LECTURE NOTES 0 FOR CAMBRIDGE PART III COURSE ON "THE RIEMANN ZETA FUNCTION", LENT 2014

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Abstract. These are rough notes explaining some preliminary details, mostly practical arrangements, basic notation, and the course synopsis, for "The Riemann Zeta Function" course.

1. Practical matters

Lecture notes. As discussed later, this course will be divided into three main chapters. I will produce notes for each block of lectures, and post them on my webpage `https://www.dpmms.cam.ac.uk/~ajh228/` at some point, probably around the end of that block of lectures. I will try to write the notes carefully, but their main purpose is for my own reference when giving the lectures. So please come to the lectures, and take your own notes whilst there!

Books. The books that correspond most closely to this course are Titchmarsh, *The Theory of the Riemann Zeta-function*, (for roughly the first half of the course), and Ivić, *The Riemann Zeta-Function. Theory and Applications* (for roughly the second half of the course). You should be able to follow the course without access to these books, but they are certainly well worth a look if possible. Other books on analytic number theory, such as Davenport, *Multiplicative Number Theory*; Iwaniec and Kowalski, *Analytic Number Theory*; Montgomery and Vaughan, *Multiplicative Number Theory*; will cover at least the introductory parts of the course. The books by Davenport, and Montgomery and Vaughan, should be quite inexpensive and give a nice general introduction to analytic number theory.

Example sheets. I expect to write three examples sheets for the course this term, and probably have two examples classes this term, and a third at the start of Easter term (together with a revision class). I will post the examples sheets on my webpage `https://www.dpmms.cam.ac.uk/~ajh228/` as I write them, and during the lectures we will agree a time for the examples classes.

Date: 15th January 2014.

2. Notation and conventions

This is an analysis course, and will involve estimating/bounding various quantities that are too complicated to understand exactly (or, sometimes, that we don't need to understand precisely). To facilitate this we will need a bit of notation.

We write $f(x) = O(g(x))$, and say that f is "big Oh" of g, if there exists a constant C such that

$$|f(x)| \leq Cg(x) \quad \forall x.$$

Here we usually want the inequality to hold either for all x for which the functions are defined, or for all sufficiently large x (i.e. all x larger than some fixed constant). Which meaning is desired should always be made clear from the context.

We will also write $f(x) \ll g(x)$, which means the same as $f(x) = O(g(x))$.

We write $f(x) \asymp g(x)$, and say that f is *of order* g, if both

$$f(x) \ll g(x) \quad \text{and} \quad g(x) \ll f(x),$$

in other words if $(1/C)g(x) \leq |f(x)| \leq Cg(x)$ for all relevant x.

For $g(x) \neq 0$, we write $f(x) = o(g(x))$ as $x \to \infty$, and say that f is "little oh" of g as $x \to \infty$, if

$$\frac{f(x)}{g(x)} \to 0 \quad \text{as } x \to \infty.$$

One can similarly say that $f(x) = o(g(x))$ as $x \to 0$, for example.

Finally, for $g(x) \neq 0$ we write $f(x) \sim g(x)$ as $x \to \infty$, and say that f is *asymptotic to* g, if

$$\frac{f(x)}{g(x)} \to 1 \quad \text{as } x \to \infty.$$

Note this is the same as saying that $f(x) = (1 + o(1))g(x)$ as $x \to \infty$.

For example, for $x \geq 1$ we have

$$100x = O(x^2), \quad \log x = o(x) \text{ as } x \to \infty, \quad x^2 - 2x + 1 = x^2 + O(x), \quad x^3 - 10 \sim x^3 \text{ as } x \to \infty.$$

We will usually write C to mean a large constant, c to mean a small constant, and ϵ to mean a parameter close to zero. To economise on notation, in long arguments it is customary to use these symbols many times with different values at each use, *provided* this does not confuse the reader.

3. What will the course be about?

This course is about the Riemann zeta function. That is a huge subject, so to focus the course we will concentrate on aspects of zeta function theory with direct applications to the distribution of primes. I hope this is a naturally appealing class of applications, and it is the class for which the zeta function was introduced and studied in the first place (by Euler, Riemann, ...).

By "the distribution of primes", I mean questions like:

- how many primes are less than x ?
- how many primes are in the short interval $[x, x + x^{0.99}]$?

Here x is an arbitrary large number.

Euclid knew that the number $\pi(x)$ of primes less than x tends to infinity with x, but it took millenia to obtain more precise information. Chebychev showed, around 1850, that the number of primes less than x is $\asymp x/\log x$ (for $x \geq 2$). Legendre and Gauss had conjectured that one should actually have the asymptotic

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty,$$

and in 1859 Riemann outlined a programme for proving this result, based on complex analysis of the zeta function $\zeta(s)$. This function had been studied earlier, by Euler, but only for fixed real s. Riemann died in 1866, before his programme could be worked out.

This course will have three main chapters.

(i) The first chapter will introduce the zeta function, and develop the theory far enough to prove that indeed $\pi(x) \sim x/\log x$. (A result called the *Prime Number Theorem*, first proved in 1896). We will see that proving the Prime Number Theorem is equivalent, roughly speaking, to showing that $\zeta(s) \neq 0$ whenever $\Re(s) \geq 1$.

(ii) In the second chapter we will expand the region for which we know $\zeta(s) \neq 0$ to obtain the best known result, due to Vinogradov and to Korobov in 1958. This is some of the deepest information we have about the zeta function, and the proofs have connections with huge parts of analytic number theory. As an application, we will improve our estimate for the difference $\pi(x) - x/\log x$ (or actually for a closely related quantity).

(iii) In view of the Prime Number Theorem, it is reasonable to expect that the interval $[x, x + x^{0.99}]$ will always contain $(1 + o(1))x^{0.99}/\log x$ primes. It was quite shocking when, in 1930, Hoheisel managed to prove a result like this. It turns out that, for this application, one doesn't need to prove that the zeta function has no zeros with real part close to 1, but only that it doesn't have many. In chapter 3 we will prove such *zero-density results*, which have been a huge focus of research in the last few decades.

During the course we will encounter many clever ideas for understanding the Riemann zeta function (and, equivalently, for understanding the primes), and I will try to explain how (I think) people think about it when researching such problems. But we will also see that many aspects of the zeta function remain mysterious and poorly understood.

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