A different proof of a finite version of Vinogradov's bilinear sum inequality (NOTES)

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Abstract

We give a different proof of a finite version of Vinogradov's bilinear sum inequality, which is perhaps simpler than the proof in a recent preprint of Bourgain, Sarnak and Ziegler (although our proof yields a poorer bound). Our proof essentially follows an argument of Katai concerning exponential sums with multiplicative coefficients.

1 Introduction

In these brief notes we shall prove the following result:

Theorem 1. *Let F and v be number-theoretic functions taking values in the complex unit disc, and suppose additionally that v is multiplicative. Let $\tau > 0$ be a small parameter, and suppose that for any distinct primes $p_1, p_2 \leq e^{1/\tau}$, and for all $M \geq M_\tau$, we have*

$$\left| \sum_{m \leq M} F(p_1 m) \overline{F(p_2 m)} \right| \leq \tau M.$$

Then

$$\left| \sum_{n \leq N} v(n) F(n) \right| \leq \frac{N}{\sqrt{\log(1/\tau) + O(1)}} + O\left(\frac{N}{\log(1/\tau)} + \sqrt{N e^{1/\tau}} + M_\tau e^{1/\tau} \right).$$

Theorem 1 corresponds closely to Theorem 2 of Bourgain, Sarnak and Ziegler [1], except that they have the stronger bound $2\sqrt{\tau \log(1/\tau)} N$ for the sum over n (if N is large enough). However, the proof given by Bourgain, Sarnak and Ziegler involves a bit fiddly decomposition of the integers less than N, whereas our proof of Theorem 1 will be fairly short and easy.

Our proof of Theorem 1 is not very new— in fact it corresponds almost exactly to an argument of Katai [2] concerning exponential sums with multiplicative coefficients, the only real difference being that we need to introduce a dyadic decomposition to obtain an acceptable bound under our hypotheses. The proof is also clearly related to an argument of Montgomery and Vaughan [3] concerning exponential sums with multiplicative coefficients, and no doubt also to many other arguments in that area. As with the proof of Bourgain, Sarnak and Ziegler [1], the aim is just to introduce a double summation in place of the single sum over n, which will allow one to use the Cauchy–Schwarz inequality to remove the unknown weight function v and apply the hypothesis about F.

2 A lemma from probabilistic number theory

To prove Theorem 1 we shall require a lemma concerning the additive function

$$\omega_\tau(n) := \sum_{\substack{p|n, \\ p \le e^{1/\tau}}} 1.$$

Lemma 1. *Define* $\mu_\tau := \sum_{p \le e^{1/\tau}} 1/p$, *and let* N *be any natural number. Then we have the following variance estimate:*

$$\sum_{n \le N} (\omega_\tau(n) - \mu_\tau)^2 \le N\mu_\tau + O(e^{1/\tau}).$$

Lemma 1 is a special case of the Turán–Kubilius inequality, but since the proof is just a short calculation we shall give it in full. Expanding the sum in the statement we obtain

$$\sum_{p,q \le e^{1/\tau}} \sum_{n \le N} \mathbf{1}_{p,q|n} - 2\mu_\tau \sum_{p \le e^{1/\tau}} [N/p] + N\mu_\tau^2,$$

and on removing the square brackets, and paying attention to the diagonal contribution in the double sum, we see that is at most

$$\sum_{p,q \le e^{1/\tau}} [N/pq] - N\mu_\tau^2 + N\mu_\tau + 2\mu_\tau \pi(e^{1/\tau}),$$

which is certainly at most $N\mu_\tau + O(e^{1/\tau})$.

3 Proof of Theorem 1

In view of Lemma 1 and the Cauchy–Schwarz inequality, we have that

$$\begin{aligned} \left| \sum_{n \le N} v(n)F(n) \right| &= \left| \frac{1}{\mu_\tau} \sum_{n \le N} v(n)F(n) \sum_{\substack{p|n, \\ p \le e^{1/\tau}}} 1 + \sum_{n \le N} v(n)F(n) \frac{\mu_\tau - \omega_\tau(n)}{\mu_\tau} \right| \\ &\le \frac{1}{\mu_\tau} \left| \sum_{n \le N} v(n)F(n) \sum_{\substack{p|n, \\ p \le e^{1/\tau}}} 1 \right| + \sqrt{N(N\mu_\tau + O(e^{1/\tau}))/\mu_\tau^2}. \end{aligned}$$

Since v is multiplicative, and $|v(mp) - v(m)v(p)| \leq 2$ in any case,

$$\left|\sum_{n\leq N} v(n)F(n) \sum_{\substack{p|n,\\ p\leq e^{1/\tau}}} 1\right| \leq \left|\sum_{\substack{mp\leq N,\\ p\leq e^{1/\tau},\\ m\geq M_\tau}} v(m)v(p)F(mp)\right| + \sum_{n\leq N} \sum_{\substack{p^2|n,\\ p\leq e^{1/\tau}}} 2 + M_\tau e^{1/\tau}$$

$$\leq \sum_{\substack{j\geq 0,\\ 2^j\geq M_\tau}} \left|\sum_{2^j\leq m<2^{j+1}} v(m) \sum_{p\leq \min\{e^{1/\tau}, N/m\}} v(p)F(mp)\right| + O(N + M_\tau e^{1/\tau}).$$

Then using the Cauchy–Schwarz inequality, for any fixed j the inner sums have size at most

$$\sqrt{2^j \sum_{2^j\leq m<2^{j+1}} \left|\sum_{p\leq \min\{e^{1/\tau}, N/m\}} v(p)F(mp)\right|^2} \leq \sqrt{2^j \sum_{p_1,p_2\leq \min\{e^{1/\tau}, N/2^j\}} \left|\sum_{\substack{2^j\leq m<2^{j+1},\\ m\leq \min\{N/p_1, N/p_2\}}} F(mp_1)\overline{F(mp_2)}\right|}.$$

Here the contribution from the diagonal terms (where $p_1 = p_2$) is at most $2^j\sqrt{\pi(\min\{e^{1/\tau}, N/2^j\})}$, and in view of the hypotheses of Theorem 1 the contribution from the other terms is at most

$$\sqrt{2^j\tau(2^j + 2^{j+1}) \sum_{\substack{p_1,p_2\leq \min\{e^{1/\tau}, N/2^j\},\\ p_1\neq p_2}} 1}.$$

If we now sum over j, using Chebychev-type estimates for the prime counting function, we find that

$$\left|\sum_{n\leq N} v(n)F(n) \sum_{\substack{p|n,\\ p\leq e^{1/\tau}}} 1\right| \ll \sum_{2^j\leq \frac{N}{e^{1/\tau}}} 2^j\tau^{3/2}e^{1/\tau} + \sum_{\frac{N}{e^{1/\tau}}\leq 2^j\leq 2N} \sqrt{\frac{N2^j}{\log(N/2^j+1)} + \frac{\tau N^2}{\log^2(N/2^j+1)}}$$
$$+N + M_\tau e^{1/\tau}$$
$$\ll \tau^{3/2}N + N + N\sqrt{\tau}\log(1/\tau) + N + M_\tau e^{1/\tau}.$$

The above is all $\ll N + M_\tau e^{1/\tau}$, and Theorem 1 follows on recalling that we must divide by $\mu_\tau \gg \log(1/\tau)$ to obtain the contribution to our final bound.

References

[1] J. Bourgain, P. Sarnak, T. Ziegler. Disjointness of Möbius from horocycle flows. *Preprint.* 2011

[2] I. Katai. A remark on a theorem of H. Daboussi. *Acta Math. Hung.*, **47**, pp 223-225. 1986

[3] H. Montgomery, R. Vaughan. Exponential sums with multiplicative coefficients. *Invent. Math.*, **43**, no. 1, pp 69-82. 1977