert spaces (see Chapter 1 in Triebel [1978]), (7.9) then holds using real interpolation in this more general setting; for a related discussion see Chapter I, Section 2.9 in Amann [1995]. See also Seeley [1971b, 1972].]

Proof. For $u = \sum_{j=1}^\infty \hat{u}_j w_j$, where $\{w_j\}$ are the eigenfunctions and (λ_j) eigenvalues of the operator A (as in Section 7.1), we have

$$K(t,u) = \inf_{(y_j)} \left[\sum_{j=1}^\infty |\hat{u}_j - y_j|^2 + t^2\lambda_j^2\,|y_j|^2\right]^{1/2}.$$

A simple minimisation over (y_j) shows that

$$K(t,u) = \left(\sum_{j=1}^{\infty} \frac{t^2\lambda_j^2\,|\hat{u}_j|^2}{1+t^2\lambda_j^2} \right)^{1/2}.$$

Indeed, if we define a function $f(y) := (\hat{u}-y)^2 + t^2\lambda^2 y^2$, then

$$f'(y) = -2(\hat{u}-y) + 2t^2\lambda^2 y, \quad \text{and} \quad f'(y) = 0 \quad \Longleftrightarrow \quad y = \frac{\hat{u}}{1+t^2\lambda^2}.$$

So, it follows that f attains its minimal value $f(y_0) = \frac{t^2\lambda^2\hat{u}^2}{1+t^2\lambda^2}$ at $y_0 := \frac{\hat{u}}{1+t^2\lambda^2}$.

Now observe that

$$\begin{aligned}
\int_0^\infty t^{-2\theta} K(t,u)^2 \, \frac{\mathrm{d}t}{t} &= \int_0^\infty \sum_{j=1}^{\infty} \frac{(t^2\lambda_j^2)^{1-\theta}}{1+t^2\lambda_j^2}\, \lambda_j^{2\theta}\,|\hat{u}_j|^2 \, \frac{\mathrm{d}t}{t} \\
&= \sum_{j=1}^{\infty} \int_0^\infty \frac{(t^2\lambda_j^2)^{1-\theta}}{1+t^2\lambda_j^2}\, \lambda_j^{2\theta}\,|\hat{u}_j|^2 \, \frac{\mathrm{d}t}{t} \\
&= \sum_{j=1}^{\infty} \lambda_j^{2\theta}\,|\hat{u}_j|^2 \int_0^\infty \frac{s^{1-2\theta}}{1+s^2}\,\mathrm{d}s \\
&= I(\theta) \sum_{j=1}^{\infty} \lambda_j^{2\theta}\,|\hat{u}_j|^2 = I(\theta)\|A^\theta u\|^2 = I(\theta)\|u\|^2_{D(A^\theta)},
\end{aligned}$$

where

$$I(\theta) = \int_0^\infty \frac{s^{1-2\theta}}{1+s^2}\,\mathrm{d}s < \infty$$

for $0 < \theta < 1$. (In fact the integral can be evaluated explicitly using contour integration to give $I(\theta) = \frac{\pi}{2}\frac{1}{\sin(\pi\theta)}$.) It follows that $u \in (H, D(A))_\theta$ if and only if $u \in D(A^\theta)$. $\square$

The following particular cases of the Reiteration Theorem 6.6 (see also Theorem 1.6.1 in Lions and Magenes [1972]) are simple corollaries of the above result.

Corollary 7.2. *In the same setting as that of Lemma* 7.1

$$(H, D(A^{1/2}))_\theta = D(A^{\theta/2}) \qquad \text{and} \qquad (D(A^{1/2}), D(A))_\theta = D(A^{(1+\theta)/2}).$$

Proof. For the first equality we apply Lemma 7.1 with A replaced by $A^{1/2}$; for the second we apply Lemma 7.1 with A replaced by $A^{1/2}$ and the 'base space' H replaced by $D(A^{1/2})$, and note that

$$D\left(A^{(1+\theta)/2}\right) = \left\{u \in H \ : \ A^{1/2}u \in D(A^{\theta/2})\right\}. \qquad \square$$

To obtain fractional powers of operators with boundary conditions, or other constraints (e.g. the divergence-free constraint associated with the Stokes operator) the following simple result will be useful: it provides one way to circumvent the fact that interpolation does not respect intersections, i.e. in general

$$(X \cap Z, Y \cap Z)_\theta \neq (X, Y)_\theta \cap Z.$$

(A related result can be found as Proposition A.2 in Rodríguez-Bernal [2017].)

Lemma 7.3. *Let $(H, \|\cdot\|_H)$ and $(D, \|\cdot\|_D)$ be Hilbert spaces, with H_0 a Hilbert subspace of H (i.e. with the same norm) and $D \subset H$ with continuous inclusion. Suppose that there exists a bounded linear map $T\colon H \to H_0$ such that $T|_{H_0}$ is the identity and $T|_D\colon D \to D \cap H_0$ is also bounded, in the sense that*

$$\|Tf\|_D \leq C\|f\|_D \qquad \text{for some } C > 0.$$

Then for every $0 < \theta < 1$

$$(H_0, D \cap H_0)_\theta = (H, D)_\theta \cap H_0,$$

with norm equivalent to that in $(H, D)_\theta$.

Proof. Since $H_0 \hookrightarrow H$ and $D \cap H_0 \hookrightarrow D$, it follows from the definition (7.8) of the interpolation spaces that

$$(H_0, D \cap H_0)_\theta \hookrightarrow (H, D)_\theta \cap H_0,$$

with

$$\|u\|_\theta \leq C\|u\|_{0,\theta},$$

where $\|\cdot\|_{0,\theta}$ denotes the norm in $(H_0, D \cap H_0)_\theta$ (and $\|\cdot\|_\theta$ is the norm in $(H, D)_\theta$).

Now suppose that $u \in (H, D)_\theta \cap H_0$; then for each $t > 0$ we can find $f(t) \in H$ and $g(t) \in D$ such that we can write

$$u = f(t) + g(t) \qquad \text{with} \qquad \|f(t)\|_H^2 + t^2\|g(t)\|_D^2 \leq 2K(t, u);$$

then

$$\int_0^\infty t^{-2\theta-1}\left(\|f(t)\|_H^2 + t^2\|g(t)\|_D^2\right)\,\mathrm{d}t \leq 2\|u\|_\theta^2.$$

Now since $u \in H_0$ and $T|_{H_0} = \mathrm{I}d$ we also have

$$u = Tu = Tf(t) + Tg(t),$$

with $Tf(t) \in H_0$ and $Tg(t) \in D \cap H_0$, so that

$$\begin{aligned}\|u\|_{0,\theta}^2 &\le \int_0^\infty t^{-2\theta-1}\left(\|Tf(t)\|_H^2 + t^2\|Tg(t)\|_D^2\right)\,\mathrm{d}t \\ &\le C^2\int_0^\infty t^{-2\theta-1}\left(\|f(t)\|_H^2 + t^2\|g(t)\|_D^2\right)\,\mathrm{d}t \le 2C^2\|u\|_\theta^2\end{aligned}$$

i.e. $\|u\|_{0,\theta} \le C'\|u\|_\theta$, from which the conclusion follows. □

7.3 Identifying fractional power spaces

In this section we will prove the following theorem, which combines the results of Lemma 7.5, Corollaries 7.6 and 7.8, and Lemma 7.10.

Theorem 7.4. *When A is the negative Dirichlet Laplacian on $\Omega \subset \mathbb{R}^d$, $d \ge 2$, we have*

$$D(A^\theta) = \begin{cases} H^{2\theta}(\Omega), & 0 < \theta < 1/4, \\ H_{00}^{1/2}(\Omega), & \theta = 1/4, \\ H_0^{2\theta}(\Omega), & 1/4 < \theta \le 1/2, \\ H^{2\theta}(\Omega) \cap H_0^1(\Omega), & 1/2 < \theta \le 1, \end{cases}$$

where $H_{00}^{1/2}(\Omega)$ consists of all $u \in H^{1/2}(\Omega)$ such that

$$\int_\Omega \rho(x)^{-1}|u(x)|^2\,\mathrm{d}x < \infty,$$

with $\rho(x)$ any C^∞ function comparable to $\mathrm{dist}(x, \partial\Omega)$*. If $\mathcal{A}$ is the Stokes operator on Ω with Dirichlet boundary conditions then the domains of the fractional powers of $\mathcal{A}$ are as above, except that all spaces are intersected with*

$$H_\sigma := \textit{completion of } \{\phi \in [C_0^\infty(\Omega)]^d \ : \ \nabla\cdot\phi = 0\} \textit{ in the norm of } L^2(\Omega).$$

We first recall how fractional Sobolev spaces are defined using interpolation, and some of their properties. It is then relatively straightforward to give explicit characterisations of the fractional power spaces of the Dirichlet Laplacian and the Stokes operator.

7.3.1 Sobolev spaces and interpolation spaces

For non-integer s the space $H^s(\Omega)$ is defined by setting

$$H^{k\theta}(\Omega) := (L^2(\Omega), H^k(\Omega))_\theta, \qquad 0 < \theta < 1,$$

for any integer k (equation (I.9.1) in Lions and Magenes [1972]); this definition is independent of k and is consistent with the standard definition whenever $k\theta$ is an integer, so we have

$$(H^{s_1}(\Omega), H^{s_2}(\Omega))_\theta = H^{(1-\theta)s_1+\theta s_2}(\Omega), \qquad s_1 < s_2,\ 0 < \theta < 1, \tag{7.10}$$

see Theorem I.9.6 in Lions and Magenes [1972]. Defined in this way $H^s(\Omega)$ is the set of restrictions to Ω of functions in $H^s(\mathbb{R}^n)$ (Theorem I.9.1 in Lions and Magenes [1972]).

For all $s \geq 0$ we define

$$H_0^s(\Omega) := \text{completion of } C_0^\infty(\Omega) \text{ in } H^s(\Omega);$$

for $0 \leq s \leq 1/2$ we have $H_0^s(\Omega) = H^s(\Omega)$ (Theorem I.11.1 in Lions and Magenes [1972]).

7.3.2 Fractional power spaces of Dirichlet Laplacian

We now consider the case when $A = -\Delta$ is the negative Dirichlet Laplacian on a bounded domain Ω; to avoid technicalities we assume that $\partial\Omega$ is smooth. From standard regularity results for weak solutions, see Theorem 8.12 in Gilbarg and Trudinger [2001] or Section 6.3 in Evans [2010], for example, we know that $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. The following result is well-known, but we provide a proof (after the discussion following Proposition 4.5 in Constantin and Foias [1988]) for the sake of completeness.

Lemma 7.5. *If A is the negative Dirichlet Laplacian on Ω then*

$$D(A^{1/2}) = H_0^1(\Omega).$$

Proof. We have

$$\langle Au, v\rangle = \langle -\Delta u, v\rangle = \langle \nabla u, \nabla v\rangle, \tag{7.11}$$

whenever $u \in D(A)$ and $v \in H_0^1(\Omega)$, see the proof of Proposition 4.2 in Constantin and Foias [1988] (their proof is given for the Stokes operator, but it works equally well in the case of the Laplacian).

If we let $\{w_j\}$ and (λ_j) be the eigenfunctions and corresponding eigenvalues of the operator A, then $\{w_j\}$ form a basis for $L^2(\Omega)$ (so also for $H^1_0(\Omega)$) and since $\lambda_j^{-1/2} w_j \in D(A) \subset H^1_0$ we can use (7.11) to write

$$\begin{aligned}\delta_{jk} = \langle \lambda_j^{-1/2} w_j, \lambda_k^{-1/2} w_k \rangle_{D(A^{1/2})} &= \langle A(\lambda_j^{-1/2} w_j), \lambda_k^{-1/2} w_j \rangle \\ &= \langle \nabla(\lambda_j^{-1/2} w_j), \nabla(\lambda_k^{-1/2} w_k) \rangle.\end{aligned}$$

It follows that $D(A^{1/2})$ is a closed subspace of H^1_0. [Recall from (7.1) that $D(A^{1/2})$ is defined as the collection of certain convergent eigenfunction expansions; the above equality shows that if this expansion converges in the $D(A^{1/2})$-norm then it also converges in the norm of H^1_0.]

If $v \in H^1_0$ with $\langle v, u \rangle_{H^1_0} = 0$ for all $u \in D(A^{1/2})$ then for every j

$$0 = \langle \nabla v, \nabla w_j \rangle = \langle v, A w_j \rangle = \lambda_j \langle v, w_j \rangle$$

and so $v = 0$, which shows that $D(A^{1/2}) = H^1_0$. $\square$

We can now appeal to results from Lions and Magenes [1972] to deal with the range $0 < \theta < 1/2$.

Corollary 7.6. *If A is the negative Dirichlet Laplacian on Ω then*

$$D(A^\theta) = \begin{cases} H^{2\theta}(\Omega) & 0 < \theta < 1/4, \\ H^{1/2}_{00}(\Omega) & \theta = 1/4, \\ H^{2\theta}_0(\Omega) & 1/4 < \theta < 1/2, \end{cases}$$

where $H^{1/2}_{00}(\Omega)$ consists of all $u \in H^{1/2}(\Omega)$ such that

$$\int_\Omega \rho(x)^{-1} |u(x)|^2 \, \mathrm{d}x < \infty,$$

with $\rho(x)$ any C^∞ function comparable to $\mathrm{dist}(x, \partial\Omega)$.

Proof. We note that

$$D(A^{\theta/2}) = (H, D(A^{1/2}))_\theta = (L^2, H^1_0)_\theta,$$

and then the expressions on the right-hand side follow immediately from Theorem I.11.7 in Lions and Magenes [1972]. $\square$

Note that the result above is relatively elementary for $\theta \neq 1/4$. Indeed, since $w_j \in D(A^r)$ is a countable sequence whose linear span is dense in $D(A^s)$, $D(A^r)$ is always dense in $D(A^s)$ for $0 \le s < r \le 1$; since Corollary 7.2 shows that $D(A^{1/2}) = H_0^1(\Omega)$, it follows that $H_0^1(\Omega)$ is dense in $D(A^\theta)$ for $\theta < 1/2$, and so, since $\|u\|_{H^{2\theta}} \le C_\theta \|u\|_{D(A^\theta)}$,

$$\begin{aligned} D(A^\theta) &= \{\text{completion of } H_0^1(\Omega) \text{ in the norm of } D(A^\theta)\} \\ &\subseteq \{\text{completion of } H_0^1(\Omega) \text{ in the norm of } H^{2\theta}(\Omega)\} \\ &= \{\text{completion of } C_0^\infty(\Omega) \text{ in the norm of } H^{2\theta}(\Omega)\} = H_0^{2\theta}(\Omega). \end{aligned}$$

To show the equivalence of the $H^{2\theta}$ and $D(A^\theta)$ norms (and hence equality of $D(A^\theta)$ and $H_0^{2\theta}$) note that functions in L^2, H^s for $0 < s < 1/2$, and H_0^s for $1/2 < s \le 1$ can be extended by zero to functions in $H^s(\mathbb{R}^n)$ without increasing their norms (Theorem I.11.4 in Lions and Magenes [1972]); an argument following that of Example 1.1.8 in Lunardi [2009] then shows that the norms in $D(A^\theta)$ and in $H^{2\theta}$ are equivalent provided that $\theta \neq 1/4$.

Since functions in $H^{1/2}(\Omega) = H_0^{1/2}(\Omega)$ *cannot* be extended by zero to functions in $H^{1/2}(\mathbb{R}^n)$ (Theorem I.11.4 in Lions and Magenes [1972]) the case of $\theta = 1/4$ is significantly more involved.

To deal with the range $1/2 < \theta < 1$ we will use the intersection lemma (Lemma 7.3) and the following simple result.

Lemma 7.7. *Let $u \in H^s(\Omega)$ with $s = 1$ or $s = 2$, and let $w \in H_0^1(\Omega)$ solve*

$$\langle \nabla w, \nabla \phi \rangle = \langle \nabla u, \nabla \phi \rangle \qquad \text{for all} \quad \phi \in H_0^1(\Omega). \tag{7.12}$$

Then $u \mapsto w$ is a bounded linear map from $H^s(\Omega)$ into $H^s(\Omega) \cap H_0^1(\Omega)$ and $w = u$ whenever $u \in H_0^1(\Omega)$.

Proof. The Riesz Representation Theorem guarantees that (7.12) has a unique solution $w \in H_0^1(\Omega)$ for every $u \in H^1(\Omega)$. That $w = u$ when $u \in H_0^1(\Omega)$ is then immediate, and the choice $\phi = w$ guarantees that $\|\nabla w\|_{L^2} \le \|\nabla u\|_{L^2}$. To deal with the $s = 2$ case, simply note that (7.12) is the weak form of the equation

$$-\Delta w = -\Delta u, \qquad w|_{\partial\Omega} = 0,$$

and standard regularity results for this elliptic problem (see e.g. Section 6.3 in Evans [2010]) guarantee that $\|w\|_{H^2} \le C\|\Delta u\|_{L^2} \le C\|u\|_{H^2}$. □

We can now characterise $D(A^\theta)$ for $1/2 < \theta < 1$.

Corollary 7.8. *If A is the negative Dirichlet Laplacian on Ω then*

$$D(A^\theta) = H^{2\theta}(\Omega) \cap H_0^1(\Omega) \qquad \text{for} \qquad 1/2 < \theta < 1.$$

Proof. Corollary 7.2 guarantees that

$$D(A^\theta) = (D(A^{1/2}), D(A))_{2\theta-1} = (H_0^1, H^2 \cap H_0^1)_{2\theta-1}.$$

Choosing $H = H^1(\Omega)$, $H_0 = H_0^1(\Omega)$, and $D = H^2(\Omega)$ in Lemma 7.3, we can let T be the map $u \mapsto w$ defined in Lemma 7.7 to deduce that

$$(H_0^1, H^2 \cap H_0^1)_{2\theta-1} = (H^1, H^2)_{2\theta-1} \cap H_0^1 = H^{2\theta} \cap H_0^1,$$

using (7.10). □

To guarantee that our approximating functions in the next chapter are smooth we will also need to consider $D(A^\theta)$ for $\theta > 1$; here an inclusion will be sufficient.

Corollary 7.9. *If A is the negative Dirichlet Laplacian on Ω then for $\theta \geq 1$*

$$D(A^\theta) \hookrightarrow H^{2\theta} \cap H_0^1, \qquad \text{with} \qquad \|u\|_{H^{2\theta}} \leq C_{D(A^\theta)\to H^{2\theta}} \|A^\theta u\|$$

for every $u \in D(A^\theta)$.

Proof. First we note that $D(A^\theta) \subseteq D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ for every $\theta \geq 1$; in particular $D(A^\theta) \hookrightarrow H_0^1(\Omega)$, so we need only show that

$$D(A^\theta) \hookrightarrow H^{2\theta}(\Omega), \qquad \text{with} \qquad \|u\|_{H^{2\theta}} \leq C_{D(A^\theta)\to H^{2\theta}} \|A^\theta u\| \tag{7.13}$$

for every $u \in D(A^\theta)$. Theorem 7.4 shows that this holds for all $0 < \theta \leq 1$.

We now use (7.3) and induction. Suppose that (7.13) holds for all $0 < \theta \leq k$ for some $k \in \mathbb{N}$; then for $\alpha = k + r$ with $0 < r \leq 1$ we have

$$D(A^\alpha) = D(A^{k+r}) = \{u \ : \ Au \in D(A^{k-1+r})\} = \{u \ : \ -\Delta u \in D(A^{k-1+r})\},$$

noting that since $u \in D(A^\alpha)$ and $\alpha \geq 1$ we have $u \in D(A)$, which guarantees that $Au = -\Delta u$.

It follows that any $u \in D(A^\alpha)$ solves the Dirichlet problem

$$-\Delta u = f, \qquad u|_{\partial\Omega} = 0, \tag{7.14}$$

for some $f \in D(A^{k-1+r}) \hookrightarrow H^{2(k-1+r)}(\Omega)$ using our inductive hypothesis. Elliptic regularity results for (7.14) (see Theorem II.5.4 in Lions and Magenes [1972], for example) now guarantee that $u \in H^{2(k+r)}(\Omega)$ with

$$\|u\|_{H^{2(k+r)}} \leq c\|f\|_{H^{2(k-1+r)}} = c\|\Delta u\|_{H^{2(k-1+r)}} = c\|Au\|_{H^{2(k-1+r)}} \leq c\|A^{k+r}u\|,$$

thanks to our inductive hypothesis. $\square$

7.3.3 Fractional power spaces of the Stokes operator

Recall the 'Leray projection' $\mathbb{P}$ defined in Chapter 2, i.e. the orthogonal projection in $L^2(\Omega)$ onto

$$H_\sigma(\Omega) := \text{completion of } \{\phi \in C_0^\infty(\Omega)\ :\ \nabla \cdot \phi = 0\} \text{ in the norm of } L^2(\Omega).$$

Since $\mathbb{P}$ is an orthogonal projection we have the symmetry property

$$\langle \mathbb{P}u, v\rangle = \langle u, \mathbb{P}v\rangle \qquad \text{for every } u, v \in L^2(\Omega). \tag{7.15}$$

We recall also the Stokes operator (note that we used different notation for the Stokes operator in the previous chapters) $\mathcal{A}$ on Ω defined as $\mathcal{A} := \mathbb{P}A$, where A is the negative Dirichlet Laplacian, and has domain

$$D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega) \cap H_\sigma(\Omega) = D(A) \cap H_\sigma(\Omega),$$

see Theorem 3.11 in Constantin and Foias [1988]. It is a positive, unbounded self-adjoint operator with compact inverse (see Chapter 4 in Constantin and Foias [1988]), so still falls within the general framework we have considered above.

Now we show that $D(\mathcal{A}^\theta) = D(A^\theta) \cap H_\sigma$. We can do this using the 'intersection lemma' (Lemma 7.3) via an appropriate choice of the mapping T: our choice is inspired by the proof of this equality due to Fujita and Morimoto [1970], who use the trace-based formulation of interpolation spaces.

Lemma 7.10. *For every $0 < \theta < 1$ we have $D(\mathcal{A}^\theta) = D(A^\theta) \cap H_\sigma$ with $\|\mathcal{A}^\theta u\|$ and $\|A^\theta u\|$ equivalent norms on $D(\mathcal{A}^\theta)$; in particular, the inclusion $D(\mathcal{A}^\theta) \subset D(A^\theta)$ is continuous.*

Proof. First observe that Lemma 7.1 gives

$$D(\mathcal{A}^\theta) = (H_\sigma, D(A) \cap H_\sigma)_\theta.$$

In order to apply the intersection result of Lemma 7.3 we consider the operator $\tilde{T}\colon D(A) \to D(\mathcal{A})$ defined by setting

$$\tilde{T} := \mathcal{A}^{-1}\mathbb{P}A.$$

As an operator from $D(A)$ into $D(\mathcal{A})$ this is bounded, due to elliptic regularity results for the Stokes operator ($\|\mathcal{A}^{-1}g\|_{H^2} \leq C\|g\|_{L^2}$ for $g \in H_\sigma$, see Theorem 3.11 in Constantin and Foias [1988], for example): for any $f \in D(A)$ we have

$$\|\tilde{T}f\|_{D(\mathcal{A})} \leq C\|\tilde{T}f\|_{H^2(\Omega)} \leq C\|\mathbb{P}Af\|_{L^2(\Omega)} \leq C\|Af\|_{L^2(\Omega)} = C\|f\|_{D(A)}.$$

We now extend $\tilde{T}$ to an operator $T\colon L^2 \to H_\sigma$: if we take $\psi \in H_\sigma$ and $\phi \in D(A)$ then, since both $\mathcal{A}$ and A are self-adjoint and $\mathbb{P}$ is symmetric (7.15),

$$\begin{aligned} |\langle \psi, \tilde{T}\phi\rangle| &= |\langle \psi, \mathcal{A}^{-1}\mathbb{P}A\phi\rangle| = |\langle \mathcal{A}^{-1}\psi, \mathbb{P}A\phi\rangle| = |\langle \mathcal{A}^{-1}\psi, A\phi\rangle| = |\langle A\mathcal{A}^{-1}\psi, \phi\rangle| \\ &\leq \|\mathcal{A}^{-1}\psi\|_{H^2}\|\phi\|_{L^2} \leq C\|\psi\|_{L^2}\|\phi\|_{L^2}, \end{aligned}$$

which shows that

$$\|\tilde{T}\phi\|_{H_\sigma} \leq C\|\phi\|_{L^2}.$$

Since $\tilde{T}$ is linear and $D(A)$ is dense in L^2 it follows that we can extend $\tilde{T}$ uniquely to an operator $T\colon L^2(\Omega) \to H_\sigma$ as claimed.

Note that T is the identity on H_σ: this can be seen by expanding $u \in H_\sigma$ in terms of the eigenfunctions of $\mathcal{A}$.

We now obtain the result by applying Lemma 7.3 choosing $H = L^2(\Omega)$, $H_0 = H_\sigma$, $D = D(A)$, and letting $T\colon L^2 \to H_\sigma$ be the operator we have just constructed. □

Embedding of the fractional power spaces of the Stokes operator via interpolation

Using the theory of real interpolation spaces (see Chapter 6) we can prove a property of $D(\mathcal{A}^\theta)$ which will be of interest in the next chapter.

Lemma 7.11. *Let $\mathcal{A}$ be the Stokes operator on $\Omega \subset \mathbb{R}^3$ with sufficiently smooth boundary and let $0 < \theta < 1$. Then there exists $r = r(\theta) > 2$ such that*

$$D(\mathcal{A}^\theta) \hookrightarrow L^r(\Omega).$$

Proof. Recall the representation for the domains of the fractional powers of the Stokes operator from the previous section

$$D(\mathcal{A}^\theta) = (H, D(\mathcal{A}))_{\theta,2;K}.$$

Taking $T := Id$ and noting that $H \hookrightarrow L^2(\Omega)$ and $D(\mathcal{A}) \hookrightarrow L^\infty(\Omega)$, by the Exact Interpolation Theorem 6.9, we have

$$D(\mathcal{A}^\theta) = (H, D(\mathcal{A}))_{\theta,2;K} \hookrightarrow (L^2(\Omega), L^\infty(\Omega))_{\theta,2;K}.$$

Using Corollary 6.11, we obtain

$$(L^2(\Omega), L^\infty(\Omega))_{\theta,2;K} = L^{r(\theta),\,2}(\Omega),$$

where $2 < r(\theta) = \frac{2}{1-\theta} < \infty$. Therefore, via monotonicity of the Lorentz spaces

$$D(\mathcal{A}^\theta) \hookrightarrow L^{r(\theta),\,2}(\Omega) \hookrightarrow L^{r(\theta),\,r(\theta)}(\Omega) = L^{r(\theta)}(\Omega),$$

as required. $\square$

Chapter 8

Simultaneous approximation in Sobolev and Lebesgue spaces

In this chapter we describe a method that allows one to use truncated (but weighted) eigenfunction expansions in order to obtain smooth approximations of functions defined on bounded domains in a way that behaves well with respect to both Lebesgue spaces and L^2-based Sobolev spaces, and that also respects the 'side conditions' that often occur in boundary value problems (e.g. Dirichlet boundary data or a divergence-free condition).

We have already mentioned in Section 3.4 that if $u \in L^2(\mathbb{T}^d)$ with

$$u = \sum_{k \in \mathbb{Z}^d} \hat{u}_k \mathrm{e}^{\mathrm{i}k \cdot x}, \tag{8.1}$$

and we set

$$u_n := \sum_{k \in \mathbb{Z}^d \,:\, |k| \leq n} \hat{u}_k \mathrm{e}^{\mathrm{i}k \cdot x},$$

where $|k|$ is the Euclidean length of k, then this truncation behaves well in L^2-based spaces:

$$\|u_n - u\|_X \to 0 \qquad \text{and} \qquad \|u_n\|_X \leq \|u\|_X$$

for $X = L^2(\mathbb{T}^d)$ or $H^s(\mathbb{T}^d)$.

However, the same is not true in $L^p(\mathbb{T}^d)$ for $p \neq 2$ if $d \neq 1$: there *is no constant* C such that

$$\|u_n\|_{L^p} \le C\|u\|_{L^p} \quad \text{for every} \quad u \in L^p(\mathbb{T}^3).$$

This follows from the result of Fefferman [1971] concerning the ball multiplier for the Fourier transform; standard 'transference' results (see Grafakos [2014] for example) then yield the result for Fourier series. There are similar problems when using eigenfunction expansions in bounded domains, see Babenko [1973].

In the periodic setting these problems can be overcome by considering the truncation over 'cubes' rather than 'spheres' of Fourier modes. If for u as in (8.1) we define

$$u_{[n]} := \sum_{|k_j|\le n} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}, \qquad \text{where } k = (k_1, \ldots, k_d),$$

then it follows from good properties of the truncation in 1D and the product structure of the Fourier expansion that

$$\|u_{[n]} - u\|_{L^p} \to 0 \qquad \text{and} \qquad \|u_{[n]}\|_{L^p} \le C_p\|u\|_{L^p}, \quad u \in L^p(\mathbb{T}^d)$$

(see Muscalu and Schlag [2013], for example). We have already used this approach in Section 3.4 to prove that all weak solutions of the 'critical' ($r = 3$) convective Brinkman–Forchheimer equations

$$\partial_t u - \mu\Delta u + (u\cdot\nabla)u + \beta|u|^2 u + \nabla p = 0, \qquad \nabla\cdot u = 0 \tag{8.2}$$

on the torus $\mathbb{T}^3$ satisfy the energy equality.

There is no known corresponding 'good' selection of eigenfunctions in bounded domains that will produce truncations that are bounded in L^p. To circumvent this we suggest two possible approximation schemes in this chapter: for one scheme we use the linear semigroup arising from an appropriate differential operator (the Laplacian or Stokes operator); for the second we combine this with a truncated eigenfunction expansion.

We discuss these methods in the abstract setting of fractional power spaces in Section 8.1. Since we have already identified these fractional power spaces explicitly for the Dirichlet Laplacian and Stokes operators in the previous chapter, these abstract results can be made more concrete to provide the properties of operators required for our approximation schemes.

We emphasise here that we require the more refined result of Proposition 8.2 (approximation in finite-dimensional eigenspaces) in our application to the critical CBF equations. The 'approximation by semigroup' method from Lemma 8.1 is not sufficient since our approximating functions do not have compact supports and therefore cannot be used as test functions in the standard weak formulation (3.6) of the equations. We circumvent this problem using the 'eigenspace approximation' and Lemma 3.5. We also think that the eigenspace approximation is interesting in its own right, and likely to prove useful in Galerkin-based methods in bounded domains.

8.1 Approximation in fractional power spaces

We want to investigate simultaneous approximation in fractional power spaces and a second space $\mathcal{L}$, which in our applications will be one of the spaces $L^p(\Omega)$ [potentially with side conditions when treating divergence-free vector-valued functions].

8.1.1 Assumptions on the space $\mathcal{L}$

Let the operator A and the space H be as in the previous chapter (see Section 7.1). Now suppose that we have a Banach space $\mathcal{L}$ such that

($\mathcal{L}$-i) For some $\gamma > 0$ either

$$D(A^\gamma) \hookrightarrow \mathcal{L} \hookrightarrow H \tag{8.3}$$

or

$$H \hookrightarrow \mathcal{L} \hookrightarrow D(A^{-\gamma}), \tag{8.4}$$

and

($\mathcal{L}$-ii) $\mathrm{e}^{-\theta A}$ is a uniformly bounded operator on $\mathcal{L}$ for $\theta \geq 0$, i.e. there exists a constant $C_{\mathcal{L}} > 0$ such that

$$\left\| \mathrm{e}^{-\theta A} u \right\|_{\mathcal{L}} \leq C_{\mathcal{L}} \left\| u \right\|_{\mathcal{L}} \quad \text{for} \quad \theta \geq 0, \tag{8.5}$$

and $\mathrm{e}^{-\theta A}$ is a strongly continuous semigroup on $\mathcal{L}$, i.e. for each $u \in \mathcal{L}$

$$\left\| \mathrm{e}^{-\theta A} u - u \right\|_{\mathcal{L}} \to 0 \quad \text{as} \quad \theta \to 0^+. \tag{8.6}$$

We assume that the inclusions in ($\mathcal{L}$-i) are continuous (so, for example, $\mathcal{L} \hookrightarrow H$ means that we also have $\|u\| \leq C_{\mathcal{L}\to H} \|u\|_{\mathcal{L}}$ for some constant $C_{\mathcal{L}\to H}$).

Note that the embedding $\mathcal{L} \hookrightarrow H$ from (8.3) ensures that the definition of the semigroup in (7.4) makes sense for $u \in \mathcal{L}$, while if instead we have (8.4) then we can use the natural definition of $\mathrm{e}^{-\theta A}$ on $D(A^{-\gamma})$ to interpret $\mathrm{e}^{-\theta A} u$ for $u \in \mathcal{L}$.

8.1.2 Approximation using the semigroup

Using the semigroup $e^{-\theta A}$ we can easily approximate any $u \in D(A^\alpha) \cap \mathcal{L}$ in a 'good way' in both $D(A^\alpha)$ and $\mathcal{L}$. The following lemma simply combines the facts above to make this more explicit.

Lemma 8.1. *Suppose that* ($\mathcal{L}$-i) *and* ($\mathcal{L}$-ii) *hold. If* $u \in D(A^\alpha) \cap \mathcal{L}$ *for some* $\alpha \in \mathbb{R}$ *and* $u_\theta := e^{-\theta A} u$ *then*

(i) $u_\theta \in D(A^\beta)$ *for every* $\beta \in \mathbb{R}$ *when* $\theta > 0$;

(ii) $\|u_\theta\|_{D(A^\alpha)} \le \|u\|_{D(A^\alpha)}$ *for all* $\theta > 0$;

(iii) $\|u_\theta\|_{\mathcal{L}} \le C_{\mathcal{L}} \|u\|_{\mathcal{L}}$ *for all* $\theta > 0$; *and*

(iv) $u_\theta \to u$ *in* $\mathcal{L}$ *and in* $D(A^\alpha)$ *as* $\theta \to 0^+$.

Note that if $u \in \mathcal{L}$ and ($\mathcal{L}$-i) holds then we can always find a value of $\alpha \in \mathbb{R}$ so that $u \in D(A^\alpha) \cap \mathcal{L}$: if we have (8.3) then $u \in \mathcal{L} \cap H$ (since $\mathcal{L} \hookrightarrow H$), while if (8.4) holds then $u \in \mathcal{L}$ means that $u \in D(A^{-\gamma})$. If we want to apply the lemma as stated assuming explicitly only that $u \in D(A^\alpha)$ then to ensure that we also have $u \in \mathcal{L}$ we need to have $\alpha \ge \gamma$ under (8.3) or $\alpha \ge 0$ under (8.4). Nevertheless, we always have (i), (ii), and (iv) for $u \in D(A^\alpha)$ for any $\alpha \in \mathbb{R}$.

Proof. Parts (i) and (ii) both follow from (7.5), (iii) is (8.5), and (iv) combines (7.6) and (8.6). □

Use of a semigroup like this can provide a natural way to produce a smooth approximation that is well tailored to the particular problem under consideration; see Robinson and Sadowski [2014] for one example in the context of the Navier–Stokes equations (a straightforward proof of local well-posedness in $L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$).

8.1.3 Approximation using eigenspaces

We now want to obtain a similar approximation result, but for a set of approximations that lie in the finite-dimensional space spanned by eigenfunctions of an operator A satisfying the conditions above. This is the key abstract result of this chapter; as with Lemma 8.1 its use in applications relies on the explicit identification of the fractional power spaces of certain common operators that we obtained in Section 7.3. Thanks to Lemma 3.5 we will be able to use this approximation method in Chapter 9 for an application to the critical convective Brinkman–Forchheimer equations.

Proposition 8.2. *Suppose that ($\mathcal{L}$-i) and ($\mathcal{L}$-ii) hold. For $\theta > 0$ set*

$$\Pi_\theta u := \sum_{\lambda_n < \theta^{-2}} \mathrm{e}^{-\theta\lambda_n} \langle u, w_n \rangle w_n.$$

Then

(i) *the range of Π_θ is the linear span of a finite number of eigenfunctions of A, so in particular $\Pi_\theta u \in D(A^\alpha)$ for every $\alpha \in \mathbb{R}$, and*

(ii) *if $X = \mathcal{L}$ or $D(A^\alpha)$ for any $\alpha \in \mathbb{R}$, then*

  (a) *Π_θ is a bounded operator on X, uniformly for $\theta > 0$, and*

  (b) *for any $u \in X$ we have $\Pi_\theta u \to u$ in X as $\theta \to 0^+$.*

Proof. Property (i) is immediate from the definition of Π_θ.

For (ii) we start with an auxiliary estimate for $u \in D(A^\beta)$, $\beta \leq \alpha$. If

$$u = \sum_{n=1}^{\infty} \langle u, w_n \rangle w_n$$

then for every $\theta > 0$ we have

$$\begin{aligned}
\left\| \Pi_\theta u - \mathrm{e}^{-\theta A} u \right\|_{D(A^\alpha)}^2 &= \sum_{\lambda_n \geq \theta^{-2}} \lambda_n^{2\alpha} \mathrm{e}^{-2\lambda_n \theta} \left| \langle u, w_n \rangle \right|^2 \\
&\leq \sum_{\lambda_n \geq \theta^{-2}} \lambda_n^{2\alpha} \mathrm{e}^{-2{\lambda_n}^{1/2}} \left| \langle u, w_n \rangle \right|^2 \\
&\leq \sum_{\lambda_n \geq \theta^{-2}} \lambda_n^{2(\alpha-\beta)} \mathrm{e}^{-2{\lambda_n}^{1/2}} \lambda_n^{2\beta} \left| \langle u, w_n \rangle \right|^2 \\
&\leq \left(\sup_{\lambda \geq \theta^{-2}} \lambda^{2(\alpha-\beta)} \mathrm{e}^{-2\lambda^{1/2}} \right) \|u\|_{D(A^\beta)}^2 .
\end{aligned}$$

If for each $\kappa \in \mathbb{R}$ we set

$$\Phi(\theta, \kappa) := \sup_{\lambda \geq \theta^{-2}} \lambda^\kappa \mathrm{e}^{-\lambda^{1/2}}$$

then we have

$$\left\| \Pi_\theta u - \mathrm{e}^{-\theta A} u \right\|_{D(A^\alpha)} \leq \Phi(\theta, \alpha - \beta) \|u\|_{D(A^\beta)} \quad \text{for} \quad \beta \leq \alpha. \tag{8.7}$$

Since

$$\Phi(\theta,\kappa) = \begin{cases} \theta^{-2\kappa}\mathrm{e}^{-1/\theta} & \kappa < 0 \text{ or } \kappa \ge 0,\ \theta \le (2\kappa)^{-1}, \\ (2\kappa)^{2\kappa}\mathrm{e}^{-2\kappa} & \kappa \ge 0,\ \theta > (2\kappa)^{-1}, \end{cases}$$

we have $\Phi(\theta,\kappa) \le M_\kappa$ for every $\theta > 0$ and

$$\Phi(\theta,\kappa) \to 0 \quad \text{as} \quad \theta \to 0^+ \quad \text{for every } \kappa \ge 0. \tag{8.8}$$

It is immediate that Π_θ is bounded on $D(A^\alpha)$ given that Π_θ only decreases the modulus of the Fourier coefficients ($\mathrm{e}^{-\theta\lambda_n} \le 1$):

$$\|\Pi_\theta u\|_{D(A^\alpha)} \le \|u\|_{D(A^\alpha)}.$$

The convergence $\|\Pi_\theta u - u\|_{D(A^\alpha)} \to 0$ as $\theta \to 0^+$, follows from (8.7) and (8.8) with $\beta = \alpha$ and the fact that $\mathrm{e}^{-\theta A}u \to u$ in $D(A^\alpha)$ as $\theta \to 0^+$; we have

$$\|\Pi_\theta u - u\|_{D(A^\alpha)} \le \left\|\Pi_\theta u - \mathrm{e}^{-\theta A}u\right\|_{D(A^\alpha)} + \left\|\mathrm{e}^{-\theta A}u - u\right\|_{D(A^\alpha)} \to 0$$

as $\theta \to 0^+$.

Now suppose that $u \in \mathcal{L}$ and (8.3) holds. Then we have $u \in H$ with $\|u\| \le C_{\mathcal{L}\to H}\|u\|_{\mathcal{L}}$, and there is a $\gamma > 0$ such that $D(A^\gamma) \hookrightarrow \mathcal{L}$ with

$$\|u\|_{\mathcal{L}} \le C_{D(A^\gamma)\to\mathcal{L}}\|u\|_{D(A^\gamma)}.$$

Then we have

$$\begin{aligned}
\|\Pi_\theta u\|_{\mathcal{L}} &= \left\|(\Pi_\theta u - \mathrm{e}^{-\theta A}u) + \mathrm{e}^{-\theta A}u\right\|_{\mathcal{L}} \\
&\le C_{D(A^\gamma)\to\mathcal{L}} \left\|\Pi_\theta u - \mathrm{e}^{-\theta A}u\right\|_{D(A^\gamma)} + \left\|\mathrm{e}^{-\theta A}u\right\|_{\mathcal{L}} \\
&\le C_{D(A^\gamma)\to\mathcal{L}}\Phi(\theta,\gamma)\,\|u\| + C_{\mathcal{L}}\,\|u\|_{\mathcal{L}} \\
&\le \left[C_{D(A^\gamma)\to\mathcal{L}}C_{\mathcal{L}\to H}\Phi(\theta,\gamma) + C_{\mathcal{L}}\right]\|u\|_{\mathcal{L}},
\end{aligned}$$

using (8.5) and (8.7) with $(\alpha,\beta) = (\gamma,0)$. It follows (since $\Phi(\theta,\gamma) \le M_\gamma$ independent of θ) that

$$\|\Pi_\theta u\|_{\mathcal{L}} \le K_{\mathcal{L}}\,\|u\|_{\mathcal{L}},$$

so $\Pi_\theta\colon \mathcal{L} \to \mathcal{L}$ is bounded for all $\theta > 0$. Convergence of $\Pi_\theta u$ to u as $\theta \to 0^+$ follows similarly, since

$$
\begin{aligned}
\|\Pi_\theta u - u\|_{\mathcal{L}} &\leq \left\|\Pi_\theta u - \mathrm{e}^{-\theta A} u\right\|_{\mathcal{L}} + \left\|\mathrm{e}^{-\theta A} u - u\right\|_{\mathcal{L}} \\
&\leq C_{D(A^\gamma)\to\mathcal{L}} \left\|\Pi_\theta u - \mathrm{e}^{-\theta A} u\right\|_{D(A^\gamma)} + \left\|\mathrm{e}^{-\theta A} u - u\right\|_{\mathcal{L}} \\
&\leq C_{D(A^\gamma)\to\mathcal{L}} \Phi(\theta,\gamma) \|u\| + \left\|\mathrm{e}^{-\theta A} u - u\right\|_{\mathcal{L}}
\end{aligned}
$$

and both terms tend to zero as $\theta \to 0^+$.

If, instead of (8.3), (8.4) holds then we have $H \hookrightarrow \mathcal{L} \hookrightarrow D(A^{-\gamma})$ and $\|f\|_{D(A^{-\gamma})} \leq C_{\mathcal{L}\to D(A^{-\gamma})} \|f\|_{\mathcal{L}}$ for any $f \in \mathcal{L}$. Then using (8.7) with $(\alpha, \beta) = (0, -\gamma)$ we obtain

$$
\begin{aligned}
\|\Pi_\theta u\|_{\mathcal{L}} &= \left\|(\Pi_\theta u - \mathrm{e}^{-\theta A} u) + \mathrm{e}^{-\theta A} u\right\|_{\mathcal{L}} \\
&\leq C_{H\to\mathcal{L}} \left\|\Pi_\theta u - \mathrm{e}^{-\theta A} u\right\| + \left\|\mathrm{e}^{-\theta A} u\right\|_{\mathcal{L}} \\
&\leq C_{H\to\mathcal{L}} \Phi(\theta,\gamma) \|u\|_{D(A^{-\gamma})} + C_{\mathcal{L}} \|u\|_{\mathcal{L}} \\
&\leq \left[C_{H\to\mathcal{L}} C_{\mathcal{L}\to D(A^{-\gamma})} \Phi(\theta,\gamma) + C_{\mathcal{L}}\right] \|u\|_{\mathcal{L}} =: K_{\mathcal{L}} \|u\|_{\mathcal{L}},
\end{aligned}
$$

and for the convergence we have

$$
\begin{aligned}
\|\Pi_\theta u - u\|_{\mathcal{L}} &\leq \left\|\Pi_\theta u - \mathrm{e}^{-\theta A} u\right\|_{\mathcal{L}} + \left\|\mathrm{e}^{-\theta A} u - u\right\|_{\mathcal{L}} \\
&\leq C_{H\to\mathcal{L}} \left\|\Pi_\theta u - \mathrm{e}^{-\theta A} u\right\| + \left\|\mathrm{e}^{-\theta A} u - u\right\|_{\mathcal{L}} \\
&\leq C_{H\to\mathcal{L}} \Phi(\theta,\gamma) \|u\|_{D(A^{-\gamma})} + \left\|\mathrm{e}^{-\theta A} u - u\right\|_{\mathcal{L}} \\
&\leq C_{H\to\mathcal{L}} C_{\mathcal{L}\to D(A^{-\gamma})} \Phi(\theta,\gamma) \|u\|_{\mathcal{L}} + \left\|\mathrm{e}^{-\theta A} u - u\right\|_{\mathcal{L}} \to 0
\end{aligned}
$$

as $\theta \to 0^+$. □

8.2 Approximation respecting Dirichlet boundary data

We can now combine the abstract approximation results from Section 8.1 with the characterisation of fractional power spaces from the previous chapter to give some more explicit approximation results. In all that follows we let Ω be a smooth bounded domain in $\mathbb{R}^d$, and by 'smooth function on Ω' we mean that a function is an element of $C^\infty(\overline{\Omega})$.

In the abstract setting of Section 8.1 we take $H = L^2(\Omega)$, we let $A = -\Delta$, where Δ is the Laplacian on Ω with Dirichlet boundary conditions, and we take $\mathcal{L} = L^p(\Omega)$ for some $p \in (1, \infty)$ with $p \neq 2$.

We need to check the assumptions ($\mathcal{L}$-i) and ($\mathcal{L}$-ii) from Section 8.1.1 on the relationship between the spaces $\mathcal{L}$ and $D(A^\alpha)$.

($\mathcal{L}$-i) If we take $\mathcal{L} = L^p(\Omega)$ with $p \in (2,\infty)$ then since we are on a bounded domain, we have

$$\mathcal{L} = L^p(\Omega) \hookrightarrow L^2(\Omega)$$

and we can choose $\gamma \geq d(p-2)/(4p)$ so that

$$D(A^\gamma) \hookrightarrow H^{2\gamma}(\Omega) \hookrightarrow L^p(\Omega) = \mathcal{L}.$$

In this case (8.3) holds. If $\mathcal{L} = L^q(\Omega)$ for some $1 < q < 2$ we have $L^2(\Omega) \hookrightarrow L^q(\Omega)$, and since $L^q(\Omega)$ is the dual space of some $L^p(\Omega)$ with $p > 2$ we have

$$\mathcal{L} = L^q \simeq (L^p)' \hookrightarrow D(A^\gamma)' = D(A^{-\gamma}),$$

where $\gamma \geq d(2-q)/(4q)$.

($\mathcal{L}$-ii) That $\mathrm{e}^{-\theta A}$ is bounded on $L^p(\Omega)$ for each $1 < p < \infty$ follows from the analysis in Section 7.3 of Pazy [1983], as does the fact that $\mathrm{e}^{-\theta A}$ is a strongly continuous semigroup on $L^p(\Omega)$.

Our first approximation result uses the semigroup arising from the Dirichlet Laplacian, and is a corollary of Lemma 8.1.

Theorem 8.3. *If $u \in L^2(\Omega)$ then, for every $\theta > 0$, $u_\theta := \mathrm{e}^{-\theta A}u$ is smooth and zero on $\partial\Omega$. If in addition $u \in X$ then*

$$\|u_\theta\|_X \leq C_X\|u\|_X, \qquad \text{and} \qquad \|u_\theta - u\|_X \to 0 \text{ as } \theta \to 0^+,$$

where we can take X to be $H^s(\Omega)$ for $0 < s < 1/2$, $H^{1/2}_{00}(\Omega)$, $H^s_0(\Omega)$ for $1/2 < s \leq 1$, $H^s(\Omega) \cap H^1_0(\Omega)$ for $1 < s \leq 2$, or $L^p(\Omega)$ for any $p \in (1,\infty)$.

Proof. By part (i) of Lemma 8.1 we have $u_\theta \in D(A^r)$ for every $r \geq 0$. In particular $u_\theta \in D(A) = H^2 \cap H^1_0$, so u_θ is zero on $\partial\Omega$. Since $D(A^r) \hookrightarrow H^{2r}(\Omega)$ (Corollary 7.9) it also follows that $u_\theta \in C^\infty(\overline{\Omega})$.

The boundedness in Sobolev spaces follows from part (ii) of Lemma 8.1 using the characterisation of $D(A^\alpha)$ in Theorem 7.4, and the convergence in Sobolev spaces from part (iv) with $X = D(A^\alpha)$. The boundedness and convergence in L^p follows from parts (iii) and (iv) of the same lemma. □

Proposition 8.2 yields a corresponding result on approximation that combines the semigroup with a truncated eigenfunction expansion.

Theorem 8.4. *Let $\{w_j\}$ denote the L^2-orthonormal eigenfunctions of the Dirichlet Laplacian on Ω with corresponding eigenvalues (λ_j), ordered so that $\lambda_{j+1} \geq \lambda_j$. For any $u \in L^2(\Omega)$ set*

$$u_\theta := \Pi_\theta u = \sum_{\lambda_n < \theta^{-2}} \mathrm{e}^{-\theta\lambda_n} \langle u, w_n \rangle w_n. \tag{8.9}$$

Then u_θ has all the properties given in Theorem 8.3, and lies in the linear span of a finite number of eigenfunctions of A for every $\theta > 0$.

8.3 Approximation respecting Dirichlet boundary conditions and zero divergence

To deal with functions that have zero divergence we take A to be the Stokes operator $\mathcal{A}$, and set $H = L^2_\sigma(\Omega)$ and $\mathcal{L} = L^p_\sigma(\Omega)$ for some $p \in (1, \infty)$, $p \neq 2$, where

$$L^p_\sigma(\Omega) = \text{completion of } \{\phi \in [C_0^\infty(\Omega)]^d \ : \ \nabla \cdot \phi = 0\} \text{ in the } L^p(\Omega)\text{-norm.}$$

Property ($\mathcal{L}$-i) from Section 8.1.1 is checked as before, using the facts that $(L^p_\sigma)' \simeq L^q_\sigma$ when (p, q) are conjugate (see Theorem 2 part (2) in Fujiwara and Morimoto [1977]) and that we have a continuous inclusion $D(\mathcal{A}^\gamma) \hookrightarrow D(A^\gamma)$ from Lemma 7.10, where A is the Dirichlet Laplacian. The properties in ($\mathcal{L}$-ii) for the semigroup $\mathrm{e}^{-\mathcal{A}t}$ on $L^p_\sigma(\Omega)$ can be found in Miyakawa [1981] (Theorem 2.1) or Giga [1981].

Theorem 8.5. *Assume that $\Omega \subset \mathbb{R}^d$ with $d \leq 4$. Take $u \in L^2(\Omega)$ and for every $\theta > 0$ let*

$$u_\theta := \mathrm{e}^{-\theta\mathcal{A}} u \qquad \text{or} \qquad u_\theta := \Pi_\theta u,$$

where Π_θ is defined as in (8.9), but now $\{w_j\}$ are the eigenfunctions of $\mathcal{A}$. Then u_θ is smooth, zero on $\partial\Omega$, and divergence free. If in addition $u \in X$ then

$$\|u_\theta\|_X \leq C_X \|u\|_X, \qquad \text{and} \qquad \|u_\theta - u\|_X \to 0 \text{ as } \theta \to 0^+,$$

where we can take X to be $H^s(\Omega) \cap L^2_\sigma(\Omega)$ for $0 < s < 1/2$, $H^{1/2}_{00}(\Omega) \cap L^2_\sigma(\Omega)$, $H^s_0(\Omega) \cap L^2_\sigma(\Omega)$ for $1/2 < s \leq 1$, $H^s(\Omega) \cap H^1_0(\Omega) \cap L^2_\sigma(\Omega)$ for $1 < s \leq 2$, or $L^p_\sigma(\Omega)$ for any $p \in (1, \infty)$.

As before, this result follows by combining Lemma 8.1, Proposition 8.2, and the identification of the fractional power spaces of the Stokes operator in Theorem 7.4. The restriction to $d \leq 4$ is to ensure that $D(\mathcal{A}) \subset H^2 \subset L^p$ for every $p \in (1, \infty)$. Without restriction on the dimension we then have to restrict to $1 < p \leq 2d/(d-4)$.

Chapter 9

Energy equality on bounded domains

In this chapter we apply the eigenspace approximation result of Theorem 8.5 to prove energy conservation for the 3D 'critical' ($r = 3$) convective Brinkman–Forchheimer equations on smooth bounded domains $\Omega \subset \mathbb{R}^3$

$$\partial_t u - \mu\Delta u + (u \cdot \nabla)u + \nabla p + \beta|u|^2 u = 0, \qquad u|_{\partial\Omega} = 0, \qquad \nabla \cdot u = 0. \tag{9.1}$$

We do not give full details of the argument that guarantees the validity of the energy equality for weak solutions of (9.1), since it follows that in Section 3.4 extremely closely. Instead, we give a sketch of the proof, showing how Theorem 8.5 allows the argument to be extended to the critical CBF equations on bounded domains.

9.1 Proof of the energy equality

In this section we sketch a proof of the following theorem.

Theorem 9.1. *Every weak solution of* (9.1) *with initial condition* $u_0 \in H$ *satisfies the* energy equality*:*

$$\|u(t_1)\|^2 + 2\mu \int_{t_0}^{t_1} \|\nabla u(s)\|^2 \, \mathrm{d}s + 2\beta \int_{t_0}^{t_1} \|u(s)\|^4_{L^4(\Omega)} \, \mathrm{d}s = \|u(t_0)\|^2 \tag{9.2}$$

for all $0 \le t_0 < t_1 < T$*. Hence, all weak solutions are continuous functions into the phase space* L^2*, i.e.* $u \in C([0,T]\,;H)$*.*

Note that to prove this result we require the more refined result of Proposition

8.2, which enable an approximation that uses only finite-dimensional eigenspaces of the Stokes operator. This approximation is not compactly-supported but Lemma 3.5 allows us to use it as a test function in the weak formulation (3.6). The 'approximation by semigroup' result of Lemma 8.1 is not sufficient since we do not have a version of Lemma 3.5 for this kind of approximations.

Proof. (Sketch) We only sketch the proof, which follows that from Section 3.4, which in turn is based on the argument presented in Galdi [2000].

We approximate $u(t)$ for each $t \in [0, T]$ in such a way that

(i) $u_n(t) \in \tilde{\mathcal{D}}_\sigma(\Omega)$,

(ii) $u_n(t) \to u(t)$ in $H_0^1(\Omega)$ with $\|u_n(t)\|_{H^1} \leq C\|u(t)\|_{H^1}$,

(iii) $u_n(t) \to u(t)$ in $L^4(\Omega)$ with $\|u_n(t)\|_{L^4} \leq C\|u(t)\|_{L^4}$, and

(iv) $u_n(t)$ is divergence free and zero on $\partial\Omega$,

with (ii)–(iv) holding for almost every $t \in [0, T]$. In (i) we want $u_n(t)$ to be in a finite-dimensional space spanned by the first n eigenfunctions of the Stokes operator; $\tilde{\mathcal{D}}_\sigma(\Omega)$ is defined similarly as $\tilde{\mathcal{D}}_\sigma(\Omega_T)$ in Section 3.2.1

$$\tilde{\mathcal{D}}_\sigma(\Omega) := \left\{ \varphi \,:\, \varphi = \sum_{k=1}^{N} \alpha_k a_k(x), \quad \alpha_k \in \mathbb{R}, \quad a_k \in \mathcal{N},\, N \in \mathbb{N} \right\}.$$

We can obtain such an approximation using Theorem 8.5 by setting

$$u_n(t) := \Pi_{1/n} u(t) = \sum_{\lambda_j < n^2} \mathrm{e}^{-\lambda_j/n} \langle u(t),\, w_j \rangle\, w_j$$

for each $t \in [0, T]$ (w_j and λ_j are the eigenfunctions and eigenvalues of the Stokes operator on Ω, as in Section 7.1).

In the proof we will need the fact that

$$\|u_n - u\|_{L^4(0,T;L^4)} \to 0 \qquad \text{as} \qquad n \to \infty, \tag{9.3}$$

which follows from (iii): since $u \in L^4(0, T; L^4)$ and $\|u_n(t) - u(t)\|_{L^4} \to 0$ for almost every $t \in [0, T]$ we can obtain (9.3) by an application of the Dominated Convergence Theorem (with dominating function $(1 + C)\|u(t)\|_{L^4}$). A similar argument (using (ii)) shows that

$$\|u_n - u\|_{L^2(0,T;H^1)} \to 0 \qquad \text{as} \qquad n \to \infty. \tag{9.4}$$

To prove the energy equality for some time $t_1 > 0$ we set

$$u_n^h(t) := \int_0^{t_1} \eta_h(t-\tau)u_n(\tau)\,\mathrm{d}\tau, \tag{9.5}$$

where η_h is an even mollifier. Since $u_n^h \in \tilde{\mathcal{D}}_\sigma(\Omega_T)$ we can use it as a test function in (3.6) (with $t = t_1$) thanks to Lemma 3.5:

$$\begin{aligned}
-\int_0^{t_1} \langle u(s), \partial_t u_n^h(s)\rangle\,\mathrm{d}s + \mu\int_0^{t_1} \langle \nabla u(s), \nabla u_n^h(s)\rangle\,\mathrm{d}s + \int_0^{t_1} \langle (u(s)\cdot\nabla)u(s), u_n^h(s)\rangle\,\mathrm{d}s \\
+ \beta\int_0^{t_1} \langle |u(s)|^2 u(s), u_n^h(s)\rangle\,\mathrm{d}s = -\langle u(t_1), u_n^h(t_1)\rangle + \langle u(0), u_n^h(0)\rangle.
\end{aligned}$$

We first take the limit as $n \to \infty$. The limits in the linear terms are relatively straightforward. In the Navier–Stokes nonlinearity we can use

$$\begin{aligned}
&\left|\int_0^{t_1} \langle (u(s)\cdot\nabla)u_n^h(s), u(s)\rangle\,\mathrm{d}s - \int_0^{t_1} \langle (u(s)\cdot\nabla)u^h(s), u(s)\rangle\,\mathrm{d}s\right| \\
&\le \int_0^{t_1} \|u(s)\|_{L^4}^2 \|\nabla u_n^h(s) - \nabla u^h(s)\|\,\mathrm{d}s \le \|u\|_{L^4(0,T;L^4)}^2\, \|u_n^h - u^h\|_{L^2(0,T;H_0^1)}.
\end{aligned}$$

In the Brinkman–Forchheimer term $|u|^2 u$ we have

$$\begin{aligned}
&\left|\int_0^{t_1} \langle\, |u(s)|^2\, u(s), u_n^h(s)\rangle\,\mathrm{d}s - \int_0^{t_1} \left\langle |u(s)|^2\, u(s),\ u^h(s)\right\rangle\,\mathrm{d}s\right| \\
&\le \int_0^{t_1} \|u(s)\|_{L^4}^3\, \|u_n^h(s) - u^h(s)\|_{L^4}\,\mathrm{d}s \le \|u\|_{L^4(0,T;L^4)}^3\, \|u_n^h - u^h\|_{L^4(0,T;L^4)}.
\end{aligned}$$

By our choice of u^h we have

$$\int_0^{t_1} \langle u(s), \partial_t u^h(s)\rangle\,\mathrm{d}s = \int_0^{t_1}\int_0^{t_1} \dot{\eta}_h(t-s)\langle u(t), u(s)\rangle\,\mathrm{d}t\,\mathrm{d}s = 0$$

and so

$$\begin{aligned}
\mu\int_0^{t_1} \langle \nabla u(s), \nabla u^h(s)\rangle\,\mathrm{d}s + \int_0^{t_1} \langle (u(s)\cdot\nabla)u(s), u^h(s)\rangle\,\mathrm{d}s \\
+ \beta\int_0^{t_1} \langle |u(s)|^2 u(s), u^h(s)\rangle\,\mathrm{d}s = -\langle u(t_1), u^h(t_1)\rangle + \langle u(0), u^h(0)\rangle.
\end{aligned}$$

Next we let $h \to 0$, for which the argument is similar; we use the facts that the mollifier η_h integrates to $1/2$ on the positive real axis and that u is weakly continuous into L^2 (see Lemma 3.4) to show that the right-hand side in the above tends to

$$-\frac{1}{2}\|u(t_1)\|^2 + \frac{1}{2}\|u(0)\|^2.$$

Then we obtain the energy equality (9.2) proceeding as in Section 3.4.

The continuity of u into L^2 now follows by combining the weak continuity into L^2 and the continuity of $t \mapsto \|u(t)\|$, which is a consequence of the energy equality. $\square$

It is worth mentioning that we can now easily obtain the existence of the strong global attractor in the phase space H for the critical convective Brinkman–Forchheimer equations on a smooth bounded domain $\Omega \subset \mathbb{R}^3$. Indeed, since we just proved that the energy equality holds also on bounded domains, we can repeat the argument used in the periodic case via Cheskidov's evolutionary systems (as in Section 3.5). Note that on bounded domains $u(t) \in H_0^1(\Omega)$, so we can use the Poincaré inequality $\|u\|_{L^2(\Omega)} \leq \lambda_1^{-1} \|\nabla u\|_{L^2(\Omega)}$ to prove the existence of the absorbing set (cf. Proposition 3.20 in Section 3.5 for the periodic case) in a similar way as it is done for the Navier–Stokes equations (see e.g. Proposition 13.1 in Constantin and Foias [1988]).

Chapter 10

Conclusions and open problems

As we have seen, the convective Brinkman–Forchheimer equations constitute mathematically interesting modification of the famous Navier–Stokes model. In a thesis we studied the influence of the absorption term $|u|^{r-1} u$ on the properties of solutions of this model. When $r > 0$ is large enough, this term guarantees the existence of global strong solutions. For the 'critical' case $r = 3$ we get some conditional results on the existence of global regular solutions. When it comes to weak solutions, in the critical case the energy equality holds for all weak solutions. The case $r \in [1, 3)$ is still open in that regard. Uniqueness of weak solutions is also open for $r \in [1, 3]$. In the light of the recent result of nonuniqueness for 'distributional' weak solutions of the NSE (see Buckmaster and Vicol [2019]), it may be worth checking if a similar method would work for the critical CBF equations. However, the main problem for this model seems to be obtaining or disproving the existence of global-in-time regular solutions for the case $r = 3$. We conjecture that this case mimics the behaviour of the Navier–Stokes equations, i.e. if there is a blow-up in a finite time for the NSE, then also the critical CBF equations exhibit a blow-up in a finite time, and if regular solutions for the NSE exist for all times then the same should hold for the critical CBF equations.

Going back to Theorem 5.2, it is natural to ask what kind of condition, if any, is required if we consider 'robustness of regularity' with respect to the absorption exponent r. To focus our attention on the dependence on the exponents let us take $u_0 \equiv v_0 \in V$ and $f \equiv g \in L^2(0, T; H)$, in such a way that u is a strong solution on the time interval $[0, T]$ of the CBF equations with initial condition u_0 and the absorption exponent $s \geq 1$ (if $s > 3$ we know that it is in fact a global-in-time strong solution) and let v be a weak solution of the CBF equations with initial condition v_0 and the absorption exponent $r \in [1, 3]$ ($r < s$). We know that v is also a strong solution on some time interval $[0,\tilde{T}]$. We want to find a condition for exponents r and s depending only on the function u which ensures that v remains a strong solution at least on the time interval $[0,T]$.

The only new obstacle in the problem described above lies in estimating the difference $C_s(u)-C_r(v)$. We observe that

$$\begin{aligned}\left||u|^{s-1}u-|v|^{r-1}v\right| &\le \left||u|^{s-1}u-|u|^{r-1}u\right|+\left||u|^{r-1}u-|v|^{r-1}v\right|\\ &\le |u|^r\left||u|^{s-r}-1\right|+\left||u|^{r-1}u-|v|^{r-1}v\right|. \end{aligned} \tag{10.1}$$

We have already seen how to deal with the second term on the right-hand side of (10.1) [see (5.10) and the following lines]. Therefore, using similar arguments to those in the proof of Theorem 5.2 we can obtain a robustness condition for the absorption exponents

$$c_0\int_0^T\left(\int_{\mathbb{T}^3}|u|^{2r}\left||u|^{s-r}-1\right|^2\,\mathrm{d}x\right)^{1/2}\mathrm{d}t<R(u,r), \tag{10.2}$$

where $R(u,r)$ is equal to the constant $R(u)$ defined in Theorem 5.2; this constant is finite because u is the strong solution of the CBF equations with the absorption exponent s and $s>r$. On the other hand, the term on the left-hand side of (10.2) tends to 0 as $s-r\to 0^+$ (provided that the integral is bounded). Fixing $r=3$ and letting $s\to r^+$ we can see from the condition (10.2) how close we have to get with s to the critical case $r=3$ in order to ensure that the weak solution v is actually a strong solution on the time interval $[0,T]$.

In the works of Chernyshenko et al. [2007] or Dashti and Robinson [2008] the robustness of regularity for the Navier–Stokes equations is used to construct a numerical algorithm which can verify in a finite time regularity of a given strong solution. The second ingredient required in that construction is convergence of the Galerkin approximations to the strong solution. As we showed in Chapter 5, robustness of regularity can be extended to the convective Brinkman–Forchheimer equations with the absorption exponent $r\in[1,3]$. Using similar methods as presented there to deal with the additional nonlinearity $|u|^{r-1}u$, it should be possible to prove also for the CBF equations that the Galerkin approximations of a strong solution converge strongly to that solution in appropriate function spaces. Consequently, it should be possible to construct a similar algorithm for numerical verification of regularity for these equations as well.

Returning to the issues discussed in Chapter 8, recall that while the 'spherical' truncation of a Fourier expansion

$$u_n := \sum_{|k|\le n} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}$$

does not behave well in terms of boundedness/convergence in L^p spaces, the 'cubic' truncation

$$u_{[n]} := \sum_{|k_j|\le n} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}, \qquad k = (k_1, \ldots, k_d),$$

does. One can expect (cf. Babenko [1973]) that there are similar problems in using a straightforward truncation of an expansion in terms of an orthonormal family of eigenfunctions:

$$P_\lambda u := \sum_{\lambda_n \le \lambda} \langle u, w_n\rangle w_n,$$

(where $Aw_n = \lambda_n w_n$). It is natural to ask if there is a 'good' choice of eigenfunctions such that the truncations

$$P_n u := \sum_{w\in E_n} \langle u, w\rangle w,$$

where E_n is some collection of eigenfunctions, is well-behaved with respect to the L^p spaces. To our knowledge this is entirely open.

10.1 Partial regularity

In famous result Caffarelli et al. [1982] proved that the set of singular points for the Navier–Stakes equations cannot be too large (in terms of the box-counting and Hausdorff dimensions). A preliminary calculation suggests that the dimension of the set of singular times for the critical convective Brinkman–Forchheimer equations (note that the critical case $r = 3$ has the same scaling as the NSE) can be even smaller due to additional regularity of solution $u \in L^4(0, T; L^4)$. More precisely, it seems that the box-counting dimension of the singular set for the critical CBF inequality may be no larger than 1 which is better than 5/3 obtained in Caffarelli et al. [1982] for the Navier–Stokes inequality. Below we give a short sketch of this calculation neglecting the pressure.

At singular points (a, s) for the Navier–Stokes Inequality (see Caffarelli et al. [1982] for more details) we have either

$$\int_{Q_r(a,s)} |u|^3 \ge \varepsilon_0 r^2 \qquad \text{or} \qquad \int_{Q_r(a,s)} |p|^{3/2} \ge \varepsilon_0 r^2, \tag{10.3}$$

where Q_r is the parabolic cylinder

$$Q_r(a,s) := \left\{(x,t) : |x-a| < r,\ s-r^2 < t < s\right\}.$$

Interpolating we have

$$\int_{Q_r} |u|^3 \leq \left(\int_{Q_r} |u|^4\right)^{3/4} \left(\int_{Q_r} 1\right)^{1/4} \lesssim r^{5/4} \left(\int_{Q_r} |u|^4\right)^{3/4},$$

and hence

$$r^{-5/3}\left(\varepsilon_0 r^2\right)^{4/3} \leq r^{-5/3}\left(\int_{Q_r} |u|^3\right)^{4/3} \lesssim \int_{Q_r} |u|^4.$$

Therefore at singular points for the critical CBF inequality we obtain

$$\int_{Q_r} |u|^4 \gtrsim r{\varepsilon_0}^{4/3}.$$

Now suppose that $d = d_B(S\cap K) > 1$, where S denotes the set of space-time singularities, K any compact subset of Ω_T, and take $1 < \delta < d$. Then there exists a sequence $\varepsilon_j \to 0$, so that for each j there is a maximal collection $\{z_n^j\}$ of at least $\varepsilon_j^{-\delta}$ points in $S\cap K$ that are $2\varepsilon_j$-separated. For each j it follows that the ε_j-balls centred at the points z_n^j are disjoint, and from (10.3) - assuming that the lower bound on the u integral holds for the majority of points - we have ($\tilde{Q}_r$ is the centred cylinder $\tilde{Q}_r(a,s) := \{(x,t) : |x-a| < r,\ |s-t| < r^2/2\}$)

$$\int_{\Omega_T} |u|^4 \geq \sum_j \left(\int_{\tilde{Q}_{\varepsilon_j}} |u|^4\right) \gtrsim \varepsilon_j {\varepsilon_j}^{-\delta} = {\varepsilon_j}^{1-\delta} \to \infty$$

as $\varepsilon_j \to 0$, which contradicts the fact that $\int_{\Omega_T} |u|^4 < \infty$.

However, the partial regularity argument involves the pressure, which in our case has the additional component q that comes from the absorption term

$$-\Delta q = \partial_i(|u|^2 u_i) = 2u_i u_j \partial_i u_j.$$

So we need more work to verify if the iteration scheme and the regularity criterion used in Caffarelli et al. [1982] can be adapted to the critical convective Brinkman–Forchheimer equations.

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