LECTURE NOTES 0 FOR CAMBRIDGE PART III COURSE ON "ELEMENTARY METHODS IN ANALYTIC NUMBER THEORY", LENT 2015

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Abstract. These are rough notes explaining some preliminary details, mostly practical arrangements, basic notation, the course synopsis, and a few background facts, for the "Elementary Methods in Analytic Number Theory" course.

1. Practical matters

Lecture notes. As discussed later, this course will be divided into three main chapters. I will produce notes for each block of lectures, and post them on my webpage `https://www.dpmms.cam.ac.uk/~ajh228/` at some point, probably around the end of that block of lectures. I will try to write the notes carefully, but their main purpose is for my own reference when giving the lectures. So please come to the lectures, and take your own notes whilst there!

Books. The book that corresponds most closely to this course is Friedlander and Iwaniec, *Opera de Cribro*. This will closely match about the first third of the course, and say quite a lot about the remaining parts. Davenport, *Multiplicative Number Theory* has a good treatment of most of the middle third of the course. Montgomery and Vaughan, *Multiplicative Number Theory* contains some material on the first and last thirds of the course. You should be able to follow the course without access to these books, but they are certainly well worth a look if possible. The books by Davenport, and Montgomery and Vaughan, should be quite inexpensive and give a nice general introduction to analytic number theory.

Example sheets. I expect to write three examples sheets for the course, and probably have three examples classes this term followed by a revision class in Easter term. I will post the examples sheets on my webpage `https://www.dpmms.cam.ac.uk/~ajh228/` as I write them, and during the lectures we will agree a time for the examples classes.

2. Notation and conventions

This is an analysis course, and will involve estimating/bounding various quantities that are too complicated to understand exactly (or, sometimes, that we don't need to understand precisely). To facilitate this we will need a bit of notation.

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We write $f(x) = O(g(x))$, and say that f is "big Oh" of g, if there exists a constant C such that

$$|f(x)| \leq Cg(x) \quad \forall x.$$

Here we usually want the inequality to hold either for all x for which the functions are defined, or for all sufficiently large x (i.e. all x larger than some fixed constant). Which meaning is desired should always be made clear from the context.

We will also write $f(x) \ll g(x)$, which means the same as $f(x) = O(g(x))$.

We write $f(x) \asymp g(x)$, and say that f is *of order* g, if both

$$f(x) \ll g(x) \quad \text{and} \quad g(x) \ll f(x),$$

in other words if $(1/C)g(x) \leq |f(x)| \leq Cg(x)$ for all relevant x.

For $g(x) \neq 0$, we write $f(x) = o(g(x))$ as $x \to \infty$, and say that f is "little oh" of g as $x \to \infty$, if

$$\frac{f(x)}{g(x)} \to 0 \quad \text{as } x \to \infty.$$

One can similarly say that $f(x) = o(g(x))$ as $x \to 0$, for example.

Finally, for $g(x) \neq 0$ we write $f(x) \sim g(x)$ as $x \to \infty$, and say that f is *asymptotic to* g, if

$$\frac{f(x)}{g(x)} \to 1 \quad \text{as } x \to \infty.$$

Note this is the same as saying that $f(x) = (1 + o(1))g(x)$ as $x \to \infty$.

For example, for $x \geq 1$ we have

$$100x = O(x^2), \quad \log x = o(x) \quad \text{as } x \to \infty, \quad x^2 - 2x + 1 = x^2 + O(x), \quad x^3 - 10 \sim x^3 \quad \text{as } x \to \infty.$$

We will usually write C to mean a large constant, c to mean a small constant, and ϵ to mean a parameter close to zero. To economise on notation, in long arguments it is customary to use these symbols many times with different values at each use, *provided* this does not confuse the reader.

3. What will the course be about?

This course is about some of the things one can do in analytic number theory that don't involve applying Cauchy's Residue Theorem to functions like the Riemann zeta function. Such methods are traditionally called "elementary", but they can prove very deep things. The course is also about some things one can do by combining zeta function type information with "elementary" arguments.

To focus the course, we will mostly be trying to count prime numbers in various sequences. For example, we would like to know:

- how many primes are less than x ?

- how many primes are in the short interval $[x, x + x^{0.99}]$?
- how many primes less than x are congruent to 1 modulo $\lfloor x^{0.01} \rfloor$?
- how many primes $p \leq x$ are such that $p + 2$ is also prime?
- for a large integer N, how many primes $p \leq 2N$ are such that $2N - p$ is also prime?

Here x is an arbitrary large number, and $\lfloor \cdot \rfloor$ denotes integer part. The above questions are connected with some famous open problems, like the Riemann Hypothesis, the Generalised Riemann Hypothesis, the Twin Prime Conjecture, and the Goldbach Conjecture. Nevertheless we will obtain partial results on all these questions. In some cases "elementary" methods are the only techniques we know that are capable of doing this.

The course will have three main chapters.

(i) The first chapter will explore whether one can detect primes by removing all the multiples of smaller primes. We will formulate a very powerful method of this kind called the *Selberg sieve*, due to Selberg in 1947, and compare its performance on some of the above questions with zeta function methods.
(ii) In the second chapter we will investigate the distribution of general sequences in arithmetic progressions, by changing basis using a kind of discrete Fourier transform. This will lead to several results called *large sieve* inequalities. By combining these with zeta function methods we will prove a powerful result on primes in arithmetic progressions (called the *Bombieri–Vinogradov Theorem*, first proved in 1965).
(iii) The most fundamental result on the number of primes less than x is called the Prime Number Theorem. In the third chapter we will use ideas coming from Selberg's sieve to give an elementary proof of this theorem. We will also see how a variant of Selberg sieve weights, combined with the Bombieri–Vinogradov theorem, can prove the recent result that there are infinitely many *bounded gaps between primes.*

4. Facts about prime numbers

Before starting the course, it will be helpful to know some basic facts about the distribution of prime numbers. Let $\pi(x)$ denote the number of primes p that are $\leq x$.

Fact 1 (Chebychev, c. 1850)**.** *For any $x \geq 2$ we have*

$$\pi(x) \asymp \frac{x}{\log x}.$$

Fact 2 (Mertens, 1874)**.** *There exists a constant c_1 such that, for any $x \geq 2$, we have*

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1), \quad \textit{and} \quad \sum_{p \leq x} \frac{1}{p} = \log\log x + c_1 + O(\frac{1}{\log x}).$$

Fact 3 (Mertens, 1874)**.** *There exists a constant c_2 such that, for any $x \geq 2$, we have*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-c_2}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

The proofs of the above facts are short and elementary (but clever)— some will appear as exercises on the first problem sheet. It should be emphasised that these were some of the first facts to be proved about the distribution of primes since Euclid proved that $\pi(x) \to \infty$ as $x \to \infty$, many millenia earlier. This is a reminder that many results that seem straightforward with hindsight require considerable work and insight to obtain initially.

The reader should also bear in mind the known result that

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty.$$

This asymptotic is called the *Prime Number Theorem*, and was first proved by Hadamard and de la Vallée Poussin, independently, in 1896. Their proofs used the Riemann zeta function, but in chapter 3 we will give an elementary proof of this as well. As we go along I will explain some other classical analytic results about the distribution of primes, both to compare with the results that we obtain, and in a few cases because we will need to use them.

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