# MA243 Geomety lecture notes

H. A. Verrill, Warwick University, Autumn 2022

Abstract. Geometry MA243 covers an introduction to Euclidean geometry, spherical geometry, hyperbolic geometry and projective geometry. The main prerequisite is first year linear algebra. Taking the Erlangen program as our philosophy of how to characterise different geometries, we use matrices to describe "motions" in the different geometries. These are the transformations which preserve the structure of the geometry. The structure is described by distance for the first three – they are metric spaces, and cross ratio for projective geometry. Motivated by the historical problem of Euclid's parallel postulate, we pay particular attention to comparing results about parallel lines and triangles in the Euclidean plane, the sphere, and the hyperbolic plane. However, we do not cover classical geometry of Euclid, and generally take a Cartesian approach, that is, points are described by their coordinates. In the last section, we briefly discuss axiomatic projective planes. The course is based on chapters one to six of the book Geometry and Topology by Reid and Szendrői.

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introduction

# 1. Lecture 1: Introduction, motivation, overview. Definition of a metric.

## 1.1. Practical matters.

- Lecturer: Helena Verrill, office: B1.35, email: H.A.Verrill@Warwick.ac.uk
- Lectures: 11am on Monday (H0.52), Tuesday (H0.52), Friday (H0.51)
- Lecture Notes: A pdf of the lecture notes will be posted at the beginning of each week. The lecture capture will be available after the lectures.
- References: This course is based on the book:

  Reid and Szendrői (chapters 1-6) Geometry and Topology.

  For more details for the parts on hyperbolic geometry we also use:

  Ratcliffe (parts of chapters 1-3) Hyperbolic Geometry.
- TAs:
  - Alvaro Gonzalez Hernandez
  - Diana Mocanu
- Support Classes: 4pm on Monday (Diana), MS.05 and 2pm on Thursday (Alvaro) in L5, starting week 2.
- Assignmnents: There are 4 assignments, contributing 15% to your final mark. The best 3 assignments count.
- Quizzes: There will be optional quizzes posted on moodle for you to check understanding. These will not contribute to your grade.
- Forum: Please post your questions to the moodle forum and join in dicusssions online.
- Goals: This course aims to introduce you to
  - **Euclidean geometry**,
  - **spherical geometry**,
  - **hyperbolic geometry**,
    - For these geometries, defining the
      - respective metrics (ie. how distance is measured),
      - proving they satisfy the triangle inequality
      - describing the groups of isometries
      - constrasting parallel lines and angles of triangles for these geometries.
  - **projective geometry**.
- Prerequisites:
  - Linear algebra: **vector space, basis, linear map, matrix, rank-nullity theorem**
  - Group theory: **group, identity, associtive, inverses**
  - Relations: **transitive, reflective, equivalence**
  - Hyperbolic function: **sinh, cosh**

  We will review these topics, but it will be useful if you can refresh your memories in advance.
- Exam: The exam counts for 85% of the mark. This will probably have a similar content to the exams for the last three years, as well as to the homework assignments.

Let me know if you find any typos in these note, or have any questions!

week 1
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introduction

**1.2. Course contents.**

MA243 Course contents
Euclidean geometry
Spherical geometry
Hyperbolic geometry
Projective geometry

**1.3. Practical motivation: What this course leads to.** In the third and fourth year, you will be able to study many kinds of geometry in detail. MA243 gives a taste of some aspects of some of these approaches to geometry. Third and fourth year geometry courses:

| | | |
|---|---|---|
| MA475 | Riemann Surfaces | Related to Euclidean, spherical, hyperbolic part of course; more analytical flavour |
| MA4C0 | Differential Geometry | |
| MA3D9 | Geometry of Curves & Surfaces | |
| MA448 | Hyperbolic Geometry | |
| MA4L7 | Algebraic Curves | Related to projective part of course; more algebraic flavour |
| MA4A5 | Algebraic Geometry | |
| MA426 | Elliptic Curves | |
| MA3J2 | Combinatorics II | |
| MA3H6 | Algebraic Topology | In the book, but not in course |
| MA4J7 | Cohomology and Poincare Duality | |
| MA3D4 | Fractal Geometry | Not in course or book |

**1.4. Historical motivation, and course content overview.** Euclid placed geometry on a logical axiomatic foundation. Euclid started with definitions, axioms, and postulates. Since we are not giving a course on Euclid's geometry, I won't go into details of Euclid's definitions and axioms, see e.g., e.g. `https://ncert.nic.in/textbook/pdf/iemh105.pdf`.
Although this course is not about Euclid's geometry, results such as Pythagoras Theorem will be implicityly assumed. In the past, A-levels and GCSEs would have included ruler and compass constructions. Where these "go wrong", or work differently in spherical and hyperbolic geometry will be pointed out in this course. However, if you don't know the basic classical results, such as parallel lines never meeting, this won't make much sense, so I'm going to talk about this a little as background motivation and context in lecture 1.

Euclid's postulates: (not examinable)

- A straight line may be drawn from any one point to any other point.
- A terminated line can be produced indefinitely.
- A circle can be drawn with any centre and any radius.
- All right angles are equal to one another.
- If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.

Easier form of fifth posutlate:

Playfair's axiom: (not examinable) Given a line and a point in the plane, there is a unique line parallel to the given line, through the given point.

For hundreds of years, mathematicians tried to prove that the fifth postulate could be deduced from the other four. In the 19th century, hyperbolic geometry was discoved by Lobachevsky 1829/1830, and Bolyai 1832, and models constructed by Eugenio Beltrami, 1868. Hyperbolic geometry satisfies all the other definitions/axioms/posutlates except the fifth, showing this can not be deduced from the others.
Euclidean, spherical, and hyperbolic geometries are all examples of geometrical systems where the notion of *distance* is defined. We are particularly interested in the dimension 2 case, but we will also define these geometries in arbitrary dimension. In a first year course like Geometry and Motion, you looked at motion

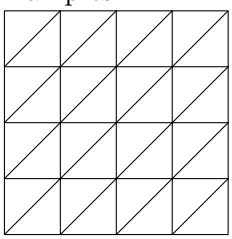
in Euclidean space. In this course, we are more concerned about the basic geometry of space. There are many other possible surfaces, and higher dimensional **Riemannian geometries** to work with, but Euclidean, spherical, and hyperbolic geometries are the most basic. E.g., they have constant curvature; zero; positive; negative (we will not discuss curvature in this course). Hyperbolic geometry turns out to have many very important applications.

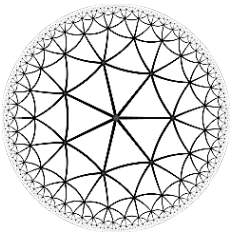
COMPARISON OF GEOMETRIES:

| | Euclidean | Spherical | Hyperbolic |
|---|---|---|---|
| Parallel lines | unique | don't exist | not unique |
| Triangle angle sum | $\pi$ | $> \pi$ | $< \pi$ |

In this course we show how to construct these geometries, and prove these results about **parallel lines** and **triangles**. The **triangle inequality** for hyperbolic space is particularly tricky.

Examples[1]:





Tilings in Euclidean, spherical, and hyperbolic 2-space: triangles with angles adding to $\pi$, $\frac{31}{30}\pi$ and $\frac{6}{7}\pi$ respectively.

**Projective geometry** is a completely different approach, where we don't automatically have a concept of distance or angles. We are interested in projections, ratios, and incidence relations between points and lines. Projective geometry is fundamental to algberaic geometry, including applications such as elliptic curve cryptography. The algebraic use of projective geometry is all about solving algebraic equations (this application is not included in this course). Projective geometry is also useful for depicting 3D images on a 2D screen. Projective geometry also can be considered in a combinatorial sense, for example, the relationship between points and lines, and using these concepts to understand problems in combinatorics, e.g., the mathematics of a game like Dobble.

In this course we show how to define projective geometries, both algebraic and axiomatic, and discuss when this gives the same result.

[1]images by Tom Ruen `https://commons.wikimedia.org/wiki/File:Icosahedral_reflection_domains.png` and Parcly Taxel `https://commons.wikimedia.org/wiki/File:Order-7_triangular_tiling.svg`

week 1
lecture 1
introduction

**Klein's Erlangen program** describes geometry in terms of groups of transformations.

GEOMETRY "=" GROUP THEORY

In this course we work out what the transformation groups of the different geometries are. So we need lots of linear algebra, to give transformations in terms of matrices.

| Geometry | Group |
|---|---|
| Euclidean | $O(n) \rtimes \mathbb{R}^n$ |
| Spherical | $O(n)$ |
| Hyperbolic | $PO(1,n)$ |
| Projective | $PGL(n+1)$ |

You already know about matrices and transformations of space. In first year algebra, you computed the symmetry group of the triangle. These are examples of group theory describing geometry.
The course text book [RS] gives further motivation for this course, especially in the preface and in Chapter 9. Note that this course is not a continuation of MA143, Geometry and Motion, which is quite analytic. This course takes a more algebraic approach to describing geometry.

**1.5. Metric Spaces.** The word "Geometry" comes from the Greek for "to measure the Earth". Euclid devised his postulates in order to describe measurment rigourously. We take the modern Cartesian approach to Euclidean geometry, using a coordinate system. Coordinates for $\mathbb{R}^n$ were introduced by Rene Descartes and Pierre de Fermat. Whereas Euclid computes distances by ruler and compass constructions, in Cartesian geometry, we can give distance in terms of the coordinates of points. Metric spaces were introduced by Frechet in his PhD dissertation on functional analysis, in 1906. This takes the Euclidean and Cartesian definitions of distance and further abstractifies, encapsulating the salient features of measuring in the definition of a metric.

DEFINITION 1. *A metric, or distance, $d$ on a set $X$ is a function*

$$d : X \times X \to [0, \infty)$$

*such that it is*

(1) *Non-degenerate:*
$$d(x,y) = 0 \iff x = y \; \forall x, y \in X$$
(2) *Symmetric:*
$$d(x,y) = d(y,x)$$
(3) *Satisfies the triangle inequality:*
$$d(x,y) \leq d(x,z) + d(z,y)$$

*The pair $(X,d)$ is called a metric space.*

Examples: Absolute value on $\mathbb{R}$, p-adic metric, Hamming distance... See homework, quiz, and support classes for more. This will be developed further in MA260 next term.

**1.6. The Euclidean metric.** Let $X = \mathbb{R}^m = \{(x_1, \ldots, x_m) \mid x_i \in \mathbb{R} \text{ for } 1 \leq i \leq m\}$
The Euclidean metric is defined by (the positive root)

$$d((x_1, x_2, \ldots, x_m), (y_1, y_2, \ldots, y_m)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_m - y_m)^2}$$

Next lecture we will show this is a metric, ie. satisfies the above three properties.
Note that whereas in Euclid's elements, the Pythagorean theorem is a culmination of a whole book of propositions, for us, we take it as a definition!

## 2. Lecture 2: The Euclidean metric. The Cauchy-Schwartz inequality. Colinearity. Euclidean space. Isometries. Euclidean Isometries.

week 1
lecture 2
Metrics

Our goal in the next few lectures is to describe transformations of Euclidean space - you already know a lot of this, but now we are framing the concepts in a context that can be extended to spherical and hyperbolic geometry. Last lecture we defined the concept of a metric space, which you will see more of in the MA260 Norms, Metrics and Topologies next term.

**2.1. The Euclidean metric.** Definitions and notation. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, we write:

- $\mathbf{x} = (x_1, x_2, \ldots, x_m)$,
- $\mathbf{y} = (y_1, y_2, \ldots, y_m)$,
- *Inner product*: $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{m} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_m y_m$
  This is the same as the dot product, $\mathbf{x} \cdot \mathbf{y}$.
- *Euclidean Norm*: $||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ (Note, $|\lambda|$ is the abolute value of a real number.)
- *Euclidean metric*: $d(\mathbf{x}, \mathbf{y}) := ||\mathbf{x} - \mathbf{y}||$

In order to prove the Euclidean metric is actually a metric (Theorem 1) – which means proving the triangle inequality – we need some results about the inner product. Later, the same strategy will be used to show the hyperbolic metric is a metric, so it's important to understand the Cauchy-Schwartz inequality in this easier situation.

LEMMA 1 (Bilinearity of the inner product). *The inner product on $\mathbb{R}^m$ is bilinear and symmetric, that is for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$, we have*

$$\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle + \langle \mathbf{b}, \mathbf{c} \rangle \text{ and } \langle \lambda \mathbf{a}, \mathbf{b} \rangle = \lambda \langle \mathbf{a}, \mathbf{b} \rangle \text{ and } \langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{a} \rangle$$

*in particular* $\langle \mathbf{a}-\mathbf{b}, \mathbf{a}-\mathbf{b} \rangle = \langle \mathbf{a}, \mathbf{a} \rangle - 2 \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{b}, \mathbf{b} \rangle$.

PROOF. Follows from the rules of vector and matrix multiplication. Details left as an exercise. □

LEMMA 2 (Cauchy-Schwartz inequality). *For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle^2 \leq ||\mathbf{x}||^2 \cdot ||\mathbf{y}||^2$ Or, equivalently,*

$$| \langle \mathbf{x}, \mathbf{y} \rangle | \leq ||\mathbf{x}|| \cdot ||\mathbf{y}||$$

PROOF. Easy case: If $\mathbf{y} = \mathbf{0}$, we get $0 \leq 0$, which is true. Now assume $\mathbf{y} \neq \mathbf{0}$. In this case $||\mathbf{y}||$ is non zero, so we can divide by it, and now our goal is to prove

$$\frac{| \langle \mathbf{x}, \mathbf{y} \rangle |}{||\mathbf{y}||} \leq ||\mathbf{x}||$$

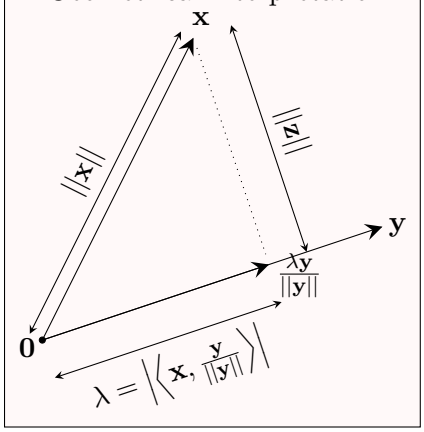
which, by bilinearity (Lemma 1) is the same as proving

$$\left| \left\langle \mathbf{x}, \frac{\mathbf{y}}{||\mathbf{y}||} \right\rangle \right| \leq ||\mathbf{x}||.$$

These quantities are the lengths of vectors, as in the diagram. The difference between these vectors is the vector

$$\mathbf{z} := \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2} \mathbf{y}$$

Geometrical interpretation




Using bilinearity, we have

$$0 \leq ||\mathbf{z}||^2 = \left\langle \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2} \mathbf{y}, \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2} \mathbf{y} \right\rangle = \langle \mathbf{x}, \mathbf{x} \rangle - 2 \left\langle \mathbf{x}, \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2} \mathbf{y} \right\rangle + \left\langle \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2} \mathbf{y}, \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2} \mathbf{y} \right\rangle$$

$$= ||\mathbf{x}||^2 - 2 \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2} \langle \mathbf{x}, \mathbf{y} \rangle + \left( \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2} \right)^2 \langle \mathbf{y}, \mathbf{y} \rangle = ||\mathbf{x}||^2 - 2 \left( \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||} \right)^2 + \left( \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||} \right)^2$$

Rearranged, this is $||\mathbf{x}||^2 \geq \left( \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||} \right)^2$, which gives the required result. □

We now prove the Euclidean metric is a metric. To save on notation, we first prove that the Euclidean metric is translation invariant.

week 1
lecture 2
Metrics

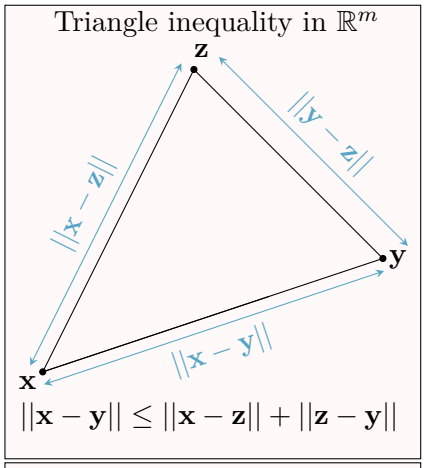
LEMMA 3 (Translation invariance of Euclidean metric). *For* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^m$ *we have* $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}-\mathbf{z}, \mathbf{y}-\mathbf{z})$.

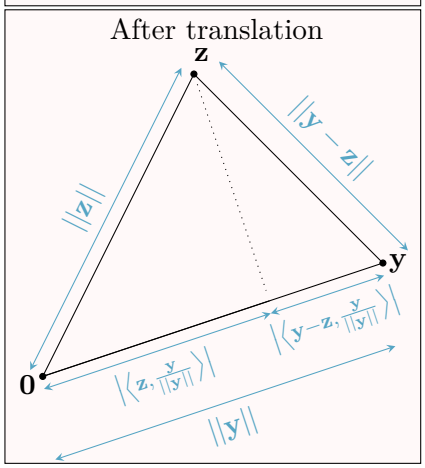
PROOF. $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = ||(\mathbf{x} - \mathbf{z}) - (\mathbf{y} - \mathbf{z})|| = d(\mathbf{x} - \mathbf{z}, \mathbf{y} - \mathbf{z})$. □

THEOREM 1 (The Euclidean metric is a metric). $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$ *on* $\mathbb{R}^m$ *is a metric on* $\mathbb{R}^m$.

PROOF. We need to show this satisfies the three properties of metrics.
Non-degenerate:
Since $x_i$ and $y_i$ are real, $(x_i - y_i)^2 = 0 \iff x_i = y_i$.
Thus $d(\mathbf{x}, \mathbf{y}) = 0 \iff x_i = y_i$ for $i = 1, \dots m \iff \mathbf{x} = \mathbf{y}$.
Symmetric:
Since $(x_i - y_i)^2 = (y_i - x_i)^2$, we have $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
Triangle inequality:
We want to show that $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$. By translation invariance of the Euclidean metric (Lemma 3), this is equivalent to $d(\mathbf{x}-\mathbf{x}, \mathbf{y}-\mathbf{x}) \leq d(\mathbf{x}-\mathbf{x}, \mathbf{z}-\mathbf{x}) + d(\mathbf{z}-\mathbf{x}, \mathbf{y}-\mathbf{x})$, so we have to prove $d(\mathbf{0}, \mathbf{y}) \leq d(\mathbf{0}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$, where I have replaced $\mathbf{y} - \mathbf{x}$ by $\mathbf{y}$ and $\mathbf{z} - \mathbf{x}$ by $\mathbf{z}$. Or in other words, by translation invariance, we can assume that $\mathbf{x} = \mathbf{0}$. (I've just done this to cut down the amount of writing, you can leave this step out if you prefer.) So, now we just have to prove that

$$||\mathbf{y}|| \leq ||\mathbf{z}|| + ||\mathbf{y} - \mathbf{z}||$$




The picture shows motivation for the proof. Just as in the proof of the Cauchy-Schwartz inequality (Lemma 2), we project to the $\mathbf{y}$ vector. You don't have to know about this motivation – the result is entirely algebraic, but this may be a useful insight to the proof structure. Now

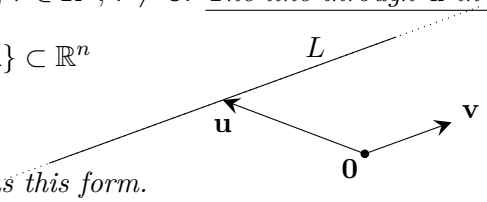
$$\begin{aligned}
||\mathbf{y}|| &= \frac{1}{||\mathbf{y}||} \langle \mathbf{y}, \mathbf{y} \rangle = \frac{1}{||\mathbf{y}||} \langle \mathbf{z} + \mathbf{y} - \mathbf{z}, \mathbf{y} \rangle \\
&= \frac{1}{||\mathbf{y}||} \Big( \langle \mathbf{z}, \mathbf{y} \rangle \ + \langle \mathbf{y} - \mathbf{z}, \mathbf{y} \rangle \Big) \\
&= \left\langle \mathbf{z}, \frac{\mathbf{y}}{||\mathbf{y}||} \right\rangle + \left\langle \mathbf{y} - \mathbf{z}, \frac{\mathbf{y}}{||\mathbf{y}||} \right\rangle \\
&\leq \left| \left\langle \mathbf{z}, \frac{\mathbf{y}}{||\mathbf{y}||} \right\rangle \right| + \left| \left\langle \mathbf{y} - \mathbf{z}, \frac{\mathbf{y}}{||\mathbf{y}||} \right\rangle \right| \\
&\leq ||\mathbf{z}|| \cdot \left|\left| \frac{\mathbf{y}}{||\mathbf{y}||} \right|\right| + ||\mathbf{y} - \mathbf{z}|| \cdot \left|\left| \frac{\mathbf{y}}{||\mathbf{y}||} \right|\right| = ||\mathbf{z}|| + ||\mathbf{y} - \mathbf{z}||,
\end{aligned}$$

where the Cauchy-Schwartz inequality is used in the last line, and the result that $\left|\left| \frac{\mathbf{y}}{||\mathbf{y}||} \right|\right| = 1$. □

### 2.2. Lines and Collinearity.

DEFINITION 2 (Definition of a line). *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n, \mathbf{v} \neq \mathbf{0}$. *The line through* $\mathbf{u}$ *in direction* $\mathbf{v}$ *is*

$$L := \{\mathbf{u} + \lambda \mathbf{v} | \lambda \in \mathbb{R}\} \subset \mathbb{R}^n$$




*A line in* $\mathbb{R}^m$ *is any subset of* $\mathbb{R}^m$ *which has this form.*

DEFINITION 3 (Collinearity). *We say that* $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *are collinear if there is a line* $L$ *with* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L$.

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lecture 2
Metrics

LEMMA 4 (Formula for collinearity). *If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are distinct collinear points on a line $L \subset \mathbb{R}^m$, then for some $\lambda \in \mathbb{R}$ we have*

$$\begin{aligned}\mathbf{z} &= \mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) \\ &= (1-\lambda)\mathbf{x} + \lambda\mathbf{y} \\ &= \lambda_1\mathbf{x} + \lambda_2\mathbf{y}\end{aligned}$$

*where $\lambda_1 + \lambda_2 = 1$ and $\lambda_2 = \lambda$. Further, $|\lambda_1| : |\lambda_2| = d_2 : d_1$.*

PROOF. By definition of $L$, for some vectors $\mathbf{u}$ and $\mathbf{v}$, all points on $L$ have the form $\mathbf{u} + \lambda\mathbf{v}$. So by definition of collinearity, for some $\lambda_x, \lambda_y, \lambda_z \in \mathbb{R}$, we have

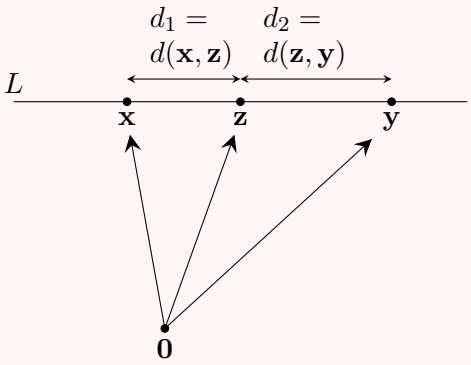
$$\begin{aligned}\mathbf{x} &= \mathbf{u} + \lambda_x\mathbf{v} \\ \mathbf{y} &= \mathbf{u} + \lambda_y\mathbf{v} \\ \mathbf{z} &= \mathbf{u} + \lambda_z\mathbf{v}\end{aligned}$$

Solving for $\mathbf{u}$ and $\mathbf{v}$, and setting $\lambda = (1-\lambda_x)/(\lambda_y - \lambda_x)$ gives the result:

$$\mathbf{z} = \frac{\lambda_y\mathbf{x} - \lambda_x\mathbf{y}}{\lambda_y - \lambda_x} + \frac{\mathbf{x} - \mathbf{y}}{\lambda_x - \lambda_y} = \left[\frac{\lambda_y - 1}{\lambda_y - \lambda_x}\right]\mathbf{x} + \left[\frac{1 - \lambda_x}{\lambda_y - \lambda_x}\right]\mathbf{y}$$

Now we have $d(\mathbf{x}, \mathbf{z}) = ||\mathbf{x} - \mathbf{z}|| = ||\mathbf{x} - \lambda_1\mathbf{x} - \lambda_2\mathbf{y}|| = |\lambda_2|||\mathbf{x} - \mathbf{y}||$, and similarly $d(\mathbf{x}, \mathbf{z}) = |\lambda_1|||\mathbf{x} - \mathbf{y}||$, which gives $|\lambda_1| : |\lambda_2| = d_2 : d_1$. □

Quiz: In this picture, $\lambda = \lambda_2 \in (0, 1)$, and $\mathbf{z}$ is between $\mathbf{x}$ and $\mathbf{y}$.
Where is $\mathbf{z}$ on $L$ if $\lambda > 1$ or $< 0$?




$\mathbf{z} = \lambda_1\mathbf{x} + \lambda_2\mathbf{y}$
$\lambda_1 + \lambda_2 = 1$
$|\lambda_1| : |\lambda_2| = d_2 : d_1$

**WARNING**: since we have the concept of vectors in $\mathbb{R}^m$ at our disposal, we are often confusing or conflating the concepts of vectors and points. For example, we write $\mathbf{x} = (1, 1)$ to mean both the point at location $(1, 1)$ and the vector in direction $(1, 1)$ from the origin. In particular, this conflation allows us to add together points, thus we write above statements such as $\mathbf{z} = \lambda_1\mathbf{x} + \lambda_2\mathbf{y}$. This only really makes sense for vectors; points can not generally be added together, and when we pass to spherical, hyperbolic, or projective geometry, we will not be able to add points together in this way, since generally there is no notion of adding points, unless it is explicitly defined. Thus, in this situation, we are fixing an origin, and identifying points with vectors from this origin, so we can add points by adding the corresponding vectors. This is not a well defined concept in Euclidean spaces in general. Euclidean space is generally identified with $\mathbb{R}^m$ for convinience, but it is not exactly the same as Euclidean space. Before going further we will now define Euclidean space, for which we will need to define isometries of metric spaces.

### 2.3. Isometries and Euclidean space.

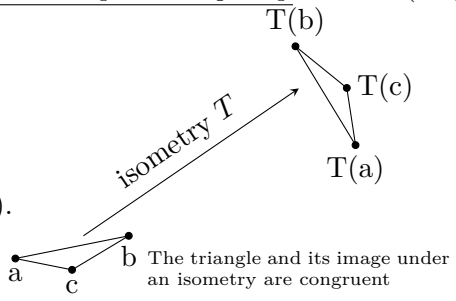
DEFINITION 4. *If $(X_1, d_1)$ and $(X_2, d_2)$ are metric spaces, then a distance preserving map between $(X_1, d_1)$ and $(X_2, d_2)$ is a map*

$$f : X_1 \to X_2$$

*such that for any points $P, Q$ in $X_1$, we have*

$$d_2(f(P), f(Q)) = d_1(P, Q).$$

*An isometry is a bijective distance preserving map.*
*If $f$ is an isometry, then $(X_1, d_1)$ and $(X_2, d_2)$ are said to be isometric.*




The triangle and its image under an isometry are congruent

DEFINITION 5. *A Euclidean space is a metric space which is isometric to $\mathbb{R}^n$, with the Euclidean metric, for some integer $n$. We use the notation $\mathbb{E}^n$ to denote the metric space $\mathbb{R}^n$ together with the Euclidean metric. If not specified otherwise, $\mathbb{R}^n$ is used to mean $\mathbb{E}^n$.*

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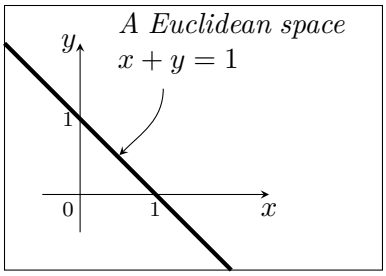
lecture 2
Metrics

EXAMPLE 1.

*The line*

$$L = \{(x, y) \in \mathbb{R}^2 : x + y = 1\},$$




*with metric given by the usual metric on $\mathbb{R}^2$ is a Euclidean space, because there is a distance preserving map to the Euclidean space $\mathbb{E}^1 = (\mathbb{R}, d(x, y) = |x - y|)$, given by*

$$f(x, y) = \sqrt{2}y.$$

*Note that this map is not unique. This map sends $(1, 0)$ to $(0, 0)$, which is a special point in $\mathbb{R}$ when considered as a vector space, but not when we "forget" about the vector space structure. This map is bijective, with inverse $f^{-1}(x) = (1 - x/\sqrt{2}, x/\sqrt{2})$. It is distance preserving, since if $(x_1, y_1), (x_2, y_2)$ are two points on the line, then $x_1 = 1 - y_1$ and $x_2 = 1 - y_2$, and the distance between these points in $\mathbb{R}^2$ is given by*

$$\begin{aligned}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} &= \sqrt{(y_1 - y_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{2(y_1 - y_2)^2} = \sqrt{2}|y_1 - y_2|,\end{aligned}$$

*which is the same as the distance between $f((x_1, y_1))$ and $f((x_2, y_2))$ in $\mathbb{R}$ with the Euclidean metric. Since $f$ is bijective and distance preserving, it is an isometry. So $L$ is a Euclidean space, since it is isometric to $\mathbb{R}$ with the Euclidean metric.*
*Note that adding points on $L$ does not make sense: If the points are added together using addition in $\mathbb{R}$, then the result is not on $L$. We can make addition on $L$ make sense, but we would have to choose a zero. Without this, addition will be undefined.*

DEFINITION 6. *A <u>motion</u> is an isometry of a Euclidean space.*

Note: Although we call an isometry of $\mathbb{E}^m$ a "motion", it might be more descriptive to call it the result of a motion. That is, there is nothing "moving" about the motion. A motion is a transformation of $\mathbb{E}^m$ that could be achieved through a continous motion through space. This is similar to how we call a matrix such as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ a rotation matrix, but the matrix describes the result of the rotation; the matrix itself is not actually the rotation, and the matrix doesn't tell you anything about the process that might have been taken to get to the result of the application of the transformation. On the other hand, being a motion rather than a stretch or shear does convey that distance is preserved.

2.3.1. *Finding all isometries of $\mathbb{E}^m$.* You may already know that all motions of Euclidean space are given by rotations, reflections, or translations. Our next goal is to prove this. Proof outline:

- Show that collinearity can be descibed in terms of the metric
- Use this to show that isometries preserve collinearity
- Use the relationship between collinearity and isometry to link linear algebra to isometry
- Express isometries of $\mathbb{E}^m$ in terms of linear maps. An isometry $T$ of $\mathbb{E}^m$ has the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for some matrix $A$ and some vector $\mathbf{b}$, which only depend on $T$, not on $\mathbf{x}$.
- Show that the matrix $A$ must be orthogonal ($A^T A = I$).
- Clasify real orthogonal matrices: Give in terms of $2 \times 2$ rotation and reflection matrices.
- we can give various different descriptions, e.g., write all isometries in terms of reflections, and write out all possible isometries for $\mathbb{R}^2$.

## 3. Lecture 3: Classifying isometries of $\mathbb{E}^m$

week 2
lecture 3
Isometries

**3.1. Collinearity in terms of the metric.** From now on, we identify $\mathbb{E}^m$ with $\mathbb{R}^m$, i.e., we choose and fix a zero. This is always possible, so the results apply to any Euclidean space. We now give two lemmas which we use to express collinearity in terms of the metric.

LEMMA 5. *Suppose* $\mathbf{y} \neq \mathbf{0}$. *There is equality in the Cauchy-Schwartz inequality, that is,*

$$|\langle \mathbf{x}, \mathbf{y} \rangle| = ||\mathbf{x}|| \cdot ||\mathbf{y}||,$$

*if and only if* $\mathbf{x} = \lambda \mathbf{y}$ *for some* $\lambda \in \mathbb{R}$.

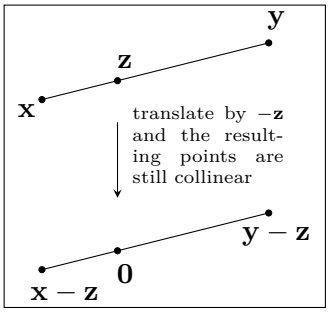
PROOF. Refering to the proof of the Cauchy-Schwartz inequality (Lemma 2), we see that equality is acheived if and only if $||\mathbf{z}|| = 0$, where $\mathbf{z} = \mathbf{x} - \frac{\langle \mathbf{x},\mathbf{y} \rangle}{||\mathbf{y}||^2}\mathbf{y}$. Because the norm is non-degenerate, this means that $\mathbf{z} = \mathbf{0}$, which is the case if and only if

$$\mathbf{x} = \lambda \mathbf{y} \quad \text{where } \lambda = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{y}||^2}$$

□

Another lemma to make life easier by showing translation invariance:

LEMMA 6 (Collinearity is preserved by translation). *Points* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^m$, *are collinear if and only if* $\mathbf{x} - \mathbf{z}, \mathbf{y} - \mathbf{z}, \mathbf{0}$ *are collinear.*




PROOF.

$$\begin{aligned} \mathbf{x}, \mathbf{y}, \mathbf{z} \text{ collinear} &\iff \mathbf{z} = (1-\lambda)\mathbf{x} + \lambda \mathbf{y} \text{ for some } \lambda \text{ ( by Lemma 4)} \\ &\iff \mathbf{0} = (1-\lambda)(\mathbf{x} - \mathbf{z}) + \lambda(\mathbf{y} - \mathbf{z}) \\ &\iff \mathbf{x} - \mathbf{z}, \mathbf{y} - \mathbf{z}, \mathbf{0} \text{ collinear} \end{aligned}$$

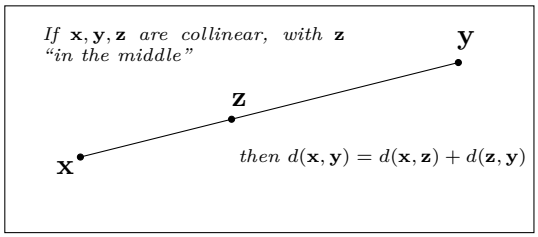
□

Now we can characterize collinearity in terms of the Euclidean metric:

PROPOSITION 1. $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *are collinear if and only if we have equality in the triangle inequality*

$$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$$

*up to permuting* $\mathbf{x}, \mathbf{y}, \mathbf{z}$.




PROOF. Easy case: if any points are equal, the result is obvious, since one term is zero, so we assume that all the points are distinct.
⇒: Suppose first that $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are collinear, with $\mathbf{z}$ in the middle, by which we mean $\mathbf{z} = \mathbf{x} + \lambda(\mathbf{y} - \mathbf{x})$ with $\lambda \in (0, 1)$. (Exercise: show this is always possible to arrange, by relabeling $\mathbf{x}, \mathbf{y}, \mathbf{z}$ if necessary).
Now we get the result by a computation:

$$\begin{aligned} d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}) = ||\mathbf{x} - \mathbf{z}|| + ||\mathbf{z} - \mathbf{y}|| &= ||\mathbf{x} - \mathbf{x} - \lambda(\mathbf{y} - \mathbf{x})|| + ||\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) - \mathbf{y}|| \\ &= |\lambda| \cdot ||\mathbf{y} - \mathbf{x}|| + |1 - \lambda| \cdot ||\mathbf{y} - \mathbf{x}|| \\ &= \lambda \cdot ||\mathbf{y} - \mathbf{x}|| + (1 - \lambda) \cdot ||\mathbf{y} - \mathbf{x}|| \text{ because } \lambda \in (0, 1) \\ &= ||\mathbf{y} - \mathbf{x}|| = d(\mathbf{x}, \mathbf{y}). \end{aligned}$$

⇐: Now assume that $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$. For simplicity, let's translate by $-\mathbf{z}$, recalling from Lemma 3 that the Euclidean metric is translation invariant. So, if $\mathbf{x}' = \mathbf{x} - \mathbf{z}$ and $\mathbf{y}' = \mathbf{y} - \mathbf{z}$, we have $d(\mathbf{x}', \mathbf{y}') = d(\mathbf{x}', \mathbf{0}) + d(\mathbf{0}, \mathbf{y}')$. If we can show that this implies that $\mathbf{x}', \mathbf{y}', \mathbf{0}$ are collinear, then by

Lemma 6, we have that $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are collinear, which is what we want to show. So, rewriting, now all we need to do is show that if $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{0}) + d(\mathbf{0}, \mathbf{y})$, then $\mathbf{x}, \mathbf{y}, \mathbf{0}$ are collinear.
By definition, $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{0}) + d(\mathbf{0}, \mathbf{y})$ means that

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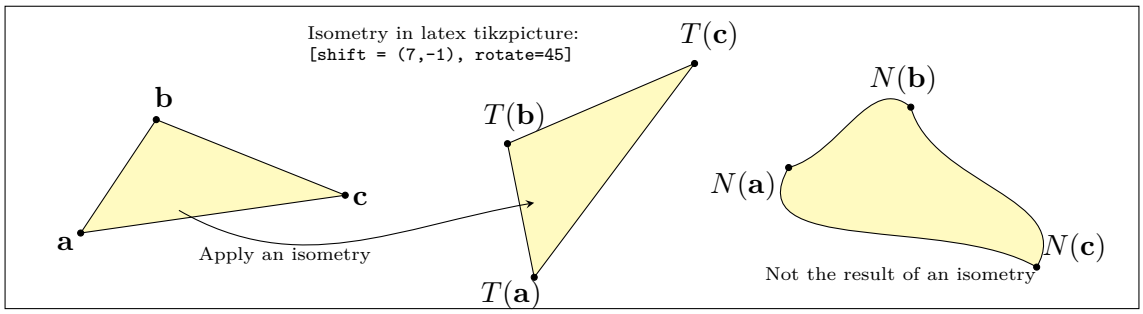
Isometries

$$
\begin{aligned}
||\mathbf{x} - \mathbf{y}|| &= ||\mathbf{x}|| + ||\mathbf{y}|| \\
\Rightarrow ||\mathbf{x} - \mathbf{y}||^2 &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}|| \cdot ||\mathbf{y}|| \\
\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}|| \cdot ||\mathbf{y}|| \text{ by definition of norm} \\
\langle \mathbf{x}, \mathbf{x} \rangle - 2 \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}|| \cdot ||\mathbf{y}|| \text{ by bilinearity of inner product} \\
||\mathbf{x}||^2 - 2 \langle \mathbf{y}, \mathbf{x} \rangle + ||\mathbf{y}||^2 &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}|| \cdot ||\mathbf{y}|| \text{ by definition of norm} \\
-2 \langle \mathbf{y}, \mathbf{x} \rangle &= 2||\mathbf{x}|| \cdot ||\mathbf{y}|| \text{ by cancelling terms} \\
\Rightarrow | \langle \mathbf{y}, \mathbf{x} \rangle | &= ||\mathbf{x}|| \cdot ||\mathbf{y}|| \\
\Rightarrow \mathbf{y} &= \lambda \mathbf{x} \text{ by Lemma 5} \\
\Rightarrow \mathbf{y} &= \lambda \mathbf{x} + (1 - \lambda)\mathbf{0} \\
\Rightarrow \mathbf{x}, \mathbf{y}, \mathbf{0} &\text{ are collinear, by Lemma 4}
\end{aligned}
$$

□

**3.2. Isometries and Collinearity.** We're going to show that isometries map lines to lines, and use this to write isometries in terms of matrices. Previously you have seen that vectors and matrices can be used to define "motions", but now we show that *all* isometries have this form.




Proposition 2. *If $T$ is an isometry of $\mathbb{E}^m$, then $T$ maps lines to lines, i.e., $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are collinear then $T(\mathbf{x}), T(\mathbf{y}), T(\mathbf{z})$ are also collinear. Moreover, if $\mathbf{z} = (1 - \lambda)\mathbf{x} + \lambda\mathbf{y}$, then $T(\mathbf{z}) = (1 - \lambda)T(\mathbf{x}) + \lambda T(\mathbf{y})$.*

Proof. By Propsition 1, and since $T$ is distance preserving,

$$
\begin{aligned}
\mathbf{x}, \mathbf{y}, \mathbf{z} \text{ are collinear } &\Rightarrow d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}) \text{ after possible permutation of } \mathbf{x}, \mathbf{y}, \mathbf{z} \\
&\Rightarrow d(T(\mathbf{x}), T(\mathbf{y})) = d(T(\mathbf{x}), T(\mathbf{z})) + d(T(\mathbf{z}), T(\mathbf{y})) \\
&\Rightarrow T(\mathbf{x}), T(\mathbf{y}), T(\mathbf{z}) \text{ are collinear by Proposition 1}
\end{aligned}
$$

If $\lambda \in [0, 1]$, the final statement follows from Lemma 4, which implies $(1 - \lambda) : \lambda = d(\mathbf{z}, \mathbf{y}) : d(\mathbf{x}, \mathbf{z}) = d(T(\mathbf{z}), T(\mathbf{y})) : d(T(\mathbf{x}), T(\mathbf{z}))$. Cases $\lambda \notin [0, 1]$ are obtained by reordering $\mathbf{x}, \mathbf{y}, \mathbf{z}$ appropriately. □

Definition 7. *A* **linear map** *is a function $L : V \to W$ between two vector spaces over $k$, such that*

$$L(\lambda \mathbf{v} + \mu \mathbf{w}) = \lambda L(\mathbf{v}) + \mu L(\mathbf{w})$$

*for all $\mathbf{v}, \mathbf{w} \in V$, and for all $\lambda, \mu \in k$. For us, the field $k$ is always $\mathbb{R}$.*

Definition 8 (**Affine maps**). *A map $T : \mathbb{R}^n \to \mathbb{R}^k$ is affine is it is of the form $T(\mathbf{x}) = L(\mathbf{x}) + \mathbf{b}$ for all $\mathbf{x} \in \mathbb{R}^n$ for some linear map $L : \mathbb{R}^n \to \mathbb{R}^k$ linear and $\mathbf{b} \in \mathbb{R}^k$. Note that $\mathbf{b} = T(\mathbf{0})$.*

We are going to show that Euclidean isometries are affine. And not just any old affine map.

## 4. Linear algebra review

This section is to review concepts from linear algebra. Some of these concepts may be new, we will spend more time on those.

**4.1. matrix of a transformation.** If $L$ is a linear map from a vector space $V$ to itself, and $E = \{\mathbf{e}_1, \dots, e_n\}$ is a basis for $V$, then the matrix of the linear map $L$ in terms of $E$ is the matrix $A$ with columns given by writing the vectors $L(\mathbf{e}_i)$ in terms of $E$. Usually we take $V = \mathbb{R}^n$, and $E$ is the standard basis:

Review: Linear algebra

$$\mathbf{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow 1 \text{ in } j^{th} \text{ row} \quad \text{and} \quad A = \begin{bmatrix} | & | & & | \\ L(\mathbf{e}_1) & L(\mathbf{e}_2) & \dots & L(\mathbf{e}_n) \\ | & | & & | \end{bmatrix}$$

Recall that the elements of $A$ are labelled as follows:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \dots & A_{1,n} \\ A_{2,1} & A_{2,2} & A_{2,3} & \dots & A_{2,n} \\ A_{3,1} & A_{3,2} & A_{3,3} & \dots & A_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & A_{n,3} & \dots & A_{n,n} \end{bmatrix}$$

So the $(i,j)^{th}$ element of $A$ is the $i^{th}$ component of $L(\mathbf{e}_j)$, that is:

$$A_{ij} = L(\mathbf{e}_j)_i$$

**4.2. Change of basis.** Matrices describe transformations with respect to a given basis. If $V$ is a vector space with basis $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $L$ is a linear map from $V$ to itself, then the matrix $A$ describing $L$ in terms of the basis $E$ has $i^{th}$ column given by $L(\mathbf{e}_i)$ as written in terms of the $\mathbf{e}_j$.
Suppose we change basis to $F = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$. Then since $\mathbf{e}_i$ is a basis, we can write each $\mathbf{f}_j$ as a column vector in terms of the $\mathbf{e}_i$. This gives a matrix $P = [\mathbf{f}_1, \mathbf{f}_2, \dots \mathbf{f}_n]_E$, where this means a matrix with $i^{th}$ column given by a column vector $(a_{i1}, a_{i2}, \dots, a_{in})^T$ such that $\mathbf{f}_i = a_{i1}\mathbf{e}_1 + \dots + a_{in}\mathbf{e}_n$, i.e., $P$ describes how to write elements of $F$ in terms of $E$. This matrix is called the change of basis matrix from $F$ to $E$. Since $F$ is also a basis, the linear map $L$ has a matrix expression describing $L$ in terms of $F$. If we call this matrix $B$, then we have

$$B = P^{-1}AP$$

Explanation: matrix mulitplication action on vectors is from right to left. $P$ takes a vector expressed in terms of the basis $F$ and outputs an expression in terms of the basis $E$. The matrix $A$ takes expressions in terms of $E$ and outputs the result of $L$ in terms of $E$. Then $P^{-1}$ converts the vector back into an expression in terms of $F$. Note that this operation is called "conjugation" of the matrix $A$ by the matrix $P$. Conjugation is a very important operation in mathematics.

**4.3. Transpose.** The **transpose** of a matrix is obtained by changing the $i, j$th element with the $j, i$ the element, i.e., flipping about the main diagonal.

**4.4. Eigen vectors/values.** Let $V$ be a vector space over $k = \mathbb{R}$ or $\mathbb{C}$, and let $L : V \to V$ be a linear map. Then $\lambda \in k$ is an eigen value of $L$ if there exists some $\mathbf{v} \in V$, $\mathbf{v} \neq \mathbf{0}$ such that:

$$L(\mathbf{v}) = \lambda\mathbf{v}$$

We call $\mathbf{v}$ an eigen vector with eigen value $\lambda$. In a linear algebra course, you prove that the eigen values are roots of the characteristic polynomial of $L$, given by:

$$p(t) := \det(L - It)$$

Each eigen value has at least one eigen vector. Eigen vectors tell you the direction a transformation "stretches" in, and the eigen values tell you how much the stretch is. Complex eigen values of real matrices can be interpreted as involving a rotation, as we will see in the next section.

**4.5. Subspaces.** If $V$ is a vector space, then a subspace $W$ of $V$ is a subset of $V$ which is also a vector space, with the same vector operations as on $V$. This requires that $W$ is closed under the vector addition and scalar multiplication.

Review: Linear algebra

**4.6. Orthogonality.** Vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are orthogonal to each other if their inner product is zero. They are normalised if their length is 1. They are orthonormal if they are orthogonal to each other and normalised. A basis of $\mathbb{R}^n$ is an orthogonal basis if it consists of normalised vectors which are mutually orthogonal. A square matrix $A$ is orthogonal if its columns form an orthonormal basis of $\mathbb{R}^n$. This occurs if and only if $A^T A = I$. Given any basis, there is an orthonormalisation algorithm to modify it to obtain an orthonormal basis. This is covered in MA251.

**4.7. Extending a linear map from $\mathbb{R}^n$ to $\mathbb{C}^n$.** Sometimes working over $\mathbb{C}$ is easier than over $\mathbb{R}$. Suppose $L$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, with matrix representation $A$. We can extend $L$ to a map $\mathbb{C}^n \to \mathbb{C}^n$ by defining

$$L(\mathbf{x} + i\mathbf{y}) := L(\mathbf{x}) + iL(\mathbf{y})$$

for $x, y \in \mathbb{R}^n$. The new map $L$ also turns out to be a linear map, but now on $\mathbb{C}^n$, over $\mathbb{C}$. This is an abuse of notation, because $L$ had domain $\mathbb{R}^n$, but now we use the same symbol to mean a map with domain $\mathbb{C}^n$. But the restriction of the new map to $\mathbb{R}^n$ is the same as the original map, and this reuse of the same symbol should not cause problems. Now the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is also a basis of $\mathbb{C}^n$ as a complex vector space, and the matrix of $L$ as a map on $\mathbb{C}^n$ is also $A$, by construction. since $L(\mathbf{x}+i\mathbf{y}) = A\mathbf{x}+iA\mathbf{y}$. If $L$ preserves the inner product as a map on $\mathbb{R}^n$, then it turns out that $L$ preserves the "sesquilinear inner product" on $\mathbb{C}^n$.

**4.8. Sesquilinear inner product.** For $\mathbf{z}$,$\mathbf{w} \in \mathbb{C}^n$ we define the sesquilinear inner product by

$$\langle \mathbf{z}, \mathbf{w} \rangle := \sum_{i=1}^{m} z_i \cdot \overline{w_i} = \mathbf{z}^T \cdot \overline{\mathbf{w}}$$

Where $\overline{a}$ is the complex conjugate of $a$. Note that $\langle \lambda\mathbf{v}, \mathbf{w} \rangle = \lambda \langle \mathbf{v}, \mathbf{w} \rangle$ and $\langle \mathbf{v}, \lambda\mathbf{w} \rangle = \overline{\lambda} \langle \mathbf{v}, \mathbf{w} \rangle$. Note that $\langle \mathbf{z}, \mathbf{z} \rangle = 0 \iff \mathbf{z} = 0$, since if $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ with $\mathbf{x}$ and $\mathbf{y}$ real, we have $\langle \mathbf{z}, \mathbf{z} \rangle = \langle \mathbf{x} + i\mathbf{y}, \mathbf{x} + i\mathbf{y} \rangle = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$, so the sesquilinear inner product also leads to a non-degenerate norm, and $||\overline{\mathbf{z}}|| = ||\mathbf{z}||$.

**4.9. Direct sum.** If $W$ is a vector space containing subspaces $U$ and $V$, then we say $W$ is the direct sum of $U$ and and $V$, written $W = U \oplus V$ if $U \cap V = \{\mathbf{0}\}$ and every element of $W$ can be written uniquely as a sum of an element of $U$ and an element of $V$.

**4.10. Orthogonal complement.** If $V$ is a vector space, with a subspace $W$, then the orthogonal complement of $W$ in $V$ is the vector subspace:

$$W^\perp := \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \ \forall \mathbf{w} \in W\}$$

If $V$ is a complex vector space, the inner product here means the sesquilinear inner product. In MA251 you will see a proof that $V = W \oplus W^\perp$. This involves finding an orthonormal basis for $W$.

## 5. Lecture 4: Isometries of $\mathbb{R}^m$ in terms of linear maps

First a more general result about affine maps between real vector spaces which may have different dimension. This result shows that affine maps send lines to lines.

PROPOSITION 3. *Given a map $T : \mathbb{R}^n \to \mathbb{R}^k$, the following are equivalent*

(1) *$T$ is affine.*

(2) *for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,*

$$T(\lambda\mathbf{x} + \mu\mathbf{y}) - T(\mathbf{0}) = \lambda(T(\mathbf{x}) - T(\mathbf{0})) + \mu(T(\mathbf{y}) - T(\mathbf{0})) \tag{†}$$

*I.e., $\mathbf{x} \to T(\mathbf{x}) - \mathbf{b}$ is a linear map, and $T$ is linear if $T(\mathbf{0}) = \mathbf{0}$.*

(3) *For all $\lambda \in \mathbb{R}$,*

$$T(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = \lambda T(\mathbf{x}) + (1-\lambda)T(\mathbf{y}) \tag{∗}$$

week 2
lecture 4
Linear maps

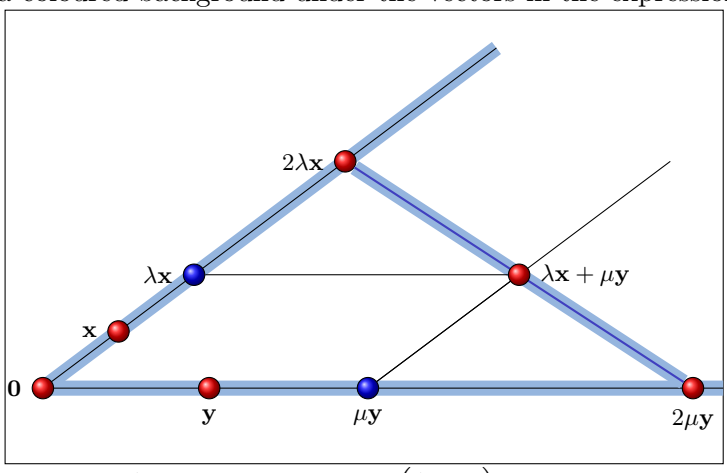
PROOF. (1) $\iff$ (2): Define a map $L$ by $L(\mathbf{x}) := T(\mathbf{x}) - T(\mathbf{0})$.

$T$ is affine $\iff$ $L$ is linear, by definition of affine

$\iff$ $L(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda L(\mathbf{x}) + \mu L(\mathbf{y})$, by defintion of linear

$\iff$ $T(\lambda\mathbf{x} + \mu\mathbf{y}) - T(\mathbf{0}) = \lambda(T(\mathbf{x}) - T(\mathbf{0})) + \mu(T(\mathbf{y}) - T(\mathbf{0}))$ since $L(\mathbf{x}) = T(\mathbf{x}) - T(\mathbf{0})$.

(2)⇒ (3): This is obtained by setting $\mu = 1 - \lambda$ in (Equation †); all terms involving $T(\mathbf{0})$ cancel.
(3)⇒ (2): By Lemma 4, (3) is telling us that $T$ maps lines to lines, since $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is on the line through $\mathbf{x}$ and $\mathbf{y}$, and $T(\mathbf{z}) = \lambda T(\mathbf{x}) + (1-\lambda)T(\mathbf{y})$ means that $T(\mathbf{z})$ is on the line through $T(\mathbf{x})$ and $T(\mathbf{y})$. This is stronger than just colinearity: it says that the relation between the points on a line is preserved. Now we need to relate linearity to conservation of collinearity. We will use the relationship between the red points on the blue lines in the figure.
Since it may be unclear whether $a\mathbf{v}$ means the scalar $a$ times the vector $\mathbf{v}$, or the vector $a\mathbf{v}$, I have put a coloured background under the vectors in the expressions of collinearity.




$2\lambda\mathbf{x}$, $\lambda\mathbf{x} + \mu\mathbf{y}$ and $2\mu\mathbf{y}$ are collinear, since

$$\lambda\mathbf{x} + \mu\mathbf{y} = \left(1 - \tfrac{1}{2}\right) 2\lambda\mathbf{x} + \tfrac{1}{2}\, 2\mu\mathbf{y}$$

So by Equation ∗, with $\lambda = \frac{1}{2}$,

(★)

$$T(\,\lambda\mathbf{x} + \mu\mathbf{y}\,) = \frac{1}{2}T(\,2\lambda\mathbf{x}\,) + \frac{1}{2}T(\,2\mu\mathbf{y}\,).$$

$\mathbf{0}, \mathbf{x}$ and $2\lambda\mathbf{x}$ are collinear, with

$$2\lambda\mathbf{x} = 2\lambda\,\mathbf{x} + (1 - 2\lambda)\,\mathbf{0}$$

So $T(\,2\lambda\mathbf{x}\,) = 2\lambda T(\,\mathbf{x}\,) + (1-2\lambda)T(\,\mathbf{0}\,)$, so

$$\tfrac{1}{2}T(2\lambda\mathbf{x}) = \lambda T(\mathbf{x}) + \left(\tfrac{1}{2} - \lambda\right) T(\mathbf{0})$$

Similarly, $\frac{1}{2}T(2\mu\mathbf{y}) = \mu T(\mathbf{y}) + \left(\frac{1}{2} - \mu\right) T(\mathbf{0})$. Substituting in Equation ★ and rearranging gives the required expression (†) of linearity of $T(\mathbf{x}) - T(\mathbf{0})$. □

COROLLARY 1. *Every Euclidean isometry $T : \mathbb{R}^m \to \mathbb{R}^n$ is affine, and $n = m$ i.e.,$T$ has the form $T(\mathbf{x}) = L(\mathbf{x}) + \mathbf{b}$ for some linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ and vector $\mathbf{b} \in \mathbb{R}^n$.*

PROOF. By Proposition 2 $T$ preserves collinearity, so (3) of Proposition 3 is satisfied, so since (1) and (3) are equivalent, $T$ is affine. Since $T$ is a bijection, $L$ must also be a bijection. From linear algebra, e.g., use the rank-nullity theorem, this implies $n = m$. □

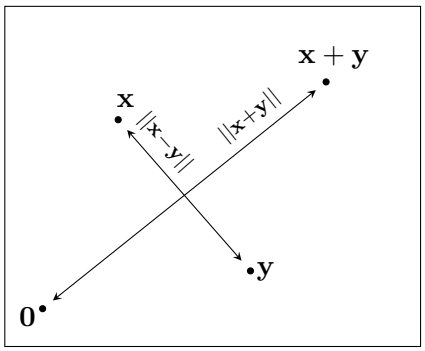
Next we work out what kinds of linear map $L$ can be. It turns out that being an isometry implies extra conditions on $L$, so we want to understand what kind of $L$ is possible.

## 6. Lecture 5: Linear isometries have orthogonal matrices

We now find a characterisation of linear isometries $L$ in terms of the matrix $A$ corresponding to $L$. We need a result, which expresses the inner product in terms of the norm:

LEMMA 7 (polarization identity for real inner products). *For $\mathbf{x}, \mathbf{y}$ in a vector space over $\mathbb{R}$ (or over a field with characteristic not 2),*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}||\mathbf{x}+\mathbf{y}||^2 - \frac{1}{4}||\mathbf{x}-\mathbf{y}||^2$$




PROOF. Exercise in manipulating the inner product, and recalling the definition of the norm. □

week 2 lecture 5 Orthogonal matrices

PROPOSITION 4. *let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with matrix $A$ with respect to the standard basis. Then the following are equivalent:*

(1) *$L$ is an isometry*
(2) *$||L(\mathbf{x})|| = ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$, i.e., $L$ is norm preserving*
(3) *$\langle L(\mathbf{x}), L(\mathbf{y}) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ i.e., $L$ preserves the inner product*
(4) *The matrix $A$ is orthogonal, i.e., $A^T A = I$*

PROOF. (1)⇒(2): Using definition of the metric, defining property of isometries, and linearity of $L$: $||\mathbf{x}|| = d(\mathbf{x}, \mathbf{0}) = d(L(\mathbf{x}), L(\mathbf{0})) = d(L(\mathbf{x}), \mathbf{0}) = ||L(\mathbf{x})||$.
(2)⇒(3): This follows from the polarisation inequality in Lemma 7.
(3)⇒(4): Now compute the elements of $A^T A$. By the formula for matrix multiplication

$$(A^T A)_{ij} = \sum_{k=1}^{n} (A^T)_{ik} A_{kj} = \sum_{k=1}^{n} A_{ki} A_{kj} = \sum_{k=1}^{n} L(\mathbf{e}_i)_k L(e_j)_k = \langle L(\mathbf{e}_i), L(e_j) \rangle$$

So the entries of $A^T A$ are the inner products of the columns of $A$. Since the $\mathbf{e}_i$ are the standard basis, and by hypothesis (3), $L$ preserves the inner product, we have

$$(A^T A)_{ij} = \langle L(\mathbf{e}_i), L(e_j) \rangle = \langle \mathbf{e}_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

These are the elements of the identity matrix, so $A$ is orthogonal.
(4)⇒ (1): Assuming (4), $A$ is invertible with inverse $A^T$, so $L$ is a bijection. So we just need to check $L$ is distance preserving. First check that $L$ is norm preserving:

$$\begin{aligned} ||L(\mathbf{x})||^2 &= \langle L(\mathbf{x}), L(\mathbf{x}) \rangle \\ &= (L(\mathbf{x}))^T L(\mathbf{x}) \text{ using matrix notation for dot product} \\ &= (A\mathbf{x})^T (A\mathbf{x}) \text{ writing } L(\mathbf{x}) = A\mathbf{x} \\ &= \mathbf{x}^T A^T A \mathbf{x} \text{ property of transpose} \\ &= \mathbf{x}^T \mathbf{x} \text{ since } A \text{ is orthogonal} \\ &= \langle \mathbf{x}, \mathbf{x} \rangle, \text{ by definition of inner product} \\ &= ||\mathbf{x}||^2, \text{ by definition of norm.} \end{aligned}$$

Now distance preservation follows from the linearity of $L$:

$$d(L(\mathbf{x}), L(\mathbf{y})) = ||L(\mathbf{x}) - L(\mathbf{y})|| = ||L(\mathbf{x}-\mathbf{y})|| = ||\mathbf{x}-\mathbf{y}|| = d(\mathbf{x}, \mathbf{y}).$$

Since $A$ has inverse $A^T$, $L$ is invertible, so $L$ is an isometry. □

Now we can characterize isometries in terms of their matrices. Proposition 4 deals with isometries which are linear maps. Corollary 1 showed how isometries are related to linear maps. So we have:

COROLLARY 2. *A Euclidean isometry (motion) $T : \mathbb{R}^n \to \mathbb{R}^n$ is of the form $T(\mathbf{x}) = A\mathbf{x}+\mathbf{b}$ for all $\mathbf{x} \in \mathbb{R}^n$ for some unique orthogonal $n \times n$ matrix $A$ and some vector $\mathbf{b} \in \mathbb{R}^n$.*
*Conversely, if $A$ is an orthogonal matrix, $A\mathbf{x} + \mathbf{b}$ is an isometry for all $\mathbf{b} \in \mathbb{R}^m$.*

PROOF.

$$\begin{aligned} T \text{ is a motion} &\iff L(\mathbf{x}) := T(\mathbf{x}) - T(\mathbf{0}) \text{ is a linear isometry} \\ &\qquad \text{(distance is translation invariant by Lemma 3)} \\ &\iff L(\mathbf{x}) = A\mathbf{x} \text{ for some orthogonal } A \text{ by Proposition 4} \\ &\iff T(\mathbf{x}) = A\mathbf{x} + \mathbf{b} \text{ for some orthogonal } A \text{ and } \mathbf{b} = T(\mathbf{0}) \end{aligned}$$

The matrix corresponding to a linear map with respect to a given basis is unique (result from linear algebra). □

EXAMPLE 2. *The following matrices are orthogonal:* $I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

week 2
lecture 5
Orthogonal matrices

## 7. Normal form Theorem

Next we go into more detail to describe exactly what the motions of $\mathbb{R}^n$ can be in terms of rotations, reflections, translations.

**7.1. Description of orthogonal matrices over $\mathbb{R}^n$.** We will see how orthogonal matrices can be written in a nice form with respect to an appropriate basis. This is called the normal form of an orthogonal matrix.

THEOREM 2 (Normal form Theorem). *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry and $A$ an orthogonal matrix such that $L(\mathbf{x}) = A\mathbf{x}$.*
*Then there is an orthonormal basis of $\mathbb{R}^n$ with respect to which $L$ has the matrix of the form*

$$\begin{bmatrix} I_k & & & & \\ & -I_m & & & \\ & & B_1 & & \\ & & & \ddots & \\ & & & & B_\ell \end{bmatrix}$$

*where $I_k$ is the $k \times k$ identity matrix, and $B_i$ is the $2 \times 2$ matrix representing a rotation of $\theta_i$ degrees counter clockwise about origin*

$$B_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$

In order to prove this result, we will work over the complex numbers, extending $L$ as described in Section 4.7. We will need the following results:

LEMMA 8. *If $L$ is an isometry of $\mathbb{R}^n$, with matrix then the extension of $L$ to $\mathbb{C}^n$ preserves the sesquilinear inner product.*

PROOF. Take $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$. Then by definition,

$$\langle L(\mathbf{z}), L(\mathbf{w})\rangle = \langle A\mathbf{z}, A\mathbf{w}\rangle = (A\mathbf{z})^T \cdot \overline{(A\mathbf{w})} = (\mathbf{z}^T A^T)(\overline{A}\overline{\mathbf{w}}) = \mathbf{z}^T A^T A\overline{\mathbf{w}} = \mathbf{z}I_n\overline{w} = \mathbf{z}^T\overline{\mathbf{w}} = \langle \mathbf{z}, \mathbf{w}\rangle$$

□

## 8. Review of Group Theory

DEFINITION 9. *A group is a set $G$ with an operation*

$$\circ : G \times G \to G$$
$$(x, y) \to g \circ h$$

*such that the following axioms hold:*

- *associativity:* $a \circ (b \circ c) = (a \circ b) \circ c$
- *unitality* $\exists e \in G$ *such that* $a \circ e = e \circ a = a > \forall a \in G$
- *inverses* $\forall a \in G \, \exists b \in G$ *such that* $a \circ b = b \circ a = e$

A group $G$ is **Abelian** if $a \circ b = b \circ a \forall a, b \in G$
Examples of groups:

- $\mathbb{Z}, +$
- $\mathbf{Q} \setminus \{0\}, \times$
- the group of permutations of $n$ objects, $S_n$ under composition of permtations.
- group of all symmetries of a square, under composition
- matrix group $M_{n\times m}(\mathbb{R}), +$ of real $n \times m$ matrices under addition
- $GL_n(\mathbb{R}), \times$ group of invertible $n \times n$ matrices under multiplication
- upper triangular invertible matrices under composition

week 3 Review: Groups

DEFINITION 10. *A* **homomorphism** *between two groups $G$ and $H$ is a map*

$$\phi : G \to H$$

*such that $\phi(gh^{-1}) = \phi(g)\phi(h)^{-1}$ for all $g, h \in G$. I.e., $\phi$ preserves multiplcation, inverses, identity.*
*An* **isomorphism** *is a group homomorphism which is a bijection.*
*Two groups $G$ and $H$ are said to be* **isomorphic** *is there is an isomorphism between them.*
*The group of* **automorphisms** *of a group $H$ is the group*

$$\mathrm{Aut}(H) := \{f : H \to H : f \text{ is a group isomorphism}\},$$

*with group law composition of functions.*

Example The cyclic group of order $n$, $C_n$ is the group isomorphic to:

- The set of integers $\{0, 1, 2, \ldots, n-1\}$ with addition modulo $n$
- The set of rotational symmetries of a regular $n$-gon, under composition of functions
- The set of complex numbers $\exp(2\pi i/n)$ for $i = 0, 1, \ldots, n-1$.

If groups are isomorphic, their multiplication tables are "the same", up to reordering and relabeling. The two groups are essentially "the same" even though they may be construced in different ways, use different notation etc. As groups, they are "the same".
If $f : G \to H$ is a group homomorpism, and $e$ is the identity of $H$, the **kernel** of $f$ is the set

$$\ker(f) = \{g \in G : f(g) = e\}$$

A subset $S$ **generates** a group $G$ if all elements of $G$ can be written in terms of elements of $S$.

**8.1. Ways to construct groups.** If $G$ is a group, a **subgroup** of $G$ is a set $H \subset G$ which is closed under the group operations, multiplcation and inversion, and contains the group identity.
If $G, \circ_G$ and $H, \circ_H$ are two groups, then the direct **product** of $G$ and $H$ is the group

$$G \times H = \{(g, h) : g \in G, h \in H$$
$$g_1, h_1) \circ (g_2, h_2) = (g_1 \circ_G g_2, h_1 \circ_H h_2)$$

There are many other ways to build groups, such as quotient groups, which we won't cover, and semi-direct products, which we will see soon.

## 9. Lecture 6: Normal Form Theorem

The normal form theorem will involve decomposing $\mathbb{R}^n$ into subspaces corresponding to the $I_k$, $-I_m$ and $B_\ell$, so we need a result about subspaces.

LEMMA 9. *If $L$ is an isometry of $\mathbb{R}^n$, and $W$ is a subspace of $\mathbb{R}^n$ such that $L(W) = W$, then also $L(W^\perp) = W^\perp$. This result also holds for $L$ extended to $\mathbb{C}^n$ and $W$ a subspace of $\mathbb{C}^n$.*

PROOF. Note that since $L$ is an isometry, it is a bijection, so has an inverse $L^{-1}$.

$$\begin{aligned}
\mathbf{v} \in L(W^\perp) &\iff L^{-1}(\mathbf{v}) \in W^\perp \\
&\iff \left\langle L^{-1}(\mathbf{v}), \mathbf{w} \right\rangle = 0 \,\forall \mathbf{w} \in W \\
&\iff \left\langle L(L^{-1}(\mathbf{v})), L(\mathbf{w}) \right\rangle = 0 \,\forall \mathbf{w} \in W, \text{ by Lemma 8} \\
&\iff \langle \mathbf{v}, L(\mathbf{w}) \rangle = 0 \,\forall \mathbf{w} \in W \\
&\iff \langle \mathbf{v}, \mathbf{w} \rangle = 0 \,\forall \mathbf{w} \in W, \text{ since } L(W) = W \\
&\iff \mathbf{v} \in W^\perp
\end{aligned}$$

□

COROLLARY 3. *Suppose $L$ is an isometry of $\mathbb{R}^n$, and $W$ is a subspace with $L(W) = W$. Now let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p$ be a basis for $W$ and $\mathbf{w}_{p+1}, \mathbf{w}_{p+2}, \ldots, \mathbf{w}_n$ be a basis for $W^\perp$. Then with respect to this basis, the matrix for $L$ has the form:*

week 3 lecture 6 Normal form

$$\begin{pmatrix} \textit{matrix of } L|_W & 0 \\ 0 & \textit{matrix of } L|_{W^\perp} \end{pmatrix}$$

PROOF. Let $L$ have matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with respect to this basis, so if $\mathbf{w}_1 \in W$, $\mathbf{w}_2 \in W^\perp$, then

$$L(\mathbf{w}_1 + \mathbf{w}_2) = A\mathbf{w}_1 + B\mathbf{w}_2 + C\mathbf{w}_1 + D\mathbf{w}_2,$$

with $A\mathbf{w}_1, B\mathbf{w}_2 \in W$, $C\mathbf{w}_1, D\mathbf{w}_2 \in W^\perp$. Then $L(W) = W \Rightarrow C = 0$ and $L(W^\perp) = W^\perp \Rightarrow B = 0$.

□

The components in the normal form theorem correspond to eigen vectors of $L$, so we need a result about these:

LEMMA 10. *Let $\lambda \in \mathbb{C}$ be an eigen value of an isometry $L$ of $\mathbb{R}^n$, with eigen vector $\mathbf{z}$ in $\mathbb{C}^n$. Then*

1. $|\lambda| = 1$
2. *The complex conjugate $\overline{\lambda}$ is also an eigenvalue, with eigenvector $\overline{\mathbf{z}}$.*
3. *If $\lambda \neq \mu$ are eigenvalues with eigen vectors $\mathbf{z}$ and $\mathbf{w}$ respectively, then $\langle \mathbf{z}, \mathbf{w} \rangle = 0$.*

PROOF. Since the norm is non degenerate, and eigen vectors are non-zero, (1) follows from:

$$\langle \mathbf{z}, \mathbf{z} \rangle = \langle A\mathbf{z}, A\mathbf{z} \rangle = \langle \lambda\mathbf{z}, \lambda\mathbf{z} \rangle = \lambda\overline{\lambda} \langle \mathbf{z}, \mathbf{z} \rangle$$

Since $A$ is real, (2) follows from:

$$A\overline{\mathbf{z}} = \overline{A}\overline{\mathbf{z}} = \overline{A\mathbf{z}} = \overline{\lambda}\overline{\mathbf{z}}$$

To prove (3), first compute:

$$\langle \mathbf{z}, \mathbf{w} \rangle = \langle A\mathbf{z}, A\mathbf{w} \rangle = \langle \lambda\mathbf{z}, \mu\mathbf{w} \rangle = \lambda\overline{\mu} \langle \mathbf{z}, \mathbf{w} \rangle$$

So, either $\langle \mathbf{z}, \mathbf{w} \rangle = 0$ or $1 = \lambda\overline{\mu}$. If $1 = \lambda\overline{\mu}$, then $\mu = \lambda\mu\overline{\mu} = \lambda|\mu|^2 = \lambda$, since by (1) $|\mu| = 1$. But this contradicts $\lambda \neq \mu$, and so $\langle \mathbf{z}, \mathbf{w} \rangle = 0$. □

LEMMA 11. *If $L$ is a linear isometry of $W \cong \mathbb{R}^2$ with eigen values $\lambda, \overline{\lambda} \in \mathbb{C} \setminus \mathbb{R}$, then $L$ has an orthnormal basis in $\mathbb{R}^2$ with respect to which it has matrix*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

*where $\lambda = \cos(\theta) - i\sin(\theta)$.*

PROOF. We extend $L$ to $\mathbb{C}^2$. Let $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ be an eigen vector with eigen value $\lambda$, with $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. By Lemma 10, $|\lambda| = 1$, so $\lambda = \cos(\theta) - i\sin(\theta)$ for some $\theta$, and we have

$$L(\mathbf{x} + i\mathbf{y}) = (\cos\theta - i\sin\theta)(\mathbf{x} + i\mathbf{y})$$

which by considering real and imaginary parts gives us

$$L(\mathbf{x}) = (\cos\theta)\mathbf{x} + (\sin\theta)\mathbf{y} \tag{1}$$

$$L(\mathbf{y}) = -(\sin\theta)\mathbf{x} + (\cos\theta)\mathbf{y} \tag{2}$$

Thus, with respect to the basis $\mathbf{x}, \mathbf{y}$, the matrix for $L$ has the given form, and by construction $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. We need to show that we can take $\mathbf{x}$ and $\mathbf{y}$ to be an orthonormal basis.
By Lemma 10 $\langle \mathbf{z}, \overline{\mathbf{z}} \rangle = 0 = \langle \overline{\mathbf{z}}, \mathbf{z} \rangle$. Since $\mathbf{x} = \frac{1}{2}(\mathbf{z} + \overline{\mathbf{z}})$ and $\mathbf{y} = \frac{1}{2i}(\mathbf{z} - \overline{\mathbf{z}})$,we have

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{2}\overline{\left(\frac{1}{2i}\right)} \langle \mathbf{z} + \overline{\mathbf{z}}, \mathbf{z} - \overline{\mathbf{z}} \rangle = \frac{-1}{4i} \langle \mathbf{z} + \overline{\mathbf{z}}, \mathbf{z} - \overline{\mathbf{z}} \rangle = \frac{i}{4}\Big( ||\mathbf{z}||^2 - ||\overline{\mathbf{z}}||^2 - \langle \mathbf{z}, \overline{\mathbf{z}} \rangle + \langle \overline{\mathbf{z}}, \mathbf{z} \rangle \Big) = 0,$$

where we have also used that $||\mathbf{z}|| = ||\overline{\mathbf{z}}||$. So $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

week 3 lecture 6 Normal form

Using $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ we now have $0 = \langle \mathbf{z}, \overline{\mathbf{z}} \rangle = \langle \mathbf{x} + i\mathbf{y}, \mathbf{x} - i\mathbf{y} \rangle = ||\mathbf{x}||^2 - ||\mathbf{y}||^2 + 2i \langle \mathbf{x}, \mathbf{y} \rangle = ||\mathbf{x}||^2 - ||\mathbf{y}||^2$. Hence $||\mathbf{x}|| = ||\mathbf{y}||$, and so we can divide (1) and (2) throughout by $||x||$, obtaining the same relationship for $\mathbf{x}/||\mathbf{x}||$ and $\mathbf{y}/||\mathbf{y}||$, and so our basis can be normalised without changing the matrix, and is already shown to be orthogonal. □

PROOF OF THE NORMAL FORM THEOREM, THEOREM 2. We extend $L$ to a map $\mathbb{C}^n \to \mathbb{C}^n$ as described in Section 4.7. We will inductively find a nice basis for $\mathbb{C}^n$.
Induction step: The eigen values of $A$ are the roots of $p(t) = \det(A - tI)$. By the fundamental theorem of algebra, there is at least one root, $\lambda$. By Lemma 10, $|\lambda| = 1$.
Case (1): $\lambda \in \mathbb{R}$, then $|\lambda| = 1 \Rightarrow \lambda = \pm 1$.
Choose an eigen vector $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, with $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Since $\lambda(\mathbf{x} + i\mathbf{y}) = L(\mathbf{x} + i\mathbf{y}) = L(\mathbf{x}) + iL(\mathbf{y})$, equating real and imaginary parts shows $\mathbf{x}$ and $\mathbf{y}$ are real eigen vectors for $\lambda$. At least one of these, wlog $\mathbf{x}$, is non-zero. Now $\frac{\mathbf{x}}{||\mathbf{x}||}$ is a basis for $W = \mathbb{C} \cdot \mathbf{x}$, and the matrix for $L$ has the form

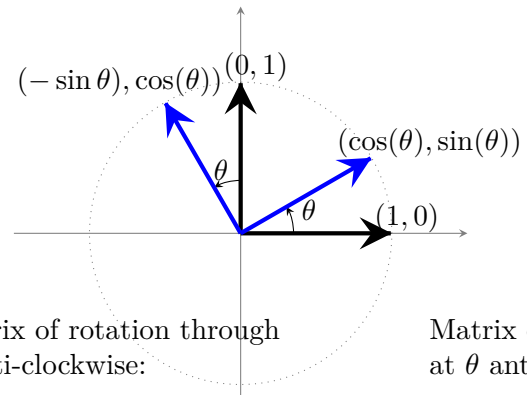
$$\begin{pmatrix} \pm 1 & 0 \\ 0 & L|_{W^\perp} \end{pmatrix}$$

Case (2): $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Choose an eigen vector $\mathbf{z}$. By Lemma 10 $\overline{\lambda}$ is also an eigen value, with eigen vector $\overline{\mathbf{z}}$. Let $W = \mathbb{C}\mathbf{z} \oplus \mathbb{C}\overline{\mathbf{z}}$. By Lemma 11 there is a real orthonormal basis for $W$ with respect to which $L|_W$ has matrix as in Lemma 11. In terms of a basis for $W \oplus W^\perp$, the matrix for $L$ is

$$\begin{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} & 0 \\ 0 & L|_{W^\perp} \end{pmatrix}$$

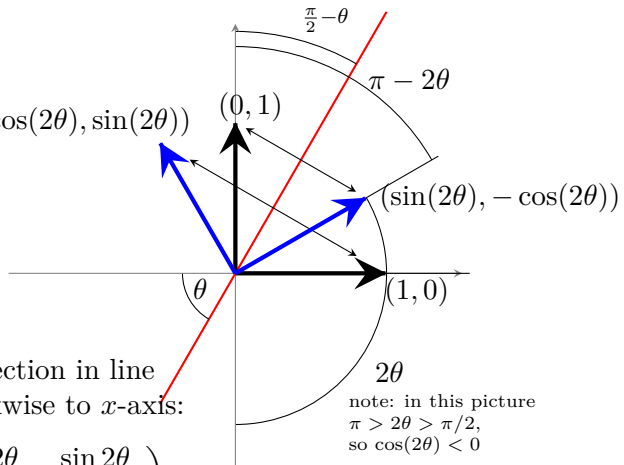
By Corollary 3 in both cases $L|_{W^\perp}$ is an isometry on $W^\perp$, so we can apply the induction step to $W^\perp$. Eventually, we will have decomposed $V$ into mutually orthogonal subspaces, $W_1, W_2, \ldots W_M$, all invariant under $L$. Either $W_i$ has dimension 2, with a real orthonormal basis with respect to which the matrix of $L|_W$ has the form as in Lemma 11, or $W_i$ has dimension 1, with normalized real basis, and $L$ acts as 1 or $-1$. Since the $W_i$ are mutually orthogonal, the basis consisting of the union of all these basis is orthonormal. By reordering the subspaces appropriately we end up with a basis such that the matrix of $L$ has the required form. □

**9.1. Isometries of $\mathbb{R}^2$.** Recall the matrices for rotations and reflections are defined as follows:




Matrix of rotation through $\theta$ anti-clockwise:

$$B_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$




Matrix of reflection in line at $\theta$ anti-clockwise to $x$-axis:

$$\rho_\theta := \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

Note that $\rho_\alpha \circ \rho_\beta = B_{2(\beta-\alpha)}$

THEOREM 3. *Let $T : \mathbb{E}^2 \to \mathbb{E}^2$ be an isometry. Then $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where, with respect to the standard basis, $A = B_\theta$ or $\rho_\theta$ and $\mathbf{b} \in \mathbb{R}^2$, so, $T$ is a rotation or reflection, followed by a translation.*

PROOF. By Corollary 2, $T(\mathbf{x}) = A\mathbf{x}+\mathbf{b}$, with $A$ orthogonal. Write $A$ as a column matrix, $A = [\mathbf{v}, \mathbf{w}]$. Then $A^TA = I$ implies $||\mathbf{v}|| = ||\mathbf{w}|| = 1$ and $\langle \mathbf{v}, \mathbf{w} \rangle = 0$. Let $\mathbf{v} = (a, b)$. Since $a^2 + b^2 = 1$, and $a, b \in \mathbb{R}$, for some $\theta$, $(a, b) = (\cos\theta, \sin\theta)$. Since $\langle \mathbf{v}, \mathbf{w} \rangle = 0$, and $||\mathbf{w}|| = 1$, $\mathbf{w} = \pm(-b, a)$. These give the two cases. □

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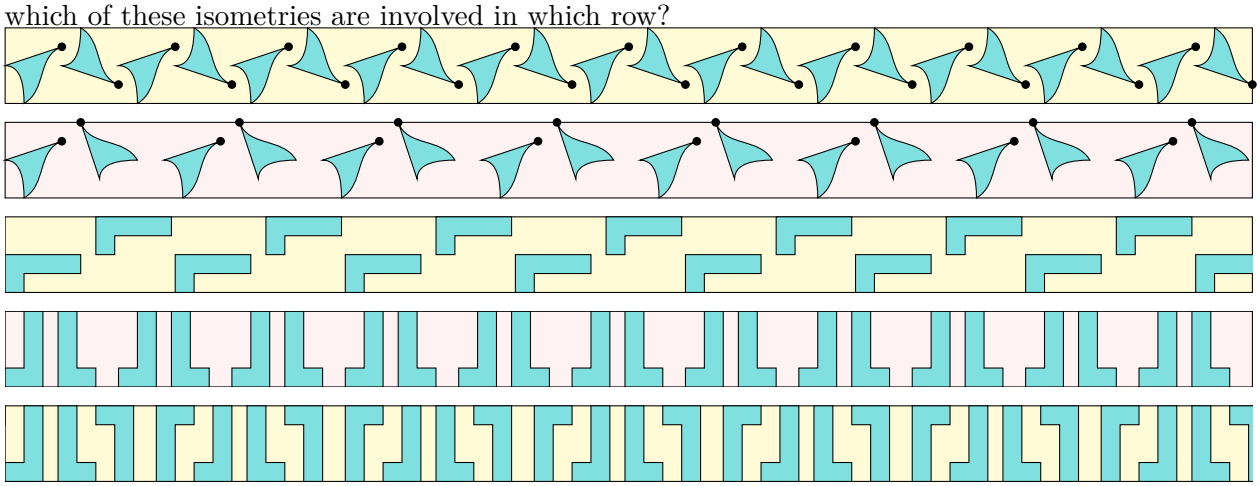
We can classify isometries in $\mathbb{E}^2$ further. We will skip the proof to save time; this can be found in [RS] Section 1.14, and is non-examinable.

THEOREM 4. *Every isometry of $\mathbb{E}^2$ has one of the following forms:*

- *Translation*
- *Rotation*
- *Reflection*
- *Glide reflection: reflection, followed by translation parallel to the line fixed by the reflection.*

**9.2. Frieze groups [non-examinable examples of isometries of $\mathbb{R}^2$].** The motives in the following frieze patterns are related by translations, rotations, reflections or glide reflections - can you identify which of these isometries are involved in which row?



## 10. Lecture 7: The group of Euclidean isometries

We know that isometries of $\mathbb{E}^n$ have the form

$$\mathbf{x} \to A\mathbf{x} + \mathbf{b}$$

where $A$ is an orthogonal matrix and $\mathbf{b}$ is a vector. I.e., we know what the set of all isometries is. But the collection of all isometries of $\mathbb{E}^n$ is not just a set. Isometries can be composed to give other isometries - the set of isometries has a group structure.
We want to understand the group structure of the group of isometries of $\mathbb{E}^n$.

DEFINITION 11 (The orthogonal group). *The group of $n \times n$ real orthogonal matrices is defined by*

$$O(n, \mathbb{R}) = \{A \text{ is an } n \times n \text{ real matrix } : A^T A = I_n\}$$

We will usually just write $O(n)$, since usually we will only consider matrices with real coefficients.

LEMMA 12. *$O(n, \mathbb{R})$ is a group.*

PROOF. Exercise. □

DEFINITION 12 (group of isometries of a metric space). *Let $(X, d)$ be a metric space. Then the group of isometries of $(X, d)$ is the set*

$$\text{Isom}(X, d) = \{f : X \to X | f \text{ is an isometry of } X\}$$

*with group operation defined by composition of functions.*

THEOREM 5. *For a metric space,* $\text{Isom}(X, d)$ *is a group*

week 3 lecture 7 Euclidean Isometry group

PROOF. Exercise: You need to prove that the composition of two isometries is also an isometry; that the identity is an isometry; that the inverse of an isometry is an isometry, and that composition is associative. □

As sets, we have shown that

$$\text{Isom}(\mathbb{R}^n, d) \cong O(n, \mathbb{R}) \times \mathbb{R}^n$$

$$T \leftrightarrow (A, \mathbf{b}) = (T - T(\mathbf{0}), T(\mathbf{0}))$$

But as a group, $\text{Isom}(\mathbb{R}^n, d)$ is not isomorphic to a direct product of these groups. In a direct product, we would have $(A, \mathbf{b}) \circ (A', \mathbf{b}') = (AA', \mathbf{b} + \mathbf{b}')$. But in $\text{Isom}(\mathbb{R}^n, d)$, if $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ and $T'(\mathbf{x}) = A'\mathbf{x} + \mathbf{b}'$, with $A$ and $A' \in O(n)$, and $\mathbf{b}, \mathbf{b}' \in \mathbb{R}^n$, we have

$$\begin{aligned}(T \circ T')\mathbf{x} &= A(T'(\mathbf{x})) + \mathbf{b} \\ &= A(A'\mathbf{x} + \mathbf{b}') + \mathbf{b} \\ &= AA'\mathbf{x} + A\mathbf{b}' + \mathbf{b}\end{aligned}$$

We can also write this in matrix notation:

$$\left(\begin{array}{ccc|c} & A & & \mathbf{b} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right)\left(\begin{array}{ccc|c} & A' & & \mathbf{b}' \\ \hline 0 & \cdots & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} & AA' & & A\mathbf{b}' + \mathbf{b} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right)$$

This kind of "twisted" multiplication is an example of a phenomena called the semi-direct product:

DEFINITION 13. *(Non-examinable) Let $G, \circ_G$ and $H, \circ_H$ be groups, and let $\phi : G \to \text{Aut}(H)$ be a group homomorphism. Then the* **semi-direct** *product of $G$ and $H$ is the group*

$$G \ltimes_\phi H := \{(g, h) : g \in G, h \in H\}$$

$$(g, h) \circ (g', h') := (g \circ_G g', h \circ_H (\phi(g)(h')) \tag{3}$$

Exercise: check $G \ltimes_\phi H$ is a group.

PROPOSITION 5. *There is a group isomorphism*

$$\psi : \mathrm{Isom}(\mathbb{R}^n, d) \to O(n, \mathbb{R}) \ltimes_\phi \mathbb{R}^n$$

*where $\phi$ is given by the natural action of $O(n)$ on $\mathbb{R}^n$, and $\psi$ is given by*

$$\psi : T \mapsto (T - T(\mathbf{0}), T(\mathbf{0}))$$

*The inverse is given by*

$$\psi^{-1} : (A, \mathbf{b}) \mapsto \Big(T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}\Big)$$

PROOF. The "natural action" of $O(n)$ on $\mathbb{R}^n$ just means taking matrices in $O(n)$ to be interpreted as maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ in the usual way, so $\phi$ in Equation 3 is the map

$$\phi : O(n) \to \mathrm{Aut}(\mathbb{R}^n)$$
$$A \mapsto \Big(\mathbf{x} \mapsto A\mathbf{x}\Big)$$

usually, we write $\phi(A) = A$, because we use the same notation to mean $A$ as a matrix and $A$ as the linear map $\mathbf{x} \mapsto A\mathbf{x}$.

Remark: $\mathrm{Aut}(R^n)$ is very big. The group structure on $\mathbb{R}^n$ is addition of vectors. An automorphism of $\mathbb{R}^n$ is a bijection $f : \mathbb{R}^n \to \mathbb{R}^n$ which is a bijection, and hence invertible, and which preserves the group structure on $\mathbb{R}^n$, i.e., $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$, $f(\mathbf{0}) = \mathbf{0}$ and $f(-\mathbf{x}) = -f(\mathbf{x})$. This includes linear maps, but also other maps. The condition $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ means that $\mathrm{Aut}(\mathbf{Q}^n) = \mathrm{GL}_n(\mathbf{Q}^n)$. But for $\mathbb{R}^n$ there are other, non-continuous weird maps, but we don't need to worry about these.

We have already seen in Corollary 2 that $\psi$ is a bijection. Showing that $\psi$ is a group homomorphism is a matter of writing out the multiplication and susbstituting in the formula for the multiplication in the semi-direct product (3), and checking the multiplication in $\mathrm{Isom}(\mathbb{R}^n, d)$ agrees with that in $O(n, \mathbb{R}) \ltimes_\phi \mathbb{R}^n$, i.e. check,

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$$(AA', A\mathbf{b}' + \mathbf{b}) = (A \circ_{O(n)} A', \mathbf{b} \circ_{\mathbb{R}^n} (\phi(A)(\mathbf{b}'))).$$

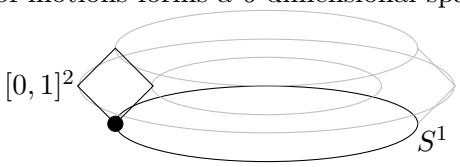
We just have to work out what all the symbols mean and make the appropriate substitutions, and the result follows. □

## 11. Topology of $\mathrm{Isom}(\mathbb{E}^2)$ [non-examinable; from [RS chapter 8]]

Now we have seen that all elements of $\mathrm{Isom}(\mathbb{E}^2)$ can be described by four parameters, $\theta \in [0, 2\pi), a \in \mathbb{R}, b \in \mathbb{R}, c \in \{1, -1\}$:

$$\mathbf{x} \mapsto \begin{pmatrix} \cos(\theta) & -c\sin(\theta) \\ \sin(\theta) & c\cos(\theta) \end{pmatrix} \mathbf{x} + \begin{pmatrix} a \\ b \end{pmatrix}$$

Since a rotation of $2\pi$ is the same as a rotation through 0, we may think of the rotations as being parameterised by points on the circle $S^1 = \{\mathbf{x} \in \mathbb{R}^2 | ||\mathbf{x}|| = 1\}$. So, the elements of $\mathrm{Isom}(\mathbb{E}^2)$ correspond to points in $S^1 \times \mathbb{R}^2 \times \{\pm 1\}$. You can think of this as two disjoint copies of a kind of an infinite solid torus. Now a continuous "motion" in $\mathbb{R}^2$ will correspond to a continuous path in the component corresponding to rotations. (called $SO(2)$.) In geometry and motion, you didn't care about the rotational part, so you would consider the path of a particle, from time 0 to time 1, in the plane to be given by a map from $[0, 1]$ into $\mathbb{R}^2$. Now, taking an object bigger than a particle, its path of motion would be parameterised by a map from $[0, 1]$ into $S^1 \times \mathbb{R}^2$. Our motions form a three dimensional topological space. For $\mathbb{R}^3$, the space of motions forms a 6 dimensional space.




sketch of $S_1 \times [0, 1]^2$, representing rotations with centre of rotation a point in $[0, 1]^2$

## 12. Lecture 8: Reflections Generate $\mathrm{Isom}(\mathbb{E}^n)$

DEFINITION 14. *A hyperplane of $\mathbb{R}^n$ is an affine subspace of dimension $n-1$, which has the form*

$$\Pi = V + \mathbf{b} = \{\mathbf{b} + \mathbf{v} : \mathbf{v} \in V\} = \{\mathbf{b} + \lambda_1\mathbf{v}_1 + \cdots + \lambda_{m-1}\mathbf{v}_{n-1} | \lambda_i \in \mathbb{R}\} = (\mathbb{R}\mathbf{v}_n)^\perp + \mathbf{b}.$$

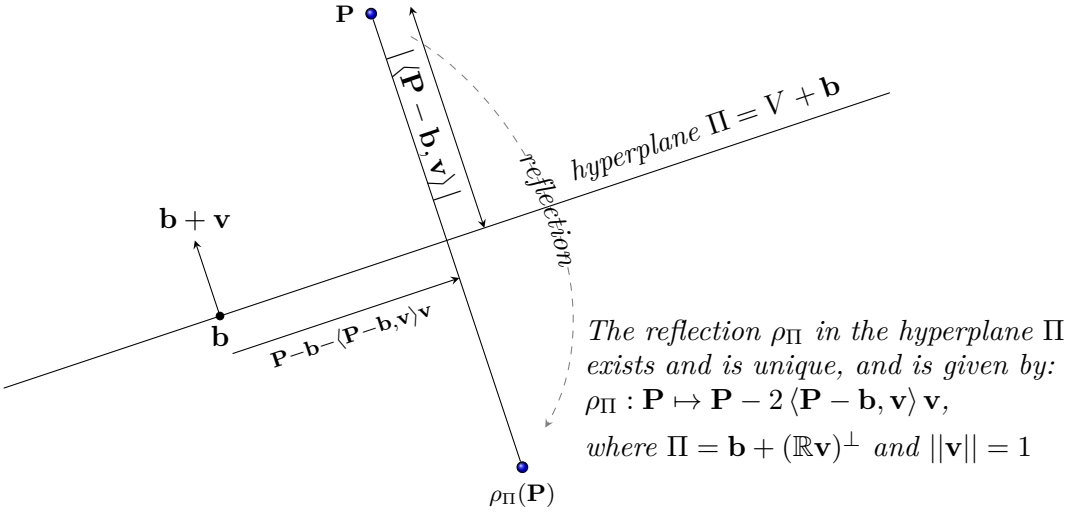
$$= \{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \mathbf{v}_n \rangle = \beta\}$$

*where $\mathbf{b} \in \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$ is a vector subspace of dimension $n-1$, with basis $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$, and $\mathbf{v}_n$ is orthogonal to $V$, and $\beta = \langle \mathbf{v}_n, \mathbf{b} \rangle$.*
*For any $\mathbf{c} \in \mathbb{R}^n$, $\Pi + \mathbf{c} = V + (\mathbf{b} + \mathbf{c})$. ($\Pi$ is a coset of $V$ as an additive subgroup of $\mathbb{R}^n$.)*

DEFINITION 15. *For a map $T : \mathbb{R}^n \to \mathbb{R}^n$, the set of fixed points of $T$ is*

$$\mathrm{Fix}(T) := \{\mathbf{x} \in \mathbb{R}^n : T(\mathbf{x}) = \mathbf{x}\}$$

DEFINITION 16. *Let $\Pi$ be a hyperplane. A* **reflection** *in $\Pi$ is a Euclidean isometry $\rho_\Pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathrm{Fix}(\rho_\Pi) = \Pi$.*




*The reflection $\rho_\Pi$ in the hyperplane $\Pi$ exists and is unique, and is given by:*
$\rho_\Pi : \mathbf{P} \mapsto \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v} \rangle \mathbf{v}$,
*where $\Pi = \mathbf{b} + (\mathbb{R}\mathbf{v})^\perp$ and $||\mathbf{v}|| = 1$*

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THEOREM 6. *The reflection $\rho_\Pi$ in the hyperplane $\Pi$ exists and is unique.*

PROOF. Let $\Pi = V + \mathbf{b}$. Pick $\mathbf{v}$ with $V^\perp = \mathbb{R}\mathbf{v}$. Take a basis of $\mathbb{R}^n$ consisting of $\mathbf{v}$ together with a basis for $V$. With respect to this basis, we define a linear map $\rho$ with matrix

$$A = \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & I_{n-1} \end{pmatrix}.$$

This fixes $V$, and is not the identity, so is a reflection. Exercise: the conjugate $\rho_\Pi := T^{-1}\rho T$ of $\rho$ by $T(\mathbf{x}) = \mathbf{x} - \mathbf{b}$, is a reflection in $\Pi$. Thus $\rho_\Pi$ exists. If $R \neq \mathrm{id}$ fixes $\Pi$, then $S := T \circ R \circ T^{-1}$ fixes $V$. By Lemma 9, $S$ fixes $V^\perp = \mathbb{R}\mathbf{v}$, so $S(\mathbf{v}) = \alpha\mathbf{v}$ for some $\alpha \in \mathbb{R}$. By Proposition 4, $|\alpha| = 1$, so since $S \neq \mathrm{id}$, $\alpha = -1$, and $S$ has matrix $A$, so $S = \rho$, so $R = \rho_\Pi$, so the reflection in $\Pi$ is unique. □

LEMMA 13. $\rho_\Pi(\mathbf{P}) = \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v} \rangle \mathbf{v}$, *where $\Pi = \mathbf{b} + (\mathbb{R}\mathbf{v})^\perp$ and $||\mathbf{v}|| = 1$.*

PROOF. Exercise. □

LEMMA 14. *Given distinct points $\mathbf{P}$ and $\mathbf{Q} \in \mathbb{R}^n$, there exists a reflection $\rho$ with $\rho(\mathbf{P}) = \mathbf{Q}$.*

PROOF. Let $\mathbf{v} = (\mathbf{P} - \mathbf{Q})/||\mathbf{P} - \mathbf{Q}||$. Let $\mathbf{b} = \frac{1}{2}\mathbf{P} + \frac{1}{2}\mathbf{Q}$. By Lemma 4, $\mathbf{b}$ is a point on the line between $\mathbf{P}$ and $\mathbf{Q}$. Let $\Pi = \mathbb{R}\mathbf{v}^\perp + \mathbf{b}$. Then $\rho_\Pi(\mathbf{P}) = \mathbf{Q}$, because using the formula in Lemma 13, $\rho(\mathbf{P}) = \mathbf{P} - 2\left\langle \mathbf{P} - \frac{1}{2}\mathbf{P} - \frac{1}{2}\mathbf{Q}, \frac{(\mathbf{P}-\mathbf{Q})}{||\mathbf{P}-\mathbf{Q}||} \right\rangle \frac{(\mathbf{P}-\mathbf{Q})}{||\mathbf{P}-\mathbf{Q}||} = \mathbf{P} - \langle \mathbf{P} - \mathbf{Q}, \mathbf{P} - \mathbf{Q} \rangle \frac{(\mathbf{P}-\mathbf{Q})}{||\mathbf{P}-\mathbf{Q}||^2} = \mathbf{Q}$. □

THEOREM 7. *The group $\mathrm{Isom}(\mathbb{E}^n)$ is generated by reflections. Moreover, any isometry of $\mathbb{E}^n$ is the product of at most $n+1$ reflections.*

PROOF. Take an isometry $T$. By Lemma 14, with $\mathbf{P} = T(\mathbf{0})$ and $\mathbf{Q} = \mathbf{0}$, there is a reflection $R$ with $R(T(\mathbf{0})) = \mathbf{0}$, so $L := R \circ T$ is a linear isometry. Choose an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ so that with respect to this basis, the matrix $A$ of $L$ has the normal form in Theorem 2:

$$A = \begin{bmatrix} I_k & & & & \\ & -I_m & & & \\ & & B_1 & & \\ & & & \ddots & \\ & & & & B_\ell \end{bmatrix}$$

We are going to decompose this into a composition of reflections. Let $J_i$ be an $n \times n$ matrix with $-1$ at the $(i,i)$ position, and otherwise 1 on the diagonal and 0 off the diagonal. This is the matrix of a reflection in $(\mathbb{R}\mathbf{v}_i)^\perp$.

$$J_i = \begin{bmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n-i} \end{bmatrix} = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}, \quad (J_i)_{s,t} = \begin{cases} 0 & \text{if } s \neq t \\ -1 & \text{if } s = t = i \\ 1 & \text{if } s = t \neq i \end{cases}$$

Recall that $B_\theta$ is the matrix of a rotation, described in Section 9.1. We can take a decomposition

$$B_i = A_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ where } A_i = \begin{pmatrix} \cos(\theta_i) & \sin(\theta_i) \\ \sin(\theta_i) & -\cos(\theta_i) \end{pmatrix}.$$

The matrix $A_i$ has eigen values 1 and $-1$, with corresponding eigen vectors $(\cos\frac{\theta_i}{2}, \sin\frac{\theta_i}{2})$ and $(\sin\frac{\theta_i}{2}, -\cos\frac{\theta_i}{2})$. Let's define matrices $\mathcal{C}_i = J_{k+m+2i}$ and

$$\mathcal{B}_i = \begin{bmatrix} I_{k+m+2i-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n-k-m-2i} \end{bmatrix}, \quad \mathcal{A}_i = \begin{bmatrix} I_{k+m+2i-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n-k-m-2i} \end{bmatrix}.$$

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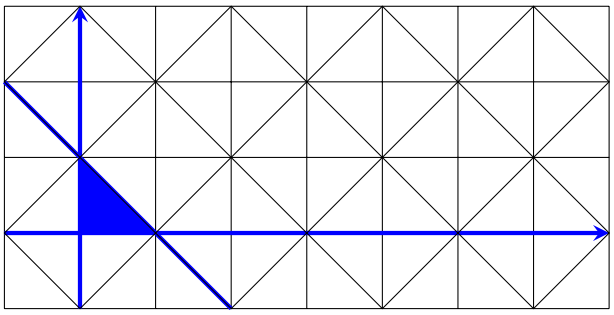
Note that $\mathcal{A}_i$ is a reflection in the plane orthogonal to $\left[\sin\frac{\theta_i}{2}\right]\mathbf{v}_{k+m+2i-1} - \left[\cos\frac{\theta_i}{2}\right]\mathbf{v}_{k+m+2i}$. Then we have

$$\begin{aligned} A &= J_{k+1} \circ J_{k+2} \circ \cdots \circ J_{k+m} \circ \mathcal{B}_1 \circ \mathcal{B}_2 \circ \cdots \mathcal{B}_\ell \\ &= J_{k+1} \circ J_{k+2} \circ \cdots \circ J_{k+m} \circ \mathcal{A}_1 \circ \mathcal{C}_1 \circ \mathcal{B}_2 \circ \mathcal{C}_2 \circ \cdots \mathcal{A}_\ell \circ \mathcal{C}_\ell, \end{aligned}$$

which is the product of at most $n$ reflections. Since $T = R^{-1} \circ L$, $T$ is the product of at most $n+1$ reflections. □

**12.1. Triangle groups [non examinable].** All the triangles in this figure can be obtained from the blue triangle by a sequence of reflections in the lines $x = 0$, $y = 0$ and $x + y = 1$. Triangle groups are discrete groups generated by reflections in the sides of a triangle. This can be in Euclidean, spherical, or hyperbolic geometry. There are only three non-isomorphic Euclidean triangle groups.



### 13. Example of Theorem 7 (Not done in class; for extra homework reading)

The indexing in the proof of Theorem 7 may be confusing, but in practice, given a matrix in normal form, it's easy to write as a product of reflection matrices.
Note that in [RS] there are (almost) two proofs given of Theorem 7. In section 2.5 an algebraic proof is given - though this doesn't quite give the product in terms only of reflections, but in reflections and rotations – and in section 2.6 a geometric proof is given. In order to save time, I have only given the algebraic proof in this class. In [RS], messy indexing is avoided by suggesting "you should convince yourself...", which maybe I should have also said! Whilst the geometric proof is very nice, I would like to get on to spherical geometry; reading through [RS] section 2.6 is recommended to those who are easily keeping up with the material.
Now for an example. Suppose we have a linear isometry already put into normal form with respect to some basis, which for this section will be denoted $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. e.g.,

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ 0 & 0 & \sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\beta) & -\sin(\beta) \\ 0 & 0 & 0 & 0 & \sin(\beta) & \cos(\beta) \end{pmatrix} \mathbf{x}$$

In this example, in the notaion of the Normal Form Theorem, $n = 6$, $k = m = 1$ and $\ell = 2$
Note that $n = k + m + 2\ell$, in this case $6 = 1 + 1 + 2 \times 2$.
I will confuse $T$ with its matrix, to save introducing more notation. Then we have

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ 0 & 0 & \sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\beta) & -\sin(\beta) \\ 0 & 0 & 0 & 0 & \sin(\beta) & \cos(\beta) \end{pmatrix}$$

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In the notation I have used in the proof of Theorem 7, this is

$$T = J_2 \circ \mathcal{B}_1 \circ \mathcal{B}_2,$$

with

$$\mathcal{B}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ 0 & 0 & \sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} I_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$

where $\mathbf{0}$ in this matrix means a submatrix of zeros, with the appropriate number of rows and columns. The 2 in the first $I_2$ is $2 = k + m + 2i - 2 = 1 + 1 + 2 - 2$, with $i = 1$. The 2 in the second $I_2$ is $2 = n - k - m - 2i = 6 - 1 - 1 - 2$. Also in notation of Theorem 7, $\alpha = \theta_1$. The matrix $B_1$ is a rotation matrix, but we can write it as a product of reflection matrices: We have

$$B_1 = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ where } A_1 = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}$$

and

$$\mathcal{B}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ 0 & 0 & \sin(\alpha) & -\cos(\alpha) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \mathcal{A}_1 \mathcal{C}_1 = \mathcal{A}_1 J_4$$

Here $\mathcal{C}_1 = J_4$, and $J_4$ means a diagonal matrix, with 1s on the diagonal, except for the fourth diagonal entry, which is $-1$. In general, $J_i$ is a reflection in the hyperplane $(\mathbb{R}\mathbf{v}_i)^\perp$. Note that

$$\mathcal{A}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ 0 & 0 & \sin(\alpha) & -\cos(\alpha) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} I_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix},$$

$\mathcal{A}_i$ fixes $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_5, \mathbf{v}_6$, since these vectors correspond to the blocks of $\mathcal{A}_i$ given by the copies of $I_2$. Let's write

$$\mathbb{R} = V_1 \oplus V_2 \oplus V_3$$

with

$$V_1 = \mathbb{R}\mathbf{v}_1 \oplus \mathbb{R}\mathbf{v}_2; \quad V_2 = \mathbb{R}\mathbf{v}_3 \oplus \mathbb{R}\mathbf{v}_4; \quad V_3 = \mathbb{R}\mathbf{v}_5 \oplus \mathbb{R}\mathbf{v}_6.$$

Note that this means $V_1$ is the subspace of $\mathbb{R}_6$ spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and so on. The block form of the matrix $\mathcal{A}_1$, means that considered as a linear map,

$$\mathcal{A}_1|_{V_1} = I_2; \quad \mathcal{A}_1|_{V_2} = A_1; \quad \mathcal{A}_1|_{V_3} = I_2.$$

Note that $A_1$ has eigen values 1 and $-1$, with corresponding eigen vectors $(\cos(\alpha/2), \sin(\alpha/2))$ and $(\sin(\alpha/2), -\cos(\alpha/2))$. The map $\mathcal{A}_1$ operates on $V_2$ by the matrix $A_1$. This means that $\mathcal{A}_1$ fixes the eigen vector

$$\mathbf{w}_1 = \cos(\alpha/2)\mathbf{v}_3 + \sin(\alpha/2)\mathbf{v}_4,$$

and muliplies

$$\mathbf{w}_2 = \sin(\alpha/2)\mathbf{v}_3 - \cos(\alpha/2)\mathbf{v}_4$$

by $-1$. So, if we let $U$ be the vector subspace (and hyperplane) of $\mathbb{R}^6$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{v}_5, \mathbf{v}_6$, i.e.

$$U = \mathbb{R}\mathbf{v}_1 \oplus \mathbb{R}\mathbf{v}_2 \oplus \mathbb{R}\mathbf{w}_1 \oplus \mathbb{R}\mathbf{v}_5 \oplus \mathbb{R}\mathbf{v}_6$$

, then $\mathcal{A}_1$ fixes $U$, and is non trivial, so it is the reflection

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$$\mathcal{A}_1 = \rho_U$$

We get a similar decomposition of $\mathcal{B}_2$, and end up with

$$T = J_2 \circ \mathcal{A}_1 \circ J_4 \circ \mathcal{A}_2 \circ J_6.$$

This is the expression for $T$ as a composition of 5 refelections.
In general, each $2 \times 2$ block $B_i$ can be written as a product of two reflections. So, $T$ is written as a product of $m + 2\ell$ reflections, and $m + 2\ell = n - k \leq n$. So, any linear isometry is a product of at most $n$ reflections.

**13.1. $-I_m$ is not a reflection unless $m = 1$.** Note that in the Normal form, of a linear isometry $T$, we decompose $\mathbb{R}^n$ into a direct sum

$$\mathbb{R}^k \oplus \mathbb{R}^m \oplus \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2,$$

where we have a subspace of $\mathbb{R}^n$ isomorphic to $\mathbb{R}^m$, on which $T$ acts by $\mathbf{x} \mapsto -\mathbf{x}$, and with $T|_{\mathbb{R}^m} = -I_m$. For example, in $\mathbb{R}^3$, with basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

acts as 1 on $V_1 = \mathbb{R}\mathbf{v}_1 \cong \mathbb{R}$ and as $-1$ on $V_2 = \mathbb{R}\mathbf{v}_2 \oplus \mathbb{R}\mathbf{v}_3 \cong \mathbb{R}^2$. This matrix is not a reflection; this example acts as a rotation of $\pi$ about the $\mathbf{v}_1$ axis. To write as a product of rotations, we have:

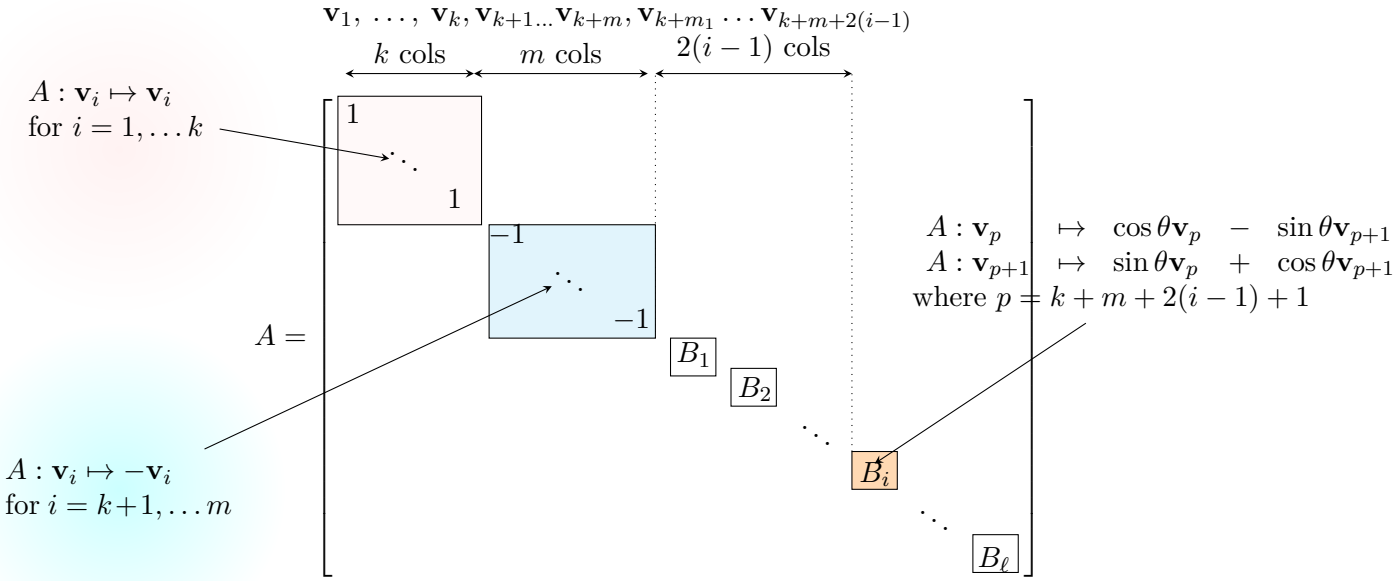
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

This is a reflection in the $\mathbf{v}_1, \mathbf{v}_3$ plane, followed by a reflection in the $\mathbf{v}_1, \mathbf{v}_2$ plane.
More generally, $-I_m$ is the product of $m$ reflections. A matrix of the form

$$C = \begin{pmatrix} I_k & 0 & 0 \\ 0 & -I_m & 0 \\ 0 & 0 & I_{2\ell} \end{pmatrix},$$

acting on $V_1 \oplus V_2 \oplus V_3$ with $V_1 \cong \mathbb{R}^k, V_2 \cong \mathbb{R}^m, V_3 \cong \mathbb{R}^{2\ell}$ fixes $V_1 \oplus V_3 \cong \mathbb{R}^{k+2\ell}$ and has matrix $-I_m$ when restricted to $V_2$. This means that for the corresponding isometry, $T_{C,\mathbf{0}}$, we have $\dim(\mathrm{Fix}(T_{C,\mathbf{0}})) = n - m$. Unless $m = 1$, this means $\mathrm{Fix}(T_{C,\mathbf{0}})$ is not a hyperplane, and so $C$ does not represent a reflection. Here I am using the notation $T_{C,\mathbf{b}}$ to mean the isometry with $T_{C,\mathbf{b}}(\mathbf{x}) = C\mathbf{x} + \mathbf{b}$.

**13.2. Note on indexing.** The indexing in the proof of Theorem 7, particularly for the index of the $B_i$ is explained by the following picture:




I have included the vectors corresponding to each column before the $B_i$ block, since this helps to show what the vector indices are for the $B_i$ block.

**13.3. Non linear case.** Now suppose that $T$ is an isometry, but not a linear isometry.
We know by Lemma 14 that there is a reflection $R$ with $R(T(\mathbf{0})) = \mathbf{0}$. Then $R \circ T$ is a linear isometry, so is given by a product of at most $n$ reflections, so $T$ is the product of at most $n+1$ reflections. Lets work with some example.
Let $T = T_{A,\mathbf{b}}$ be an isometry of $\mathbb{R}^6$ with a somewhat randomly choosen orthogonal matrix. Unfortunately, a randomly chosen example will highly likely produce not the most easy solution, but here goes. A further exercise would be to start with "nice" reflections and multiply them together, so that you are guaranteed to get a nice solution when the process is reversed. But would that be cheating? This example does not turn out particularly nice, but it's not too bad either.

$$T(\mathbf{x}) = A\mathbf{x} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \\ 0 \\ 4 \end{pmatrix} \quad \text{where} \quad A = \frac{1}{3}\begin{pmatrix} -2 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -2 & 1 & 1 \\ 1 & 1 & 1 & 1 & -2 & 1 \\ -1 & -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

The first step is to find a reflection with $R(T(\mathbf{0})) = \mathbf{0}$. Using the recipe given in Lemma 14, taking $\mathbf{P} = \mathbf{0}$ and $Q = T(\mathbf{0})$ the isometry $R$ is

$$\rho_\Pi \quad \text{where} \quad \Pi = (\mathbb{R}\ \mathbf{v})^\perp + 3\mathbf{v} \quad \text{where} \quad \mathbf{v} = \frac{1}{3}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}.$$

Here we have used that $(T(\mathbf{0}) - \mathbf{0})/||T(\mathbf{0}) - \mathbf{0}|| = \mathbf{v}$ and $\frac{1}{2}(T(\mathbf{0}) + \mathbf{0}) = 3\mathbf{v}$, and $||\mathbf{v}|| = 1$. Note that $\mathbf{b} = 6\mathbf{v}$.
Let $S$ be the translation $S(\mathbf{x}) = \mathbf{x} + 3\mathbf{v}$. Let $\rho$ be the reflection in $(\mathbb{R}\mathbf{v})^\perp$. Then as discussed in the proof of Theorem 6

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$$\rho_\Pi = S \circ \rho \circ S^{-1}$$

In the $T_{A,\mathbf{b}}$ notation, $S = T_{I_6,3\mathbf{v}}$. If we take an orthormal basis $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ for $\mathbb{R}^6$, then with respect to this basis, the matrix for $\rho$ is

$$\begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & I_5 \end{pmatrix}.$$

In order to combine the matrix expressions for $T(\mathbf{x})$ and for $R$, we will have to put them in terms of the same basis. Let's put everything in terms of the standard basis, which by assumption, the matrix in the expression for $T$ is already written with respect to.
With respect to the standard basis, the matrix for $\rho$ is

$$M\begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & I_5 \end{pmatrix} M^{-1}$$

where $M$ is the change of basis matrix. It doesn't matter exactly which basis we choose for $(\mathbb{R}\mathbf{v})^\perp$, they will all give the same result. The columns of $M$ are $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$. Let's take

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

So, computing with your favourite algebra package, or by hand, you get that the matrix for $\rho$ with respect to the standard basis is:

$$B := M\begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & I_5 \end{pmatrix} M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{9} & 0 & -\frac{4}{9} & 0 & -\frac{4}{9} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{4}{9} & 0 & \frac{1}{9} & 0 & -\frac{8}{9} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{4}{9} & 0 & -\frac{8}{9} & 0 & \frac{1}{9} \end{pmatrix}$$

You can check that $B\mathbf{v} = -\mathbf{v}$. ($B$ is not a rotation matrix, even though the symbol $B$ was used in the previous section as a rotation matrix; this is a reflection matrix.)
Now we have computed the reflection in $(\mathbb{R}\mathbf{v})^\perp$, but we want to compute the reflection in $(\mathbb{R}\mathbf{v})^\perp + 3\mathbf{v}$. This will be

$$R = T_{I_6,3\mathbf{v}} \circ T_{B,\mathbf{0}} \circ T_{I_6,-3\mathbf{v}}.$$

Recall from the discussion following Theorem 5 that $T_{A,\mathbf{b}} \circ T_{A',\mathbf{b}'} = T_{AA',A\mathbf{b}'+\mathbf{b}}$, so the isometry $R$ is given by

$$R = T_{I_6,3\mathbf{v}} \circ T_{B,\mathbf{0}} \circ T_{I_6,-3\mathbf{v}} = T_{I_6,3\mathbf{v}} \circ T_{B,-3B\mathbf{v}} = T_{B,3\mathbf{v}-3B\mathbf{v}}$$

Now $R \circ T$ is given by

$$R \circ T(\mathbf{x}) = T_{B,3\mathbf{v}-3B\mathbf{v}} \circ T_{A,6\mathbf{v}} = T_{BA,6B\mathbf{v}+3\mathbf{v}-3B\mathbf{v}} = T_{BA,\mathbf{0}}.$$

Note that by construction, $B\mathbf{v} = -\mathbf{v}$, so $6B\mathbf{v} + 3\mathbf{v} - 3B\mathbf{v} = \mathbf{0}$. We have

$$C := BA = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{9} & 0 & -\frac{4}{9} & 0 & -\frac{4}{9} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{4}{9} & 0 & \frac{1}{9} & 0 & -\frac{8}{9} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{4}{9} & 0 & -\frac{8}{9} & 0 & \frac{1}{9} \end{pmatrix}\begin{pmatrix} -2 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -2 & 1 & 1 \\ 1 & 1 & 1 & 1 & -2 & 1 \\ -1 & -1 & -1 & -1 & -1 & 2 \end{pmatrix}$$

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$$= \frac{1}{27}\begin{pmatrix} -18 & 9 & 9 & 9 & 9 & 9 \\ 7 & -14 & 7 & 19 & 7 & -5 \\ 9 & 9 & -18 & 9 & 9 & 9 \\ 5 & 17 & 5 & 2 & 5 & -19 \\ 9 & 9 & 9 & 9 & -18 & 9 \\ -13 & -1 & -13 & 11 & -13 & -10 \end{pmatrix}$$

Finally we have a linear isometry, $R \circ T\mathbf{x} = BA\mathbf{x}$. We want to put it into normal form. We have to find the characteristic polynomial. This turns out (using Pari-gp, but you can use whatever computer algebra package you want, or compute by hand) to be

$$27x^6 + 76x^5 + 71x^4 - 71x^2 - 76x - 27$$
$$= (x-1)(x+1)^3(27x^2 + 22x + 27)$$

So, there are four eigen values,

$$1, -1, \frac{-11 \pm 4i\sqrt{38}}{27}$$

There is a one dimensional space $V_1$ with eigen value 1, a three dimensional space $V_2$ with eigen value $-1$, (corresponding to a composition of 3 reflections) and perpendicular to these, there is a 2 dimensional space $V_3$ left over, on which $C$ acts by rotation.
In the proof of the normal form theorem, we saw that if the complex eigen value is $\lambda = \mu_1 + i\mu_2$, then since this has length 1, for some $\theta$, we have $\lambda = \cos(\theta) + i\sin(\theta)$, and the rotation matrix, with respect to an orthonormal basis for $V_3$ is:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \frac{1}{27}\begin{pmatrix} -11 & -4\sqrt{38} \\ 4\sqrt{38} & -11 \end{pmatrix}$$

Note, that in this matrix we could choose $\pm\sqrt{38}$, the sign choice depends on the order the basis elements are taken; switching them changes the sign.
Computer algebra can find bases for these spaces, by computing a basis for the kernel of the matrices corresponding to the factors of the characteristic polynomial; these can be taken as follows. Note that there are inifinitely many choices of orthonomal basis for a space which is larger than 1 dimensional, so these are not unique. These have been chosen to be orthonormal.

$$V_1 \text{ basis: } \mathbf{w}_1 := \frac{1}{\sqrt{38}}(2,3,2,4,2,-1)$$

$$V_2 \text{ basis: } \mathbf{w}_2 := \frac{1}{\sqrt{2}}(0,0,-1,0,1,0)$$
$$\mathbf{w}_3 := \frac{1}{2\sqrt{2}}(-1,-2,1,1,1,0)$$
$$\mathbf{w}_4 := \frac{1}{4}(3,-2,-1,1,-1,0)$$

$$V_3 \text{ basis: } \mathbf{w}_5 := \frac{1}{\sqrt{37}\sqrt{38}}(2,3,2,4,2,37)$$
$$\mathbf{w}_6 := \frac{1}{4\sqrt{37}}(11,-2,11,-15,11,0)$$

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Which way round should $\mathbf{w}_5$ and $\mathbf{w}_6$ be? We can compute:

$$(\oplus) \qquad C\mathbf{w}_5 = \frac{1}{27}\frac{1}{\sqrt{37}\sqrt{38}}\begin{pmatrix} -18 & 9 & 9 & 9 & 9 & 9 \\ 7 & -14 & 7 & 19 & 7 & -5 \\ 9 & 9 & -18 & 9 & 9 & 9 \\ 5 & 17 & 5 & 2 & 5 & -19 \\ 9 & 9 & 9 & 9 & -18 & 9 \\ -13 & -1 & -13 & 11 & -13 & -10 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 2 \\ 4 \\ 2 \\ 37 \end{pmatrix} = \frac{1}{27}\frac{1}{\sqrt{37}\sqrt{38}}\begin{pmatrix} 396 \\ -109 \\ 396 \\ -614 \\ 396 \\ -407 \end{pmatrix}$$

The first coordinate of $C\mathbf{w}_5$ is positive (equals 396), so, since the first coordinates of $\mathbf{w}_5$ and $\mathbf{w}_6$ are both positive, we can't have $C\mathbf{w}_5 = (-11\mathbf{w}_5 - 4\sqrt{38})/27\mathbf{w}_6$, so we expect that we should either change the order of $\mathbf{w}_5$ and $\mathbf{w}_6$, or change the sign of $\sqrt{38}$ in the given $2\times 2$ rotation matrix. Let's just write $\mathbf{w}_6$ before $\mathbf{w}_5$. Now we can check that

$$(*) \qquad T_C\left(\begin{pmatrix}\mathbf{w}_6 \\ \mathbf{w}_5\end{pmatrix}\right) = \frac{1}{27}\begin{pmatrix} -11 & -4\sqrt{38} \\ 4\sqrt{38} & -11 \end{pmatrix}\begin{pmatrix}\mathbf{w}_6 \\ \mathbf{w}_5\end{pmatrix} = \frac{1}{27}\begin{pmatrix} -11\mathbf{w}_6 - 4\sqrt{38}\mathbf{w}_5 \\ 4\sqrt{38}\mathbf{w}_6 - 11\mathbf{w}_5 \end{pmatrix}$$

Let's just check that $C\mathbf{w}_5 = \frac{1}{27}(4\sqrt{38}\mathbf{w}_6 - 11\mathbf{w}_5)$. We've computed the left hand side of this expression above in ($\oplus$), so now we compute the righthand side:

$$\frac{1}{27}\left(4\sqrt{38}\frac{1}{4\sqrt{37}}\begin{pmatrix}11\\-2\\11\\-15\\11\\0\end{pmatrix} - 11\frac{1}{\sqrt{37}\sqrt{38}}\begin{pmatrix}2\\3\\2\\4\\2\\37\end{pmatrix}\right) = \frac{1}{27\sqrt{38}\sqrt{37}}\left(38\begin{pmatrix}11\\-2\\11\\-15\\11\\0\end{pmatrix} - 11\begin{pmatrix}2\\3\\2\\4\\2\\37\end{pmatrix}\right)$$

$$= \frac{1}{27}\frac{1}{\sqrt{37}\sqrt{38}}\begin{pmatrix}396\\-109\\396\\-614\\396\\-407\end{pmatrix}$$

So, $C$ does act on $\mathbf{w}_5$ as described in (*), and $\mathbf{w}_6$ then has to work too.
We write the rotation as a composition of two reflections:

$$\frac{1}{27}\begin{pmatrix}-11 & -4\sqrt{38}\\ 4\sqrt{38} & -11\end{pmatrix} = \frac{1}{27}\begin{pmatrix}-11 & 4\sqrt{38}\\ 4\sqrt{38} & 11\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

The second matrix here corresponds to a reflection in the plane orthogonal to $\mathbf{w}_5$. The first corresponds to a reflection in the plane orthogonal to $\sin(\theta/2)\mathbf{w}_6 - \cos(\theta/2)\mathbf{w}_5$. We can use the half angle formulas:

$$\sin(\theta/2) = \sqrt{\frac{1-\cos(\theta)}{2}} = \sqrt{\frac{1+11/27}{2}} = \sqrt{\frac{19}{27}}$$

$$\cos(\theta/2) = \sqrt{\frac{1+\cos(\theta)}{2}} = \sqrt{\frac{1-11/27}{2}} = \sqrt{\frac{8}{27}}$$

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Finally, we can write out the initial matrix in terms of a sequence of reflections:

(1) Reflect in the hyperplane orthogonal to $\mathbb{R}\mathbf{w}_5 = \mathbb{R}(2,3,2,4,2,37)$
(2) Reflect in the hyperplane orthogonal to $\sqrt{\frac{19}{27}}\mathbf{w}_6 - \sqrt{\frac{8}{27}}\mathbf{w}_5$
(3) Reflect in the hyperplane orthogonal to $\mathbb{R}\mathbf{w}_4 = \mathbb{R}(3,-2,-1,1,-1,0)$
(4) Reflect in the hyperplane orthogonal to $\mathbb{R}\mathbf{w}_3 = \mathbb{R}(-1,-2,1,1,1,0)$
(5) Reflect in the hyperplane orthogonal to $\mathbb{R}\mathbf{w}_2 = \mathbb{R}(0,0,-1,0,1,0)$
(6) Reflect in the hyperplane $(\mathbb{R}\mathbf{v})^\perp + 3\mathbf{v}$ where $\mathbf{v} = (0,1,0,2,0,2)$

We could write out these reflections as matrix transformations, but I think this is enough for now.
Note that step (1) and (2) have to be carried out in that order, and (6) has to be done last, but the other reflections can be permuted, e.g., you could do these are (3), (4), (1), (5), (2), (6). This is because reflections in orthogonal planes commute.
There are infinitely many ways to write $T$ as a product of reflections; we could have applied $R$ first instead of last, and then computed the normal form of $AB$ instead of for $BA$. Or we could have just computed the normal form for $A$, and then composed with two reflections to obtain $T$, since a translation can be achieved by two reflections; this last method might not give the minimal number of reflections possible.

## 14. Lines and Triangles in the Euclidean plane.

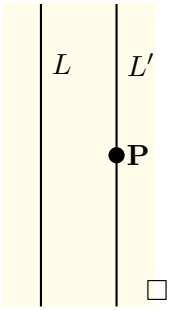
**14.1. Euclid's fifth postulate.** As mentioned in the introduction, Euclid's fifth postulate, the **parallel postulate**, was an assumption made about parallel lines. For Euclid, this was part of the system of axioms. For Euclidean space, we are working with a given metric space, and the result is a theorem. Euclidean space is a model satisfying Euclid's axioms and postulates. In this subsection, we work in $\mathbb{R}^2$, and identify points with vectors so we can add them.
Since the results of this section are probably familiar from school, I may skip this section, and take this as known results. You should be able to find the equation of a line parallel to a given line and though a given point, and you should be able to find the coordinates of the intersection point of two lines in the plane.

PROPOSITION 6. *Given a line $L$ and a point $\mathbf{P} \in \mathbb{R}^2$, with $\mathbf{P} \notin L$, there is a unique line $L'$ with $\mathbf{P} \in L$ and $L \cap L' = \varnothing$.*

PROOF. Given $L = \mathbb{R}\mathbf{v} + \mathbf{b}$, let $L' = \mathbb{R}\mathbf{v} + \mathbf{P}$. This contains $\mathbf{P}$ and is a line by construction.
Suppose $\mathbf{w} \in L \cap L$. Then $\mathbf{w} = \alpha\mathbf{v} + \mathbf{b} = \beta\mathbf{v} + \mathbf{P}$, which implies $\mathbf{P} = (\alpha - \beta)\mathbf{v} + \mathbf{b} \in L$, a contradiction. Hence $L \cap L' = \varnothing$, and the required $L'$ exists.
Suppose there was another line $L'' = \mathbb{R}\mathbf{v}' + \mathbf{P}$ also through $\mathbf{P}$ with $L \cap L' = \varnothing$.
To find $\mathbf{w} \in L \cap L''$, we want to solve $\mathbf{w} = \mathbf{b} + \alpha\mathbf{v} = \mathbf{P} + \beta\mathbf{v}'$, which means solving

$$\mathbf{b} - \mathbf{P} = \begin{pmatrix}\mathbf{v} & \mathbf{v}'\end{pmatrix}\begin{pmatrix}-\alpha \\ \beta\end{pmatrix},$$

which is possible if the matrix $[\mathbf{v}\mathbf{v}']$ is invertible, which is the case if $\mathbf{v}$ and $\mathbf{v}'$ are independent.
So, if $L \cap L'' = \varnothing$, then $\mathbf{v}'$ is a multiple of $\mathbf{v}$, and so $L' = L''$. Hence $L'$ is unique. □




**14.2. Angles and Triangles.** We now define angles and recall that the sum of angles in a Euclidean triangle is $\pi$. This is equivalent to the parallel postulate, given Euclid's other axioms and postulates.

DEFINITION 17. *If $\mathbf{A}, \mathbf{B}$ are points in $\mathbb{R}^n$, the* **line segment** *between $\mathbf{A}$ and $\mathbf{B}$ is defined by*

$$[\mathbf{A}, \mathbf{B}] := \{(1 - \lambda)\mathbf{A} + \lambda\mathbf{B} : \lambda \in [0, 1]\}$$

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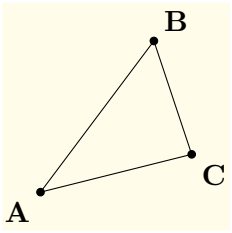
Reflections

*If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are points in $\mathbb{R}^n$, then their* **triangle** *is*

$$\Delta(\mathbf{ABC}) := [\mathbf{A}, \mathbf{B}] \cup [\mathbf{B}, \mathbf{C}] \cup [\mathbf{C}, \mathbf{A}].$$

*We define the (measure of the)* **angle** *$\angle(BAC)$ to be $\theta \in [0, \pi)$ such that*

$$\cos(\theta) := \frac{\langle \mathbf{B} - \mathbf{A}, \mathbf{C} - \mathbf{A}\rangle}{||\mathbf{B} - \mathbf{A}||||\mathbf{C} - \mathbf{A}||} \in [-1, 1]$$




This is well defined by the Cauchy-Schwartz inequality, Lemma 2.
There are other possible ways to define angle, for example, as defined by Euclid.

LEMMA 15. *The angles in a triangle are preserved by an isometry.*

PROOF. This follows from Proposition 4, that an isometry is norm and inner product preserving. □

PROPOSITION 7. *The angles of a triangle $\Delta\mathbf{ABC}$ sum to $\pi$.*

PROOF. There are many proofs of this result, some fundamentally different if other definitions and axioms are used, as for example by Euclid. In [RS], the result is proved by using preservation of angles under isometry, Lemma 15, and the result that angles aligning to form a straight line sum to $\pi$. However, to prove this result requires some more work, for example, showing that the definition of the measure of an angle in terms of the cosine is the same as a certain arc length of a circle of radius 1. This is all fairly straightforwards, but requires more space and time than I want to allocate in this lecture course. □

## 15. Spherical Geometry

Goals of this chapter:

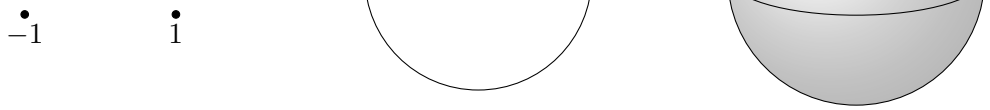
(1) Metric: Define a sphere $S^n$ and the spherical metric; prove it is a metric
(2) Lines: Describe straight lines as great circles, which always intersect
(3) Isometries: Show that $\mathrm{Isom}(S^n) \cong O(n+1)$
(4) Triangles:
   (a) Find the sum of angles in a spherical triangle, show this is greater than $\pi$
   (b) Show how the area of a spherical triangle is determined by the angles

## 16. Lecture 9: The sphere and the spherical metric

DEFINITION 18. *The $n$-dimensional sphere of radius $r \geq 0$ is defined by*

$$\begin{aligned} S^n_r &:= \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = r^2\} \\ &= \{\mathbf{x} \in \mathbb{R}^{n+1} : ||\mathbf{x}|| = r\} \\ S^n &:= S^n_1 \end{aligned}$$

EXAMPLE 3. *Spheres $S^n$ for $n = 0, 1, 2$:*




$S^0$ is two points $S^1$ is a circle $S^2$ is a sphere

DEFINITION 19. *A spherical line or great circle is the intersection of $S^n_r$ with a 2-dimensional vector subspace of $\mathbb{R}^{n+1}$.*

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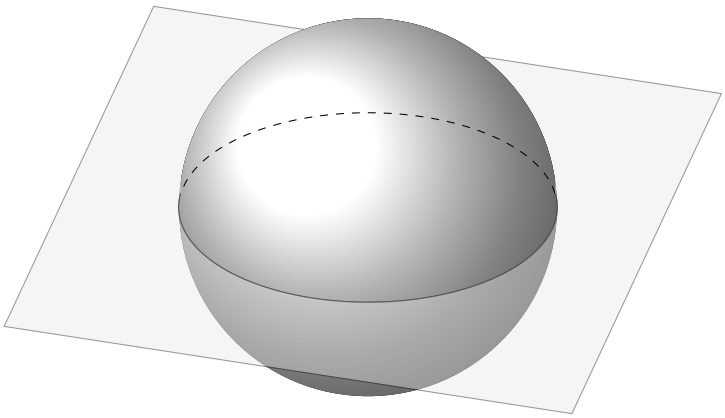
The following picture[2] shows the great circle given by intersecting the $xy$ plane with the sphere. In $\mathbb{R}^2$, a hyperplane through the origin is the same as a 2-dimensional subvector space.

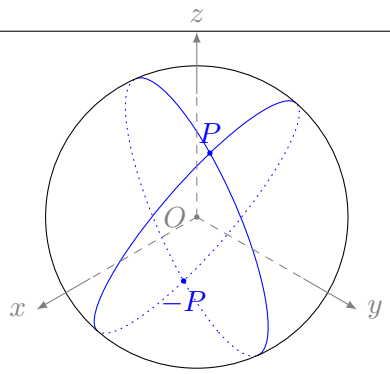


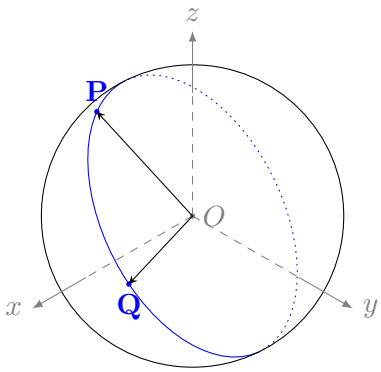
We will use these great circles to define the metric. Actually the metric can be defined in terms of measuring paths restricted to the sphere, and using e.g., integration to compute path length, using methods from calculus/analysis, based on the Euclidean metric in $\mathbb{R}^{n+1}$. However, we will assume without proof that these give the same answer. We could call these great circles "straight lines" on the sphere, as in the shortest distance between two points.

[2]Pictures from https://tex.stackexchange.com/questions/569452/ and /590928, /228197 and https://www.geogebra.org/3d/z6dhmv4n

DEFINITION 20. $\mathbf{P}, \mathbf{Q} \in S_r^n$ *are* antipodal *if* $\mathbf{Q} = -\mathbf{P}$.

In the figure (right), $\mathbf{P}$ and $-\mathbf{P}$ are shown on two great circles, with 4 possible paths along great circles. These lines all have the same length. Antipodal points have infinitely many "shortest paths" between them, and so are a special case.




LEMMA 16. *If* $\mathbf{P}$ *and* $\mathbf{Q} \in S_r^n$ *are not antipodal, then there is a unique great circle containing both of them*

PROOF. $\mathbf{P}$ and $\mathbf{Q}$ have length $r$ and so are non-zero. If they are not antipodal, then they are not multiples of each other, since a line $\mathbb{R}\mathbf{v}$ only intersects $S_r^n$ at the points $\lambda\mathbf{v}$ where $||\lambda\mathbf{v}|| = |\lambda|||\mathbf{v}|| = R$, which has only two solutions, $\lambda = \pm R/||\mathbf{v}||$. Thus the only two dimensional vector subspace containing $\mathbf{P}$ and $\mathbf{Q}$ is

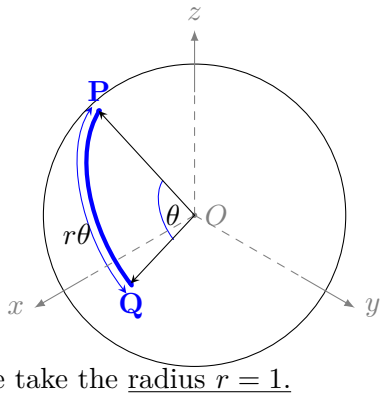
$$\mathbb{R}\mathbf{P} \oplus \mathbb{R}\mathbf{Q} = \{\alpha\mathbf{P} + \beta\mathbf{Q} : \alpha, \beta \in \mathbb{R}\}$$

□

DEFINITION 21. *The* spherical distance *between two points* $\mathbf{P}, \mathbf{Q} \in S_r^n$ *is the length of the shortest arc of a great circle joining them. We will show that this distance is a metric, called the* spherical metric*, which we can also define by*

$$d_{S_r^n}(\mathbf{P}, \mathbf{Q}) := r \cos^{-1}\left(\frac{\langle \mathbf{P}, \mathbf{Q} \rangle}{r^2}\right)$$

*where we take* $\cos^{-1}$ *to be in* $[0, \pi]$.




**IMPORTANT CONVENTION:** For simplicity, from now on, we take the radius $r = 1$.
In order to prove that $d_{S^n}$ is a metric, we need to prove the triangle inequality.

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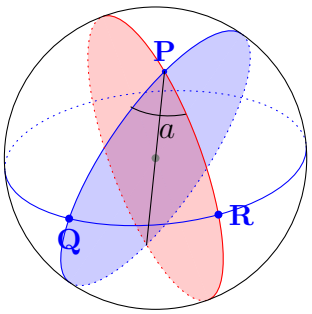
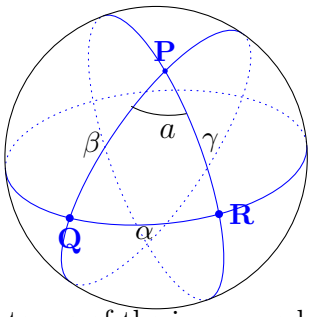
DEFINITION 22. *A* spherical triangle *a in* $S_r^n$ *consists of three distinct vertices,* $\mathbf{P}, \mathbf{Q}, \mathbf{R} \subset S^n$ *and three arcs of great circles joining them, which do not intersect except at the vertices, and a specified area enclosed by these arcs. Note that there are* 8 *triangles with vertices* $\mathbf{P}, \mathbf{Q}, \mathbf{R}$.
*The* spherical angle *at* $\mathbf{P}$ *between the lines* $\mathbf{PQ}$ *and* $\mathbf{PR}$ *is the dihedral angle between the planes containing these lines. In the figure, this is the angle between the red and blue planes. This angle can also be computed as the angle between the tangent lines to the sphere that lie in these two planes and touch the sphere at* $\mathbf{P}$.




Our pictures are for $S^2$. But since any three points on $S^n$ lie on a 2-sphere obtained by intersecting $S^n$ with the subvector space spanned by $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$, these definitions hold for any $n \geq 2$.



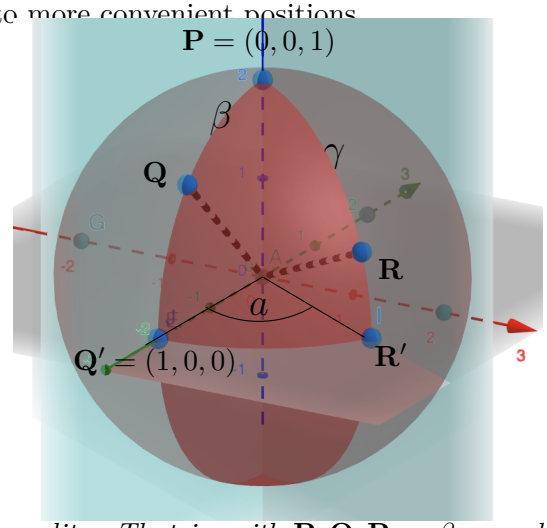

PROPOSITION 8 (Main formula of spherical trigonometry). *Let $\alpha, \beta, \gamma$ be the side lengths of a spherical triangle with vertices $\mathbf{P}, \mathbf{Q}, \mathbf{R} \in S^2$ on the unit sphere.*

$$\alpha := d(\mathbf{Q}, \mathbf{R}), \quad \beta := d(\mathbf{P}, \mathbf{Q}), \quad \gamma := d(\mathbf{P}, \mathbf{R}),$$

*where $d = d_{S^2}$ is the spherical metric. Let $a$ be the spherical angle between arcs $\mathbf{PQ}$ and $\mathbf{PR}$. Then*

$$\cos\alpha = \cos\beta \cdot \cos\gamma + \sin\beta \cdot \sin\gamma \cdot \cos a$$

*Proof.* Note that the distance on the sphere, in Definition 21, is given in terms of the inner product. By Proposition 4, any linear isometry of $\mathbb{R}^3$ preserves the inner product, so, if we apply a linear isometry of $\mathbb{R}^3$ to the sphere and the points on it, the distances between the points will be unchanged. Therefore, we may apply an isometry to map the vertices of the triangle to more convenient positions.




There is an isometry $L$ of $\mathbb{R}^3$ which maps $\mathbf{P}$ to $(0,0,1)$, since these points are equidistant from $\mathbf{0}$. This may be followed by a rotation about the $z$ axis to map $\mathbf{Q}$ to a point in the $xz$ plane (the plane where $y = 0$). Thus, without loss of generality, we may assume $\mathbf{P} = (0,0,1)$, and $\mathbf{Q}$ has $y$ coordinate 0. We can use this to write the Euclidean coordinates of $\mathbf{Q}$ and $\mathbf{R}$ in terms of $a, \beta$ and $\gamma$, as in the figure:

$$\begin{array}{rcccc} \mathbf{Q} & = & (\sin\beta, & 0, & \cos\beta) \\ \mathbf{R} & = & (\sin\gamma\cos a, & \sin\gamma\sin a, & \cos\gamma) \end{array}$$

By definition,

$$\cos\alpha = \langle \mathbf{Q}, \mathbf{R} \rangle = \sin\beta\sin\gamma\cos a + \cos\beta\cos\gamma$$

which gives the required result. □

LEMMA 17. *The spherical distance satisfies the triangle inequality. That is, with $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \alpha, \beta, \gamma$ and $a$ as in Proposition 8, we have $\alpha \le \beta + \gamma$, with equality, or with $\alpha + \beta + \gamma = 2\pi \iff a = \pi$. Furthermore, $a = 0 \iff \alpha = |\beta - \gamma|$.*

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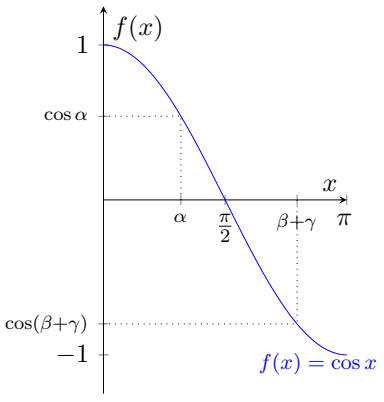
*Proof.* Since $\mathbf{P}$, $\mathbf{Q}, \mathbf{R}$ are contained in a vector space $V$ of dimension 3, and so lie on the unit sphere $S^2 \cong V \cap S^n$, it is enough to prove the result for $n = 2$. We may assume $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are distinct, since otherwise the result is easy.

By definition, $\alpha, \beta, \gamma \in [0, \pi]$, so $\sin\alpha, \sin\beta \ge 0$. Since $\cos a \ge -1$, by Proposition 8

$$(\dagger\dagger) \qquad \begin{aligned} \cos\alpha &= \cos\beta \cdot \cos\gamma + \sin\beta \cdot \sin\gamma \cdot \cos a \\ &\ge \cos\beta \cdot \cos\gamma - \sin\beta \cdot \sin\gamma = \cos(\beta + \gamma) \end{aligned}$$



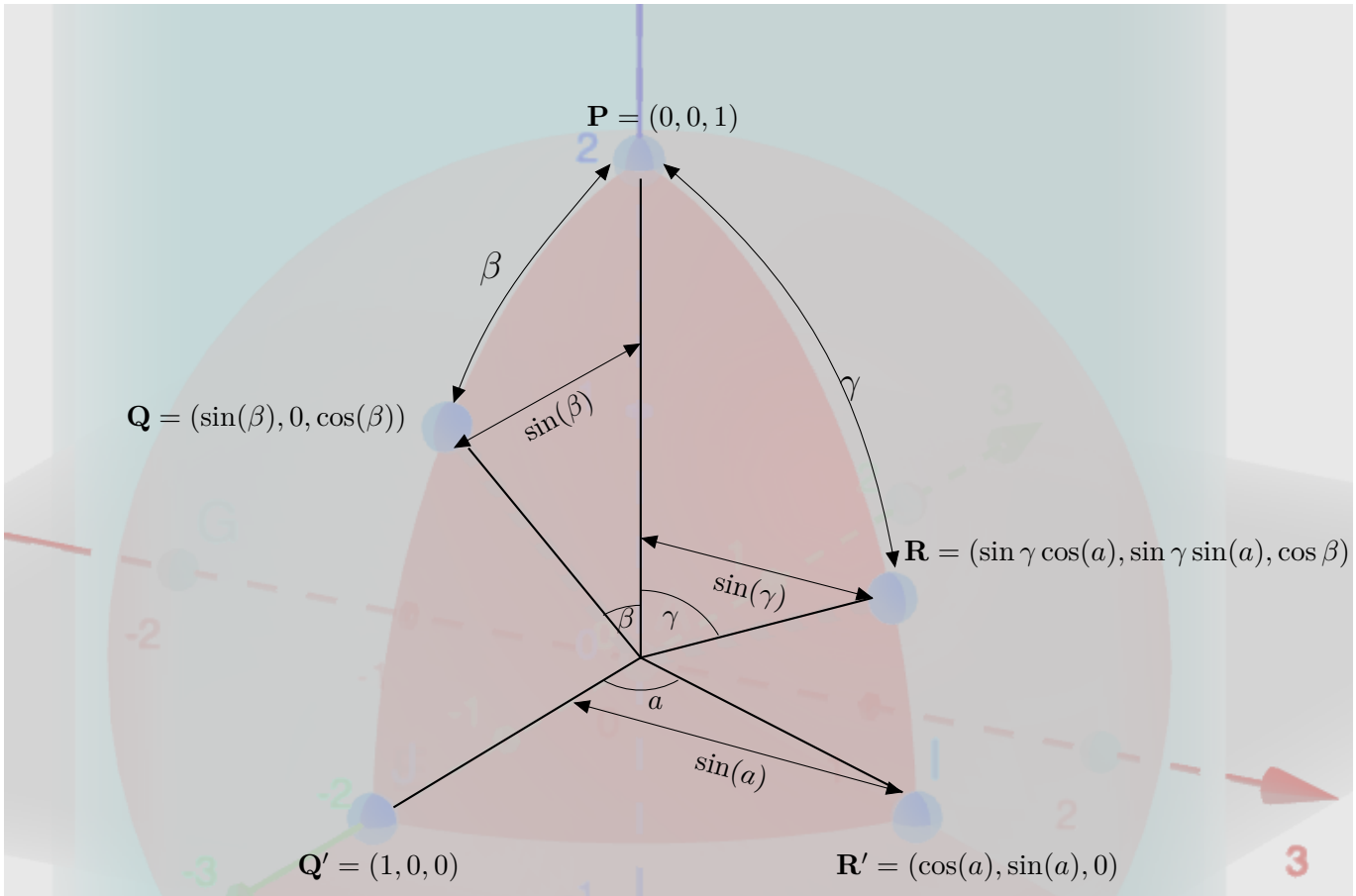

Case $\beta + \gamma \le \pi$: By definition, $\alpha \in [0, \pi]$. On $[0, \pi]$, $\cos(x)$ is strictly decreasing, so $\cos\alpha \ge \cos(\beta + \gamma) \Rightarrow \alpha \le \beta + \gamma$. There is equality in $(\dagger\dagger)$ if and only if $\cos(a) = -1$, which is if and only if $a = \pi$.

Case $\beta + \gamma > \pi$: Since $\alpha \in [0, \pi]$, the inequality is immediate. Equality in $(\dagger\dagger)$ occurs $\iff$ $a = \pi$, but now $\beta + \gamma > \pi$ so $\cos(\beta + \gamma) = \cos(\alpha) \iff$ $\alpha = 2\pi - (\beta + \gamma) \iff \alpha + \beta + \gamma = 2\pi$. The final statement follows similarly, since $a = 0 \iff \cos(\alpha) = \cos(\beta - \gamma)$. □

PROPOSITION 9. *The spherical distance $d_{S^n}$ is a metric on $S^n$*

*Proof.* Symmetry follows from symmetry of the inner product on $\mathbb{R}^n$. Non-degeneracy follows since $\cos^{-1} \in [0, \pi]$, so $d(\mathbf{P}, \mathbf{Q}) = 0 \iff \langle \mathbf{P}, \mathbf{Q} \rangle = 1$, and since $||\mathbf{P}|| = ||\mathbf{Q}|| = 1$, this implies $\mathbf{P} = \mathbf{Q}$. The triangle inequality is proved in Lemma 17. □

**16.1. Larger picture for proof of Main theorem for spherical trigonometry.** The geometric proof of this theorem requires a carefully drawn diagram. Here is a large diagram to see more clearly what is going on:



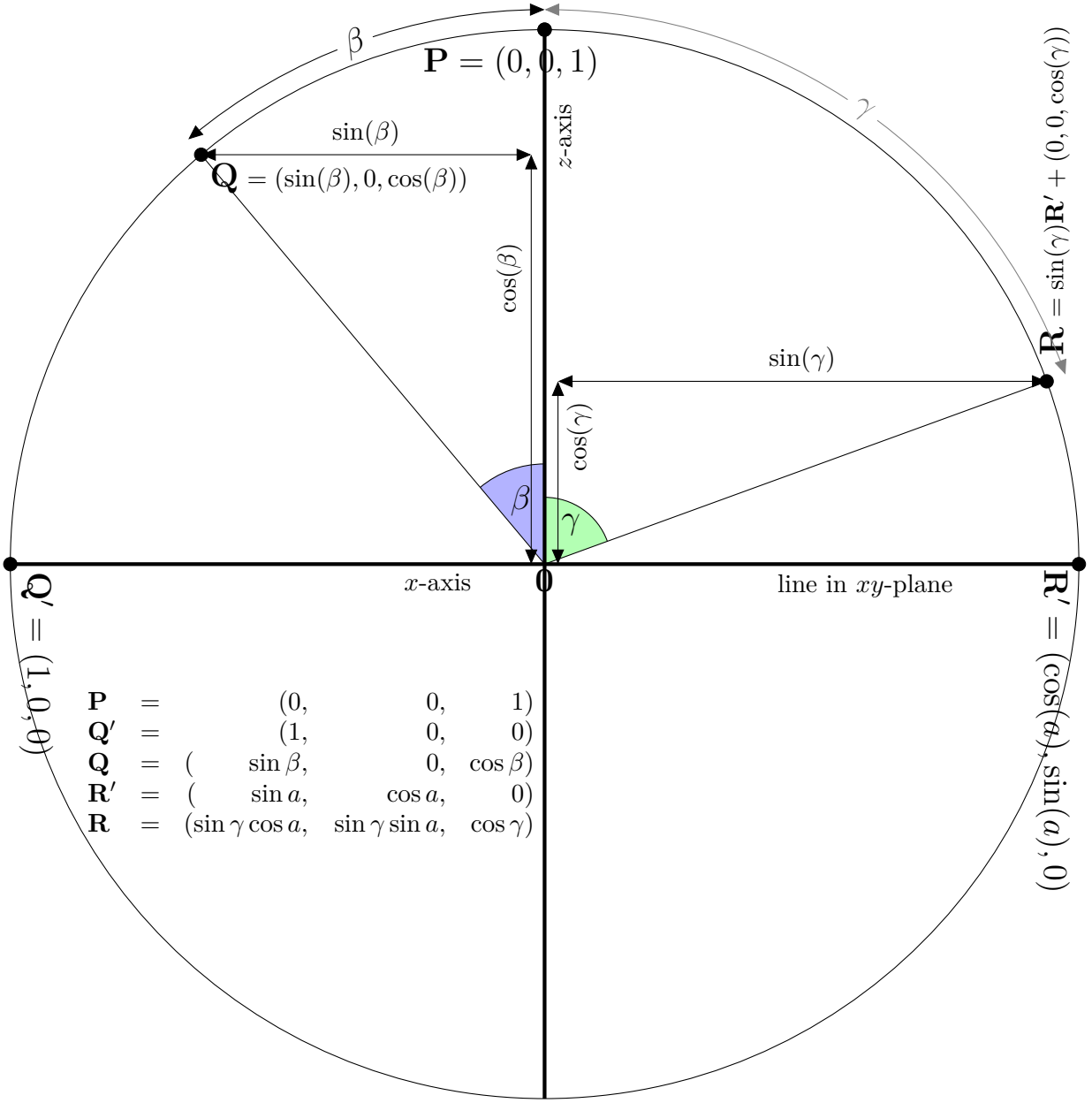

An isometry of $\mathbb{R}^3$ is applied to take $\mathbf{P}$ to $(0, 0, 1)$, and $\mathbf{Q}$ to a point on $y = 0$.
$\mathbf{Q}'$ and $\mathbf{Q}$ are both in the plane $y = 0$.
$\mathbf{R}'$ is the point of intersection between the great circle containing $\mathbf{P}$ and $\mathbf{R}$ and the plane $z = 0$. There are two such points; $\mathbf{R}'$ is chosen so that the dihedral angle between at $\mathbf{P}$ is the same as the angle $\angle(1, 0, 0)(0, 0, 0)\mathbf{R}'$. This means that the image of the spherical arc from $\mathbf{P}$ to $\mathbf{R}$ projects under the map $(x, y, z) \mapsto (x, y, 0)$ to a subset of the line from $\mathbf{0}$ to $\mathbf{R}'$.
Since the projection of $\mathbf{R} = (\mathbf{R}_x, \mathbf{R}_y, \mathbf{R}_z) = (\sin(a), \cos(a), 0)$ to the $xy$ plane lies in the same direction as $\mathbf{R}' = (\mathbf{R}'_x, \mathbf{R}'_y, \mathbf{R}'_z)$, we have that $(\mathbf{R}_x, \mathbf{R}_y) = \lambda(\mathbf{R}'_x, \mathbf{R}'_y)$ for some $\lambda$. Since the length of the line from $\mathbf{R}$ to the $z$ axis which is perpendicular to that axis has length $\sin(\beta)$, we must have $\lambda = \sin(\beta)$. The $z$ coordinate of $\mathbf{R}$ is $\cos(\beta)$, and so

$$\mathbf{R} = (\sin(\beta)\sin(a), \sin(\beta)\cos(a), \cos(\beta))$$

$\alpha$ is the spherical distance from $\mathbf{Q}$ to $\mathbf{R}$. It is not drawn in this figure for less clutter.

**16.2. "Cut out'n'keep" proof of Main theorem for spherical trigonometry.** Cut out the following circle, and fold along the $z$-axis. Adjust so that the two halves of the circle are at an angle $a$ to each other. Use this to convince yourself that

$$\cos\alpha = \cos(a)\sin(\beta)\sin(\gamma) + \cos(\beta)\cos(\gamma)$$




$$\begin{array}{lcrrr}
\mathbf{P} & = & (0, & 0, & 1) \\
\mathbf{Q}' & = & (1, & 0, & 0) \\
\mathbf{Q} & = & (\sin\beta, & 0, & \cos\beta) \\
\mathbf{R}' & = & (\sin a, & \cos a, & 0) \\
\mathbf{R} & = & (\sin\gamma\cos a, & \sin\gamma\sin a, & \cos\gamma)
\end{array}$$

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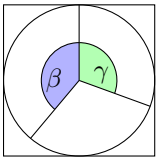
Note that as you open or close the circle, the angles $a$, which is the dihedral angle between the two planes (the plane containing $\mathbf{0}, \mathbf{P}$ and $\mathbf{R}$ and the plane containing $\mathbf{0}, \mathbf{P}, \mathbf{Q}$), and the angle $\alpha$, which is the angle between the two vectors $\mathbf{Q}$ and $\mathbf{R}$ will vary.

The minimum possible value of $\cos(a)$ is $-1$, which is obtained before the paper is folded, ie., lying flat, unfolded. In this case, either $\alpha = \beta + \gamma$, when $\beta + \gamma \leq \pi$, or, if $\beta + \gamma > \pi$, as in this little picture, then if $a = \pi$, we still have $\cos(\alpha) = \cos(\beta + \gamma)$, so $\beta + \gamma = 2\pi - \alpha$.

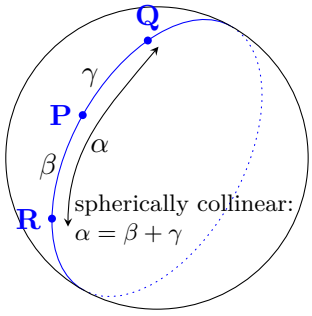


## 17. Lecture 10: Spherical isometries

In this section we write $d := d_{S^n}$. The goal is to show that $\mathrm{Isom}(S^n, d) \cong O(n+1)$.
We will show that spherical isometries preserve spherical lines, so we need a result about these.

LEMMA 18. *If $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are points on $S^n$, and $\alpha, \beta, \gamma$ are as in Proposition 8, then $\alpha = \beta \pm \gamma$ (possibly after permutation of points), or $\alpha + \beta + \gamma = 2\pi$ if and only if $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ lie on a great circle.*

PROOF. By Lemma 17, we have $\alpha = \beta + \gamma$ (up to reordering) or $\alpha + \beta + \gamma = 2\pi$ or $\alpha = |\beta - \gamma|$ if and only if $a = 0$ or $\pi$, which is the case if and only if $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ lie on a great circle, after possibly permuting the points. □




DEFINITION 23. *We say $\mathbf{P}, \mathbf{Q}, \mathbf{R} \in S^n$ are collinear on $S^n$ if and only if they are all contained on the same great circle $\mathcal{C}$.*

Note that since $S^n \subset \mathbb{R}^{n+1}$, the points $\mathbf{P}, \mathbf{Q}, \mathbf{R} \in S^n$ could be collinear in $S^n$, or in $\mathbb{R}^{n+1}$. These are different concepts. The latter only happens if two of them coincide.

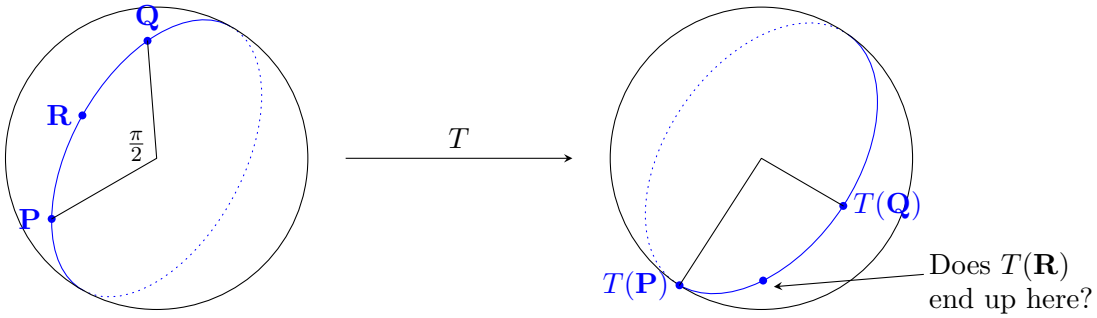
LEMMA 19. *An isometry $T : S^n \to S^n$ preserves antipodal points.*

*Proof.*
$$\begin{aligned}
&\mathbf{P} \text{ and } \mathbf{Q} \text{ are antipodal} \iff \mathbf{Q} = -\mathbf{P} \\
&\iff \langle \mathbf{P}, \mathbf{Q} \rangle = -1, \text{ using } |\langle \mathbf{P}, \mathbf{Q} \rangle| = ||\mathbf{P}||||\mathbf{Q}|| \iff \mathbf{P} \in \mathbb{R}\mathbf{Q} \\
&\iff d(\mathbf{P}, \mathbf{Q}) = \pi \\
&\iff d(T(\mathbf{P}), T(\mathbf{Q})) = \pi \\
&\iff T(\mathbf{P}) \text{ and } T(\mathbf{Q}) \text{ are antipodal}
\end{aligned}$$

□

PROPOSITION 10. *An isometry $T : S^n \to S^n$ preserves great circles.*

PROOF. We need to show that if $\mathcal{C}$ is a great circle, then so is $T(\mathcal{C})$.
Take two points $\mathbf{P}$ and $\mathbf{Q}$ on $\mathcal{C}$ with $d(\mathbf{P}, \mathbf{Q}) = \frac{\pi}{2}$. By Lemma 16, $\mathcal{C}$ is the unique great circle through $\mathbf{P}$ and $\mathbf{Q}$. Let $\mathcal{D}$ be the unique great circle through $T(\mathbf{P})$ and $T(\mathbf{Q})$. We know this is unique by Lemma 16, since $\mathbf{P}$ are choosen to be not antipodal, and so by Lemma 19, neither are $T(\mathbf{P})$ and $T(\mathbf{Q})$. Thus we need to show that for all $\mathbf{R} \in \mathcal{C}$, we have $T(\mathbf{R}) \in \mathcal{D}$.

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Let $\alpha, \beta, \gamma$ be the distances as in Proposition 8. Then either $\alpha + \beta + \gamma = 2\pi$ or $\alpha = \beta \pm \gamma$, up to reordering. In each case, since $T$ is an isometry, we will get the same corresponding relationship between the points $T(\mathbf{P})$, $T(\mathbf{Q})$ and $T(\mathbf{R})$; for example, if $\alpha = \beta + \gamma$, i.e., $d(\mathbf{Q}, \mathbf{R}) = d(\mathbf{P}, \mathbf{Q}) + d(\mathbf{P}, \mathbf{R})$, then we also have $d(T(\mathbf{Q}), T(\mathbf{R})) = d(T(\mathbf{P}), T(\mathbf{Q})) + d(T(\mathbf{P}), T(\mathbf{R}))$, and so by Lemmma 18, $T(\mathbf{Q}), T(\mathbf{R}), T(\mathbf{P})$ are also spherically collinear. The other cases are similar, and so the result follows. □

## 18. Lecture 11: Isometries are given by linear maps

LEMMA 20. *A bijective map $f : \mathbb{R}^k \to \mathbb{R}^k$, which preserves the standard inner product on $\mathbb{R}^k$ is a linear isometry of $\mathbb{R}^k$.*

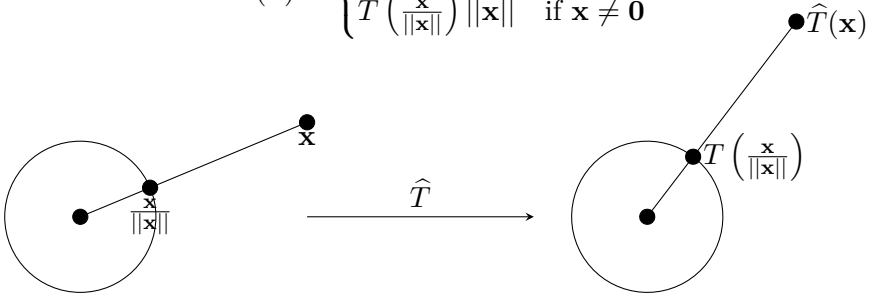
PROOF. If the inner product is preserved, then since $d(\mathbf{x},\mathbf{y}) = \sqrt{\langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle}$, then $f$ must preserve distance: $||f(\mathbf{x}) - f(\mathbf{y})||^2 = \langle f(\mathbf{x}) - f(\mathbf{y}), f(\mathbf{x}) - f(\mathbf{y})\rangle = \langle f(\mathbf{x}), f(\mathbf{x})\rangle - 2\langle f(\mathbf{x}), f(\mathbf{y})\rangle + \langle f(\mathbf{y}), f(\mathbf{y})\rangle = \langle \mathbf{x},\mathbf{x}\rangle - 2\langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{y}\rangle = ||\mathbf{x}-\mathbf{y}||$. Given that $f$ is bijective, $f$ must be a Euclidean isometry. Since the inner product is preserved, $f(\mathbf{0}) = \mathbf{0}$. By Corollary 1 isometries of $\mathbb{R}^k$ fixing $\mathbf{0}$ are given by linear maps. □

**Remark:** We can drop the assumption that $f$ is a bijection in Lemma 20, as this follows since if $\sum a_i f(\mathbf{e}_i) = \mathbf{0}$, then by taking the inner product with $f(\mathbf{e}_j)$, we find $a_j = 0$ for $1 \le j < k$.

PROPOSITION 11. *If $T$ is an isometry of $S^n$, then $T$ extends to a Euclidean isometry $\widehat{T}$ of $\mathbb{R}^{n+1}$, that is, there is a Euclidean isometry $\widehat{T}$ of $\mathbb{R}^{n+1}$, and $\widehat{T}|_{S^n} = T$.*

PROOF. Given an isometry $T$ of $S^n$, define $\widehat{T}$ by radial extension, i.e., by

$$\widehat{T}(\mathbf{x}) = \begin{cases} \mathbf{0} & \text{if } \mathbf{x} = \mathbf{0} \\ T\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)||\mathbf{x}|| & \text{if } \mathbf{x} \neq \mathbf{0} \end{cases} \tag{4}$$




We will show that $\widehat{T}$ is an isometry of $\mathbb{R}^{n+1}$. First, $\widehat{T}$ is a bijection, because $\widehat{T^{-1}}$ is the inverse of $\widehat{T}$: First, note that since $T$ and $T^{-1}$ preserve the $S^n$, only $\mathbf{0}$ maps to $\mathbf{0}$, so we assume $\mathbf{x} \neq \mathbf{0}$. Then, using the fact that $||T(\mathbf{y})|| = 1$ for all $\mathbf{y} \in S^n$, we have

$$\begin{aligned} \widehat{T^{-1}}\left(\widehat{T}(\mathbf{x})\right) &= \widehat{T^{-1}}\left(T\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)||\mathbf{x}||\right) = T^{-1}\left(\frac{\mathbf{z}}{||\mathbf{z}||}\right)||\mathbf{z}||, \text{ where } \mathbf{z} = T\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)||\mathbf{x}|| \\ &= T^{-1}\left(T\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)\right)||\mathbf{x}||, \text{ since } ||\mathbf{z}|| = ||\mathbf{x}|| \\ &= \left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)||\mathbf{x}|| = \mathbf{x}. \end{aligned}$$

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Now we show $\widehat{T}$ preserves the inner product on $\mathbb{R}^{n+1}$. Since $T$ preserves the spherical metric, for $\mathbf{x},\mathbf{y} \in S^n$, $\langle \mathbf{x},\mathbf{y}\rangle = \cos\left(d_{S^n}(\mathbf{x},\mathbf{y})\right) = \cos(d_{S^n}(T(\mathbf{x}),T(\mathbf{y})) = \langle T(\mathbf{x}),T(\mathbf{y})\rangle$, so $T$ preserves the inner product restricted to $S^n$. Suppose $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$. Assume $\mathbf{x},\mathbf{y} \neq \mathbf{0}$, since otherwise the result is easy.

$$\begin{aligned} \left\langle \widehat{T}(\mathbf{x}), \widehat{T}(\mathbf{y}), \right\rangle &= \left\langle T\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)||\mathbf{x}||, T\left(\frac{\mathbf{y}}{||\mathbf{y}||}\right)||\mathbf{y}|| \right\rangle \\ &= \left\langle T\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right), T\left(\frac{\mathbf{y}}{||\mathbf{y}||}\right) \right\rangle ||\mathbf{x}||||\mathbf{y}||, \text{ since the inner product is bilinear} \\ &= \left\langle \frac{\mathbf{x}}{||\mathbf{x}||}, \frac{\mathbf{y}}{||\mathbf{y}||} \right\rangle ||\mathbf{x}||||\mathbf{y}||, \text{ since } T \text{ preserves the inner product on } S^n \\ &= \langle \mathbf{x},\mathbf{y}\rangle \text{ by bilinearity} \end{aligned}$$

Hence by Lemma 20 $\widehat{T}$ is linear Euclidean isometry. □

COROLLARY 4. *Every isometry $T : S^n \to S^n$ is of the form $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in S^n$ for some orthogonal matrix $A \in O(n+1)$.*

PROOF. This follows from Proposition 11, which tells us that $T$ is a restriction of a linear Euclidean isometry to $S^n$, and Proposition 4. □

Now we prove a converse of Proposition 11

LEMMA 21. *If $T$ is a linear Euclidean isometry of $\mathbb{R}^{n+1}$, then the restriction $T|_{S^n}$ to $S^n$ is a spherical isometry.*

PROOF. Since $T$ preserves the Euclidean distance, and fixes $\mathbf{0}$, it must map $S^n$ to $S^n$, since this is the set of points distance 1 from $\mathbf{0}$. Since $T$ has an inverse which is also a Euclidean isometry, and so also maps $S^n$ to $S^n$, then $T$ restricted to $S^n$ is a bijection. By Proposition 4 $T$ preserves the inner product, and thus preserves the spherical distance $\cos^{-1}(\langle \mathbf{x}, \mathbf{y} \rangle)$ on $S^n$. Hence $T_{S^n}$ is a spherical isometry □

COROLLARY 5. *There is a group isomorphism*

$$\mathrm{Isom}(S^n, d_{S^n}) \cong O(n+1)$$

PROOF. From Corollary 4, there is a map:

$$\begin{aligned} \phi : \mathrm{Isom}(S^n, d_{S^n}) &\to O(n+1) \\ T &\mapsto A \end{aligned}$$

where $A$ is the orthogonal matrix of the linear Euclidean isometry $\widehat{T}$.
From Lemma 21, there is a map

$$\begin{aligned} \psi : O(n+1) &\to \mathrm{Isom}(S^n, d_{S^n}) \\ A &\mapsto T_A|_{S^n} \end{aligned}$$

where $T_A$ is the linear map defined by the matrix $A$. We need to check that these maps are inverse to each other, and that the resulting bijection of sets is also a group isomorphism.
Given $T$, if $A$ is the matrix of $\widehat{T}$, then $\widehat{T} = T_A$ and by construction, $\widehat{T}|_{S^n} = T$, so we have $\psi(\phi(T)) = \psi(A) = T_A|_{S^n} = T$.
Conversely, given an orthogonal matrix $A$, then the map $T_A$ is a linear map, so determined by the values of $A\mathbf{e}_i$ for the standard basis $\mathbf{e}_i$. Thus, since all $\mathbf{e}_i$ have length 1, and are so on $S^n$, the map $\widehat{T_A|_{S^n}}$ is uniquely determined by the values of $T_A|_{S^n}$, so we must have $\phi(\psi(A)) = \phi(T_A|_{S^n}) = A$.
So $\phi$ and $\psi$ are bijections.
Now we show that $\phi$ is a group homomorphism. We need to show that $\phi(T \circ T') = \phi(T)\phi(T')$ for any two spherical isometries $T$ and $T'$. Since $\phi(T \circ T')$ and $\phi(T)\phi(T')$ are matrices, which are determined by their values on a basis of $\mathbb{R}^{n+1}$, we have $\phi(T \circ T') = \phi(T)\phi(T')$ if and only if $\phi(T \circ T')\mathbf{e}_i = \phi(T)\phi(T')\mathbf{e}_i$ for $i = 1, \ldots, n+1$. If $\phi(T \circ T') = A$ and $\phi(T) = B$ and $\phi(T') = C$ for orthogonal matrices $A, B, C$, then for all $\mathbf{x} \in S^n$, we have

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$$\begin{aligned} \phi(T \circ T')\mathbf{x} &= A\mathbf{x} = T \circ T'(\mathbf{x}) \\ \phi(T)\mathbf{x} &= B\mathbf{x} = T(\mathbf{x}) \\ \phi(T')\mathbf{x} &= C\mathbf{x} = T'(\mathbf{x}), \end{aligned}$$

Since $A, B, C$ preserve $S^n$, we have

$$A\mathbf{x} = T \circ T'(\mathbf{x}) = T(T'(\mathbf{x})) = B(C\mathbf{x}) = (BC)\mathbf{x},$$

for all $\mathbf{x} \in S^n$, and in particular for all the $\mathbf{e}_i$, and so $A = BC$, i.e., $\phi(T \circ T') = \phi(T)\phi(T')$, and the map $\phi$ is a group isomorphism.

□

## 19. Lecture 12: lines and triangles on the 2-sphere

In this section we restrict to $S^2$, in order to compare spherical geometry with Euclid's geometry in the plane.

## 20. Lines

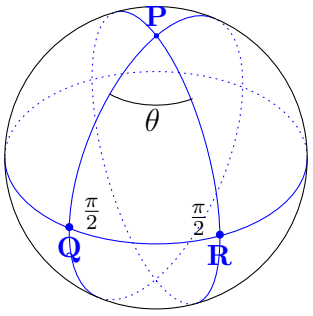
There are no parallel lines in $S^2$. Here, by lines we mean great circles, and by parallel, we mean lines with no intersection. Thus Euclid's parallel postulate does not hold for $S^n$. More precisely:

PROPOSITION 12. *If $\mathcal{C}$ and $\mathcal{D}$ are two distinct great circles on $S^2$, then the intersection $\mathcal{C} \cap \mathcal{D}$ is equal to a pair of antipodal points.*

PROOF. By definition, there are 2-dimensional subspaces $V$ and $W$ in $\mathbb{R}^3$ such that $\mathcal{C} = S^2 \cap V$ and $\mathcal{D} = S^2 \cap W$. Since $\mathcal{C} \neq \mathcal{D}$, we must have $V \neq W$. This implies that $W \cap V$ is one dimensional, since if it was zero dimensional, the space $V \oplus W$ would be four dimesional, but this is a subpace of $\mathbb{R}^3$, which is three dimensional. Since $W \neq V$, $W \cap V$ has dimension strictly less than 2. Hence $\dim(V \cap W) = 1$ and for some $\mathbf{v}$, with $||\mathbf{v}|| = 1$, we have $W \cap V = \mathbb{R}\mathbf{v}$, and so $\mathcal{C} \cap \mathcal{D} = S^2 \cap V \cap W = S^2 \cap \mathbb{R}\mathbf{v} = \{\mathbf{v}, -\mathbf{v}\}$, which is a set of two antipodal points. □

## 21. Triangles

In the Euclidean plane, the angles of a triangle sum to $\pi$.
We now show that the angles of a (non-trivial) spherical triangle sum to a number strictly greater than $\pi$.
First, note that the angle sum for a spherical triangle is not constant. For example, the triangle in the figure, where $\mathbf{P} = (0, 0, 1)$ and $\mathbf{Q} = (1, 0, 0)$ and $\mathbf{R} = (\cos\theta, \sin\theta, 0)$ can have angle sum $\pi + \theta$ for any $\theta \in (0, 2\pi)$.
We will show that the angle sum is determined by the area of the triangle.




We will make some assumptions about area, in particular, assume that you will learn about a rigourous definition of area in terms of integration in analysis.

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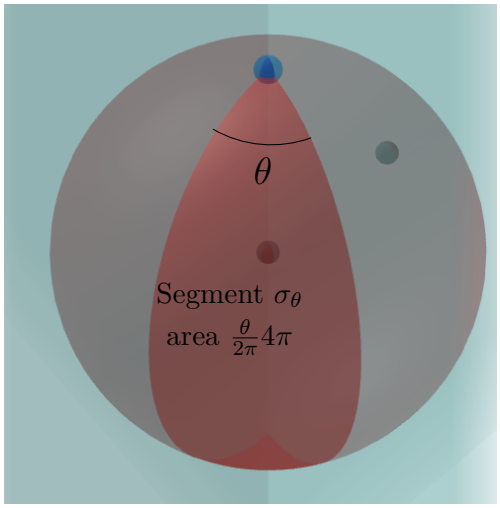
**Assumptions:**

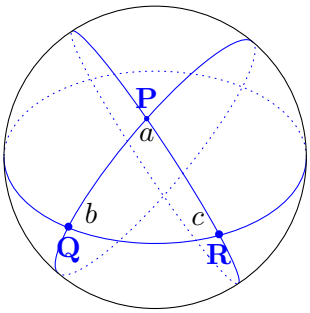
(1) The total surface area of a unit sphere is $4\pi$.
(2) The area of a region of $S^2$ is invariant under application of any isometry of $S^2$.
(3) If $X$ and $Y$ are two non intersecting regions on $S^2$, then $\text{area}(X \cup Y) = \text{area}(X) + \text{area}(Y)$
(4) The segment of a sphere, $\sigma_\theta$ that lies between two semi-circular segments of great circles, from $\mathbf{P}$ to $-\mathbf{P}$, meeting at angle $\theta$ has area $2\theta$.
Let $\Sigma_\theta$ be the union of $\sigma_\theta$ and $-\sigma_\theta$. So $\Sigma_\theta$ has area $4\theta$.




THEOREM 8. *(Girard's theorem) Let $\Delta\mathbf{PQR}$ be a spherical triangle on $S^2$, with $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ distinct points, and such that the interior of $\Delta\mathbf{PQR}$ does not intersect the great circles which contain the sides of $\Delta\mathbf{PQR}$,*

*and with non-zero area. Then*

$$\textit{area}\,(\Delta\mathbf{PQR}) = a + b + c - \pi$$




*Proof.* Notice that area($\Delta\mathbf{PQR}$) = area($-\Delta\mathbf{PQR}$), since $\mathbf{x} \mapsto -\mathbf{x}$ is an isometry, and area is preserved by isometries. Also notice that since we assume that the triangle does not contain any of the extended edges in its interior, these triangles are disjoint. Notice that $S^2$ is the union of $\Sigma_a$, $\Sigma_b$ and $\Sigma_c$, and that

$$\sigma_a \cap \sigma_b = \sigma_b \cap \sigma_c = \sigma_a \cap \sigma_b = \Delta\mathbf{PQR}$$

and similarly for the images of these areas under $\mathbf{x} \mapsto -\mathbf{x}$.
Let

$$S_a = \sigma_a \setminus \Delta\mathbf{PQR}$$
$$S_b = \sigma_b \setminus \Delta\mathbf{PQR}$$
$$S_c = \sigma_c \setminus \Delta\mathbf{PQR}$$

Then we have a disjoint union:

$$S^2 = S_a \cup S_b \cup S_c \cup (-S_a) \cup (-S_b) \cup (-S_c) \cup \Delta\mathbf{PQR} \cup (-\Delta\mathbf{PQR}).$$

Since we have a disjoint union $\sigma_a = S_a \cup \Delta\mathbf{PQR})$, we have $\text{area}(\sigma_\theta) = \text{area}(S_\theta) + \text{area}(\Delta\mathbf{PQR})$, for $\theta = a, b, c$, so

$$\text{area}(S^2) = \text{area}(\Sigma_a) + \text{area}(\Sigma_a) + \text{area}(\Sigma_a) - 4\,\text{area}(\Delta\mathbf{PQR})).$$

By assumption $\Sigma_\theta$ has area $2\theta$ so

$$4\pi = 4(a + b + c) - 4\,\text{area}(\Delta\mathbf{PQR}),$$

and the result follows. □

**Remark:** This result also holds if we do not make the restriction that the triangle does not contain the extensions of its edges in it's interior, but then $\Delta\mathbf{PQR}$ and $(-\Delta\mathbf{PQR})$ can not be guaranteed to be disjoint, and so we need to consider several different cases. To save time, I will not prove the general case, which is not needed for the following result.

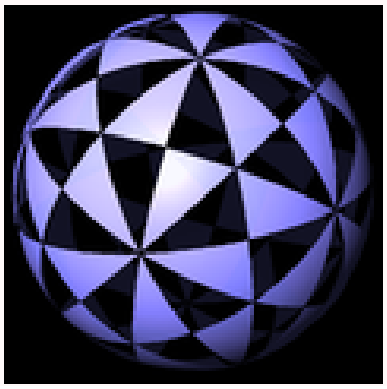
Corollary 6. *The sum of the angles of a spherical triangle, satisfying the hypothesis of Theorem 8, is strictly greater than $\pi$.*

Proof. This follows from Theorem 8. □

**Schwarz triangles:** Suppose $\Delta$ is a spherical triangle, and $G$ is the group generated by reflections in the sides of $\Delta$. Suppose these reflections result in a tiling of the sphere – this means that $S^2 = \cup_{T\in G} T(\Delta)$ and $T_1(\Delta)^\circ \cap T_2(\Delta)^\circ \neq \varnothing \iff T_1 = T_2$, where $X^\circ$ means the interior of $X$. Then the angles must divide $\pi$. This means that if $\Delta$ tiles $S^2$ through reflections, then the angles are $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ for positive integers $p, q, r$ such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

The only solutions are $(p, 2, 2), (3, 3, 2), (4, 3, 2), (5, 3, 2)$. The later is shown on the right.



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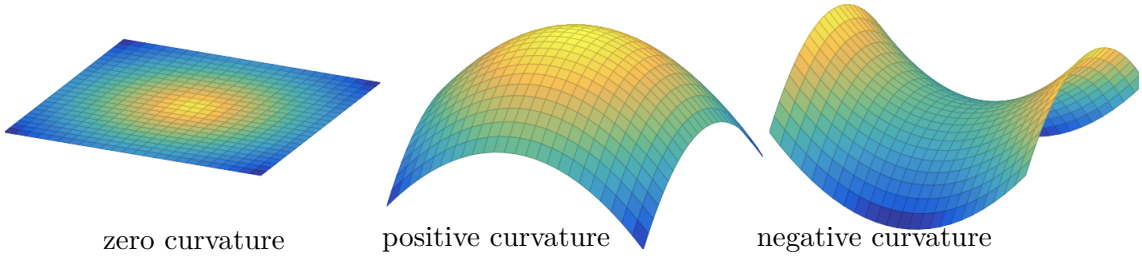
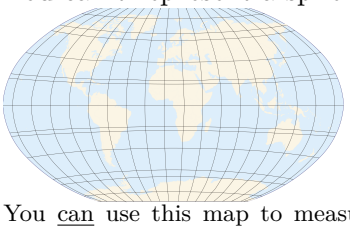
## 22. Lecture 13: Hyperbolic Geometry

**22.1. Motivation: curvature (non-examinable).** Hyperbolic geometry completes the study of the basic different Riemannian manifolds. It's beyond the scope of the course to go into what these are in general, but basically they look pretty much like Euclidean space if you look at a small enough bit of them, and you can measure distance – they have a metric. You can also measure curvature. The Euclidean plane, sphere, and hyperbolic plane are all surfaces with constant curvature: zero, positive or negative respectively. If the surface lives isometrically in $\mathbb{R}^3$, then the curvature at a point $\mathbf{P}$ is just the product of the reciprocal of the radii of two circles touching the surface at $\mathbf{P}$. We don't have time to go into what this really means, but here are some pictures:



zero curvature positive curvature negative curvature

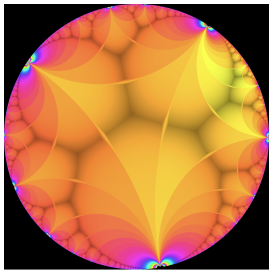
These pictures show the basic cases. (The non-flat examples have non-constant curvature.) [3]
You can't represent a sphere in $\mathbb{R}^2$ while preserving distance, meaning spherical distance on the sphere, and Euclidean distance on the plane. The hyperbolic plane can not even be embedded in $\mathbb{R}^3$. So we will work with a "model" of hyperbolic plane that lives in $\mathbb{R}^3$, but length can not be computed using the Euclidean infinitesimal distance $\sqrt{dx^2+dy^2+dz^2}$.



You can use this map to measure distance, using the curved grid, not a regular Cartesian grid.

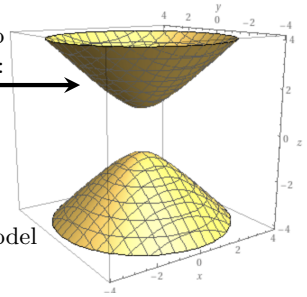
A sphere "lives in" $\mathbb{R}^3$.

The Poincare's disk, in $\mathbb{C}$ is a model of the hyperbolic plane; so is the "sphere of radius $i$":



Poincaré disk model

top sheet of two sheeted hyperboloid: $\sqrt{x^2+y^2-z^2}=i$



attempt at embedding hypberbolic space isometrically in $\mathbb{R}^3$

One of these looks flat, and the other looks like it has positive curvature. So what does it mean that hyperbolic space has negative curvature? Instead of measuring curvature in terms of the ambient space $\mathbb{R}^3$, it can be measured intrinsically, just in terms of the metric on the surface. For example, a circle with radius $r$ drawn on a plane, sphere, or on hyperbolic space, will have circumference length $2\pi r$, or greater than $2\pi r$ or less than $2\pi r$ respectively. A circle in a metric space $(X,d)$ with centre $\mathbf{P}$ is the set of all points $\mathbf{Q}$ in $X$ with $d(\mathbf{P},\mathbf{Q})=r$. So, whilst we will use the hyperboloid model of the hyperbolic plane, which lives in $\mathbb{R}^3$, we need to be careful not to think of distances as being measured with respect to the usual distance, they are not!

week 5 lecture 13 Hyperbolic geometry

[3] Figures from `https://commons.wikimedia.org/wiki/File:World_map_longlat.svg`; `https://commons.wikimedia.org/wiki/File:United_Kingdom_on_the_globe_(Europe_centered).svg`; Wolfram alpha; shutterstock

## 23. Hyperbolic space

We will use the model of hyperbolic space defined as follows. An advantage of this definition is that results are very similar to results in the case of the sphere, just with a "twist", which is to replace the usual Euclidean metric $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ with the the Lorentz inner product. We define

$$J_{n+1} := \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & I_n \end{pmatrix}$$

Note that $J_m$ has $m$ rows and columns and acts on $\mathbb{R}^m$. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, then $\mathbf{x}^T J \mathbf{y}$ means $\mathbf{x}^T J_m \mathbf{y}$.

DEFINITION 24. *The Lorentz inner product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined by:*

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle_L &:= -x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots x_n y_n \\ &= \mathbf{x}^T J \mathbf{y} \end{aligned}$$

DEFINITION 25. *The Lorentz norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is given by*

$$||\mathbf{x}||_L := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_L}$$

We always take this root to be positive or positive imaginary, or zero. WARNING: $\langle \mathbf{x}, \mathbf{x} \rangle$ may be negative, so $||\mathbf{x}||_L$ may be imaginary, so is not a norm.

DEFINITION 26. *The $n$-dimensional hyperbolic space is the metric space with underlying set*

$$\begin{aligned} \mathcal{H}^n &:= \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : -x_1^2 + x_2^2 + x_3^2 + \cdots + x_{n+1}^2 = -1 \text{ and } x_1 > 0 \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{R}^{n+1} : ||\mathbf{x}||_L = i \text{ and } x_1 > 0 \right\} \end{aligned}$$

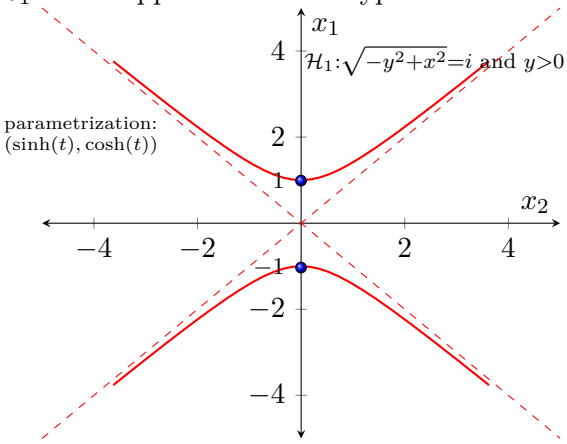
*and with metric defined by*

$$d_{\mathcal{H}^n}(\mathbf{x}, \mathbf{y}) := \cosh^{-1}\left( \frac{\langle \mathbf{x}, \mathbf{y} \rangle_L}{||\mathbf{x}||_L ||\mathbf{y}||_L} \right) = \cosh^{-1}\left( - \langle \mathbf{x}, \mathbf{y} \rangle_L \right)$$

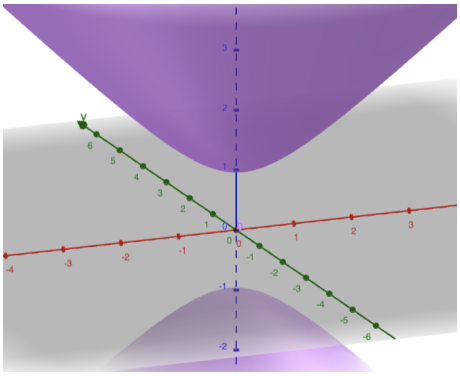
where we have used that on $\mathcal{H}^n$, $||\mathbf{x}||_L = i$. We will review the hyperbolic cosine cosh shortly.

**Examples:** The $x_1$-axis is usually drawn pointing up.

$\mathcal{H}_1$ is the upper half of the hyperbola:




$\mathcal{H}_2$ is the upper sheet of a hyperboloid:[a]



[a] https://www.geogebra.org/m/xqnc3sdm

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Hyperbolic geometry

The Goal of this chapter is to

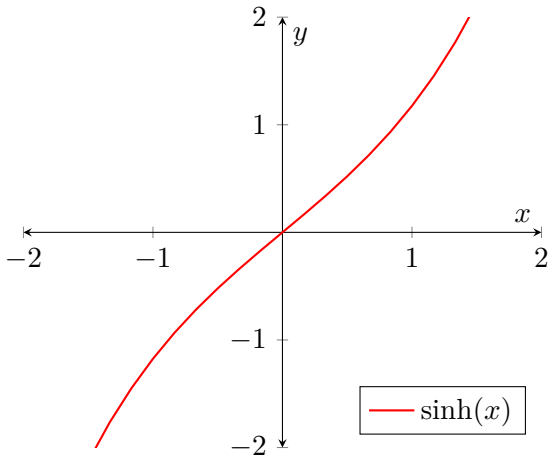
(1) show that the hyperbolic metric is a metric
(2) determine the group of hyperbolic isometries
(3) investigate the failure of the parallel postulate
(4) show that triangles have angles sum less than $\pi$

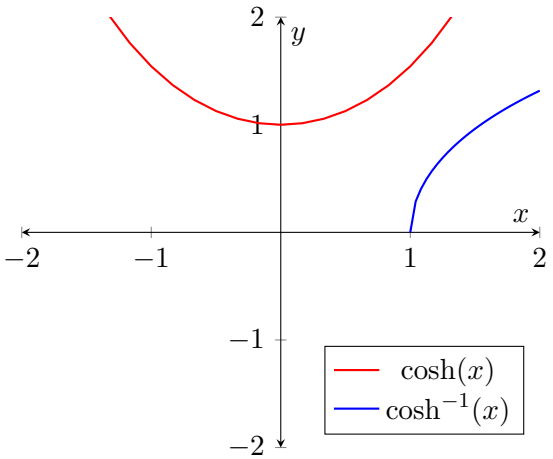
First we need to understand hyperbolic sine and cosine, and check $d_{H^n}$ is well defined.

## 24. Hyperbolic trigonometry

DEFINITION 27. *The hyperbolic sine and cosine,* sinh *and* cosh *are defined by*

$$\sinh(\theta) = \frac{\exp(\theta)-\exp(-\theta)}{2} = -i\sin(i\theta) \qquad \cosh(\theta) = \frac{\exp(\theta)+\exp(-\theta)}{2} = \cos(i\theta)$$




$$\sinh^{-1} : \mathbb{R} \to \mathbb{R} \qquad y \mapsto \ln(y + \sqrt{y^2+1})$$

$$\cosh^{-1} : [1,\infty) \to [0,\infty) \qquad y \mapsto \ln(y + \sqrt{y^2-1})$$

**Formulae:** These can be derived from properties of the exponential, or the relationship with cos and sin, and formulae you already know

**hyperbolic**

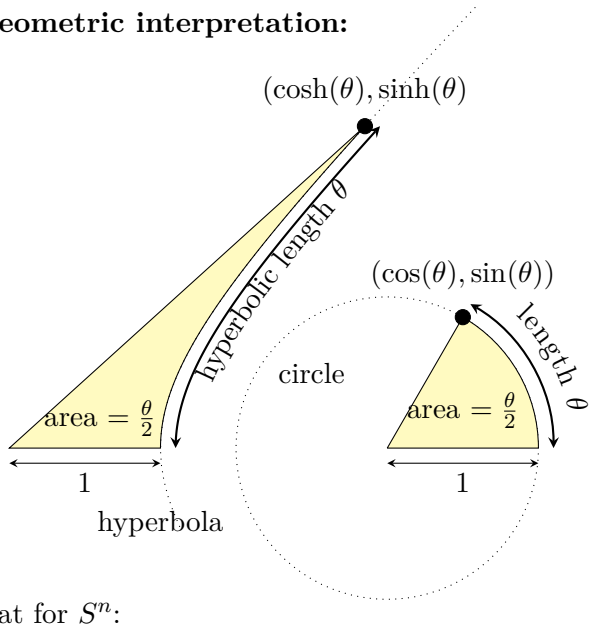
$\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$
$\sinh(a+b) = \sinh(a)\cosh(b) + \cosh(a)\sinh(b)$
$\cosh'(a) = \sinh(a) \qquad \sinh'(a) = \cosh(a)$
$\cosh^2(a) - \sinh^2(a) = 1$

**comparison with sine and cosine:**

**trigonometric**

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
$\cos'(a) = -\sin(a) \qquad \sin'(a) = \cos(a)$
$\cos^2(a) + \sin^2(a) = 1$

**Geometric interpretation:**




The definition of the metric for $\mathcal{H}^n$ is comparable to that for $S^n$:

$$d_{S^n}(\mathbf{x},\mathbf{y}) = \cos^{-1}(\mathbf{x},\mathbf{y}) = \left(\frac{\langle \mathbf{x},\mathbf{y}\rangle}{||\mathbf{x}||||\mathbf{y}||}\right) \qquad d_{\mathcal{H}^n}(\mathbf{x},\mathbf{y}) = \cosh^{-1}(\mathbf{x},\mathbf{y}) = \left(\frac{\langle \mathbf{x},\mathbf{y}\rangle_L}{||\mathbf{x}||_L||\mathbf{y}||_L}\right)$$

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So results about the hyperbolic metric are analagous to results for the spherical metric.
To show that $d_{\mathcal{H}^n}(\mathbf{x},\mathbf{y})$ is well defined, we need to prove that if $\mathbf{x},\mathbf{y} \in \mathcal{H}^n$, then $\frac{\langle \mathbf{x},\mathbf{y}\rangle_L}{||\mathbf{x}||_L||\mathbf{y}||_L} \in [1,\infty)$. We will follow [R3.7–3.10], and find an isometry which makes these results easier to prove. First we need to investigate the Lorentz inner product, and the maps of $\mathbb{R}^{n+1}$ which preserve it. Eventually we will prove that these are all linear. We follow [RS, Appendix B] for Lorentz linear algebra.

## 25. Lecture 14: Lorentzian linear algebra

### 25.1. The hyperbolic metric is well defined.

LEMMA 22. *For $\mathbf{x}, \mathbf{y} \in \mathcal{H}^n$ we have $-\langle \mathbf{x}, \mathbf{y} \rangle_L \geq 1$, and $\langle \mathbf{x}, \mathbf{y} \rangle > 0$.*

PROOF. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. Let $\widehat{\mathbf{x}} = (1, x_2, \ldots, x_n)$ and $\widehat{\mathbf{y}} = (1, y_2, \ldots, y_n)$. Then since $\mathbf{x}, \mathbf{y} \in \mathcal{H}^n$, we have

$$\begin{aligned} x_1^2 &= 1 + x_2^2 + x_3^2 + \cdots + x_n^2 = ||\widehat{\mathbf{x}}||^2 \\ y_1^2 &= 1 + y_2^2 + y_3^2 + \cdots + y_n^2 = ||\widehat{\mathbf{y}}||^2, \end{aligned} \tag{5}$$

where the norm on the right is the Euclidean norm. So we have by the Cauchy-Schwartz inequality for the Euclidean norm, Lemma 2,

$$\langle \widehat{\mathbf{x}}, \widehat{\mathbf{y}} \rangle^2 \leq ||\widehat{\mathbf{x}}||^2 ||\widehat{\mathbf{y}}||^2 = (x_1 y_1)^2$$

i.e.,

$$|1 + x_2 y_2 + x_3 y_3 + \ldots x_n y_n| \leq |x_1 y_1| \tag{6}$$

so

$$1 + x_2 y_2 + x_3 y_3 + \ldots x_n y_n \leq |x_1 y_1| = x_1 y_1$$

where we have used that $x_1, y_1$ are positive, since $\mathbf{x}, \mathbf{y} \in \mathcal{H}^n$. Rearranged, we get

$$\langle \mathbf{x}, \mathbf{y} \rangle_L = -x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots + x_n y_n \leq -1,$$

which multiplying through by $-1$ gives the first inequality.
By the Cauchy-Schwartz inequality, we have

$$\begin{aligned} |x_2 y_2 + x_3 y_3 + \cdots + x_n y_n|^2 &\leq (x_2^2 + \cdots x_n^2)(y_2^2 + \cdots y_n^2) = (x_1^2 - 1)(y_1^2 - 1) \\ &= x_1^2 y_1^2 - y_1^2 - x_1^2 + 1 \\ &< x_1^2 y_1^2, \end{aligned}$$

so we have

$$x_1 y_1 > |x_2 y_2 + x_3 y_3 + \cdots + x_n y_n|$$

where we have used that $x_1 \geq 1$ and $y_1 \geq 1$, which follows from Equation 5. So

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= x_1 y_1 + (x_2 y_2 + x_3 y_3 + \ldots x_n y_n) \\ &\geq x_1 y_1 - |x_2 y_2 + x_3 y_3 + \ldots x_n y_n| \geq 0, \end{aligned}$$

which gives the second inequality. □

COROLLARY 7. *$d_{\mathcal{H}^n}$ is well defined.*

PROOF. This is immediate from Lemma 22 and the definition of $d_{\mathcal{H}^n}$. □

### 25.2. Terminology inspired by special relativity: Time, space, and light like vectors.

DEFINITION 28.

$$\textit{A point } \mathbf{x} \in \mathbb{R}^n \textit{ is } \begin{cases} \underline{\textit{space-like}} \textit{ if} & ||\mathbf{x}||_L \in (0, \infty), \ \textit{i.e., if } \langle \mathbf{x}, \mathbf{x} \rangle_L > 0 \\ \underline{\textit{light-like}} \textit{ if} & ||\mathbf{x}||_L = 0, \quad\ \ \textit{i.e., if } \langle \mathbf{x}, \mathbf{x} \rangle_L = 0 \\ \underline{\textit{time-like}} \textit{ if} & ||\mathbf{x}||_L \in i(0, \infty), \ \textit{i.e., if } \langle \mathbf{x}, \mathbf{x} \rangle_L < 0 \\ \underline{\textit{positive}} \textit{ if} & x_1 > 0 \\ \underline{\textit{negative}} \textit{ if} & x_1 < 0 \end{cases}$$

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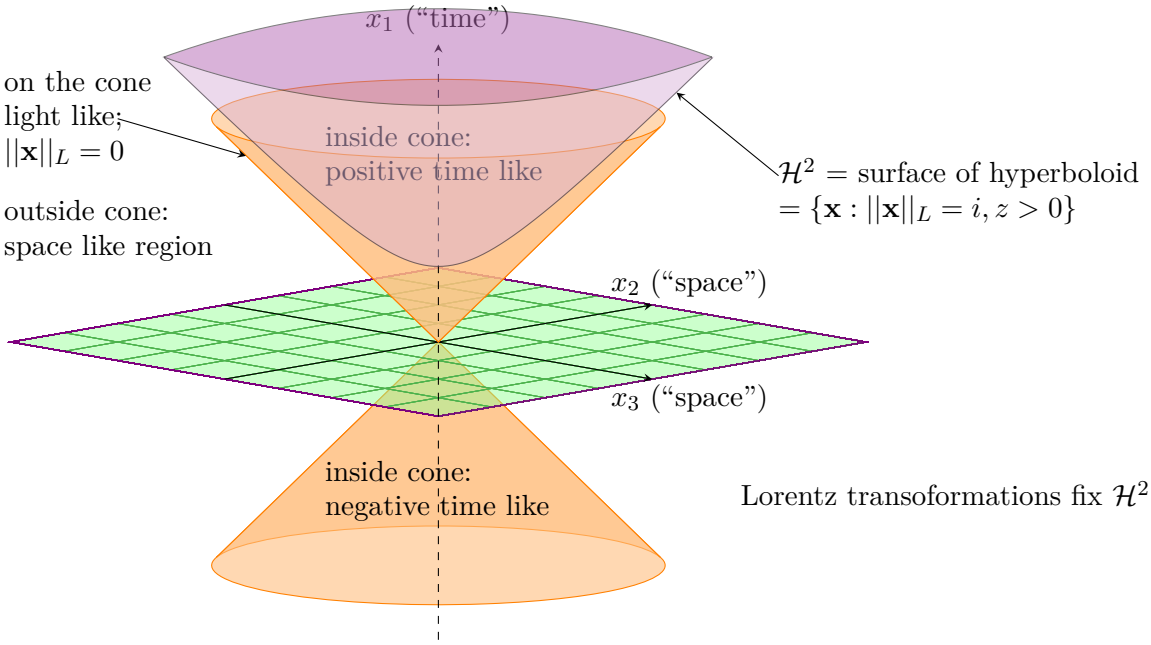
*A time-like or space-like vector $\mathbf{x} \in \mathbb{R}^n$ is positive if $x_1 > 0$ and negative if $x_1 < 0$.*

Note that vectors in $\mathcal{H}^n$ are all time-like.




DEFINITION 29. *A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transformation if it preserves the Lorentz inner product:*

$$\langle T(\mathbf{x}), T(\mathbf{y})\rangle_L = \langle \mathbf{x}, \mathbf{y}\rangle_L \; \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

*We say that $T$ is positive if $\mathbf{x}$ is positive time-like if and only if $T(\mathbf{x})$ is positive time-like.*

LEMMA 23. *If $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a positive, bijective Lorentz transformation, then $T|_{\mathcal{H}^n} \in \mathrm{Isom}(\mathcal{H}^n, d_{\mathcal{H}^n})$.*

PROOF. Since $\mathcal{H}^n = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{x}\rangle_L = -1$ and $\mathbf{x}$ is positive$\}$, $T$ preserves $\mathcal{H}^n$. Since $T$ is bijective, it has a bijective inverse, which is also a positive Lorentz transformation, and so $T|_{\mathcal{H}^n}$ is a bijection. Distance on $\mathcal{H}^n$ is given in terms of the Lorentz inner product, so $T$ is distance preserving. Hence $T|_{\mathcal{H}^n}$ is an isometry. □

Euclidean isometries correspond to orthogonal matrices. The Lorentzian analogy is given by:

DEFINITION 30. *An $n \times n$ real matrix $A$ is Lorentz orthogonal if*

$$A^T J A = J := \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & I_{n-1} \end{pmatrix}$$

DEFINITION 31. *The Lorentz group is the group of Lorentz orthogonal $n+1 \times n+1$ matrices, denoted*

$$O(1, n) = \{A \in M_{n+1\times n+1} | A^T J A = J\}$$

*The positive Lorentz group is the subgroup of $O(1, n)$ which maps positive time like vectors bijectively to positive time like vectors. This is denoted $O^+(1, n)$.*

LEMMA 24. *$O(1, n)$ and $O^+(1, n)$ are groups.*

PROOF. Exercise. □

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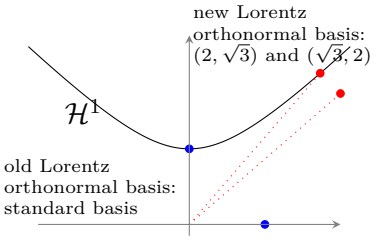
**Example:**

$$\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} \in O(1, 1)$$



We check this by showing that:

$$\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} = \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}\begin{pmatrix} -2 & -\sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

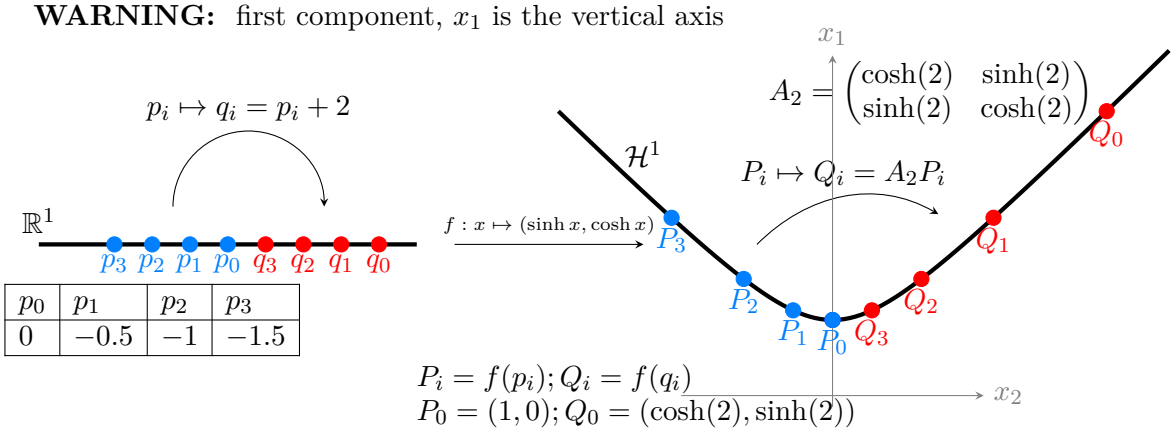
By Lemma 25 this means that the columns of this matrix form a Lorentz orthonormal basis for $\mathbb{R}^2$.

More generally for any $\gamma \in \mathbb{R}$, $A_\gamma = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix}$ defines an element of $O^+(1,1)$, which we can think of as inducing a translation on $\mathcal{H}^1$, since we have

$$\begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix}\begin{pmatrix} \cosh x \\ \sinh x \end{pmatrix} = \begin{pmatrix} \sinh\gamma\sinh x + \cosh\gamma\cosh x \\ \cosh\gamma\sinh x + \sinh\gamma\cosh x \end{pmatrix} = \begin{pmatrix} \cosh(x+\gamma) \\ \sinh(x+\gamma) \end{pmatrix}$$

The spaces $(\mathbb{E}^1, d_{\mathbb{E}^1})$ and $(\mathcal{H}^1, d_{\mathcal{H}^1})$ are isometric, via the paramterisation $x \mapsto (\sinh x, \cosh x)$.

**WARNING:** first component, $x_1$ is the vertical axis



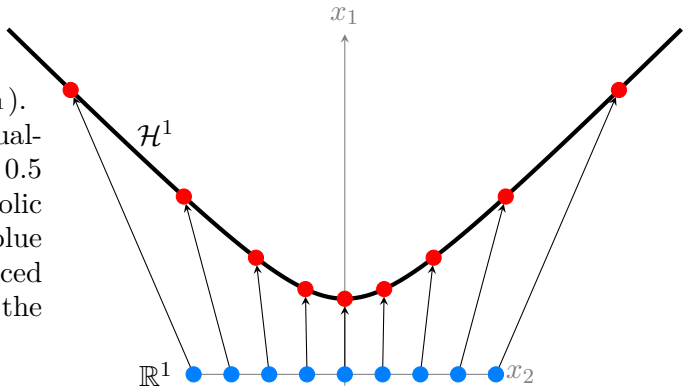

| $p_0$ | $p_1$ | $p_2$ | $p_3$ |
|---|---|---|---|
| 0 | $-0.5$ | $-1$ | $-1.5$ |

Since $\cosh(x) \in [1, \infty)$, the image of $A_\gamma$ applied to any element of $\mathcal{H}^1$ is in $\mathcal{H}^1$, and by linearity, $A_\gamma$ preserves all positive elements of $\mathbb{R}^2$, and so $A_\gamma \in O^+(1,n)$.

**Quiz:** What do you think a "reflection" matrix for $\mathcal{H}^1$ looks like?

Note that the parameterisation is not a projection to the $x_2$ axis:

illustration of the isometry from $(\mathbb{R}^1, d_{R^1})$ to $(\mathcal{H}^1, d_{\mathcal{H}^1})$. The red dots are equally spaced, distance 0.5 apart in the hyperbolic metric, and the blue dots are equally spaced distance 0.5 apart in the Euclidean metric.

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## 26. Lecture 15: { Lorentz Transformations } $= O(1, n)$

DEFINITION 32. *A set of vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ *is* Lorentz orthogonal *if* $\langle \mathbf{v}_i, \mathbf{v}_j \rangle_L = 0$ *for* $i \neq j$. *It is* Lorentz orthonormal *if*

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle_L = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } 2 \leq i = j \leq n \\ -1 & \text{if } i = j = 1 \end{cases}$$

LEMMA 25. (1) *The canonical basis* $\mathbf{e}_1, \ldots, \mathbf{e}_n$ *is Lorentz orthonormal.*
(2) *If* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *are Lorentz orthonormal, then they form a basis of* $\mathbb{R}^n$.
(3) $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *is a Lorentz orthonormal basis if and only if* $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = J_{i,j}$, *the element in the* $i^{th}$ *row,* $j^{th}$ *column of* $J$.

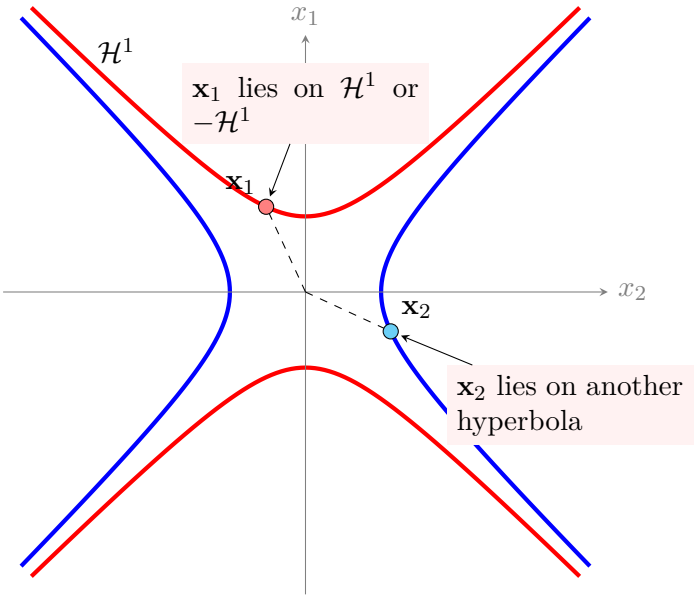
PROOF. Exercise. □

For the Euclidean norm, the orthonormal vectors were on a sphere. For the Lorentz metric, we have the first element of a Lorentz orthonormal basis is in $\mathcal{H}^n$ or in $-\mathcal{H}^n$. The other elements of a Lorentz orthonormal basis satisfy

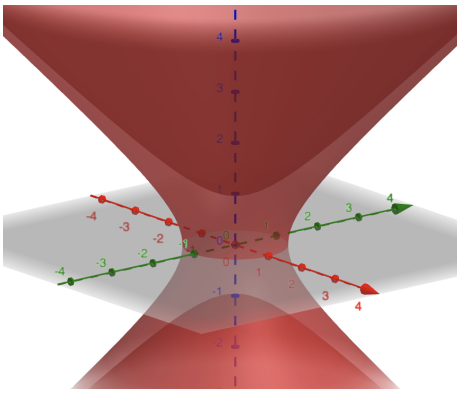
$$\langle \mathbf{x}, \mathbf{x} \rangle_L = 1 = -x_1^2 + x_2^2 + \cdots + x_n^2,$$

which means they lie on another hyperoloid, as in the figure:

Suppose $\mathbf{x}_1, \mathbf{x}_2$ is a Lorentz orthnormal basis for $\mathbb{R}^2$




surfaces where elements of a Lorentz orthonomal basis of $\mathbb{R}^3$ live:



**Quiz.** If $\mathbf{x}$ is any element of $\mathcal{H}^n$, can it be the first element of a Lorentz orthonormal basis for $\mathbb{R}^{n+1}$? If $\mathbf{x}$ is any element in the hyperboloid $1 = -x_1^2 + x_2^2 + \cdots + x_{n+1}^2$, can it be the second element in a Lorentz orthonormal basis for $\mathbb{R}^{n+1}$?
We will see that the answer is yes when we get to Lemma 30.

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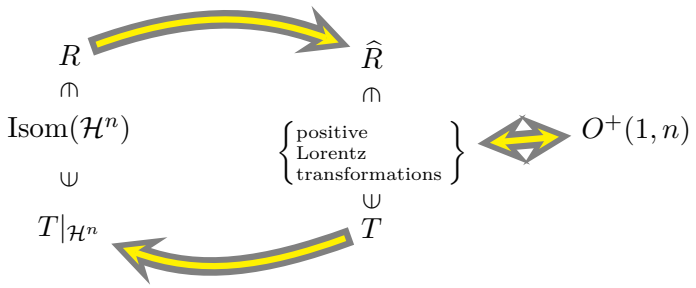
**Goal:**
Now we have defined the ingredients, here is what we are after: a concrete description of the isometries of hyperbolic space. We are following Klein's Erlangen program – we understand a geometric space by understanding the group of structure preserving transformations of that space.




**Plan:**

$T$ is a Lorentz transformation $\iff$ $T$ is linear and $T(\mathbf{e}_i)$ are Lorentz orthonormal $\iff$ $T \in O(1,n)$

$T|_{\mathcal{H}^n} \in \mathrm{Isom}(\mathcal{H}^n)$ $\impliedby$ $T$ is a positive Lorentz transformation $\iff$ $T \in O^+(1,n)$

LEMMA 26. *A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$, with matrix $A$ is a Lorentz transformation if and only if it maps the standard basis to a Lorentz orthonormal basis.*

PROOF. If $T$ is a Lorentz transformation, then by definition it preserves the inner product, and so the elements $T(\mathbf{e}_i)$ satisfy the requirement of Definition 32, and by Lemma 25 are a basis.
Conversely, suppose $T$ is a linear map, and $T(\mathbf{e}_i)$ is a Lorentz orthonormal basis. Then for any $\mathbf{x} = \sum x_i \mathbf{e}_i$ and $\mathbf{y} = \sum y_i \mathbf{e}_i$. Since $T$ is linear $T(\mathbf{x}) = \sum x_i T(\mathbf{e}_i)$ and $T(\mathbf{y}) = \sum y_i T(\mathbf{e}_i)$, so we have

$$\begin{aligned}\langle T(\mathbf{x}), T(\mathbf{y})\rangle_L = T(\mathbf{x})JT(\mathbf{y}) &= \left(\sum_{i=1}^n x_i T(\mathbf{e}_i)\right) J \left(\sum_{j=1}^n y_j T(e_j)\right) \\ &= \sum_{i=1}^n \sum_{j=1}^n x_i y_j T(\mathbf{e}_i) J T(\mathbf{e}_j) = \sum_{i=1}^n \sum_{j=1}^n x_i y_j \mathbf{e}_i J \mathbf{e}_j \\ \left(\sum_{i=1}^n x_i \mathbf{e}_i\right) J \left(\sum_{j=1}^n y_j \mathbf{e}_j\right) &= \langle \mathbf{x}, \mathbf{y}\rangle_L\end{aligned}$$

□

PROPOSITION 13. *Let $A$ be a real $n \times n$ matrix. The following are equivalent:*

(1) *The map $T(\mathbf{x}) := A\mathbf{x}$ is a Lorentz transformation.*
(2) *$||T(\mathbf{x})||_L = ||\mathbf{x}||_L$ for all $\mathbf{x} \in \mathbb{R}^n$.*
(3) *$A$ is Lorentz orthogonal*

PROOF. (1)⇒(2) Exercise.
(2)⇒(1) This follows from the following formula, similar to the polarisation identity (Lemma 7):

$$\langle \mathbf{x}, \mathbf{y}\rangle_L = \frac{1}{4}||\mathbf{x}+\mathbf{y}||_L^2 - \frac{1}{4}||\mathbf{x}-\mathbf{y}||_L^2.$$

To prove this it is useful to note that the Lorentz inner product is a bilinear map.

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(1) $\iff$ (3) Let $A$ have columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, so that $T(\mathbf{e}_i) = \mathbf{a}_i$. We have

$$A^T JA = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{pmatrix} J \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{pmatrix} \begin{pmatrix} J\mathbf{a}_1 & J\mathbf{a}_2 & \ldots & J\mathbf{a}_n \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{a}_1^T J\mathbf{a}_1 & \mathbf{a}_1^T J\mathbf{a}_2 & \ldots & \mathbf{a}_1^T J\mathbf{a}_n \\ \mathbf{a}_2^T J\mathbf{a}_1 & \mathbf{a}_2^T J\mathbf{a}_2 & \ldots & \mathbf{a}_2^T J\mathbf{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n^T J\mathbf{a}_1 & \mathbf{a}_n^T J\mathbf{a}_2 & \ldots & \mathbf{a}_n^T J\mathbf{a}_n \end{pmatrix} = \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle_L & \langle \mathbf{a}_1, \mathbf{a}_2 \rangle_L & \ldots & \langle \mathbf{a}_1, \mathbf{a}_n \rangle_L \\ \langle \mathbf{a}_2, \mathbf{a}_1 \rangle_L & \langle \mathbf{a}_2, \mathbf{a}_2 \rangle_L & \ldots & \langle \mathbf{a}_2, \mathbf{a}_n \rangle_L \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{a}_n, \mathbf{a}_1 \rangle_L & \langle \mathbf{a}_n, \mathbf{a}_2 \rangle_L & \ldots & \langle \mathbf{a}_n, \mathbf{a}_n \rangle_L \end{pmatrix}$$

So, we can see that $A^T JA = J \iff T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ are Lorentz orthonormal. By lemma 26, this is if and only if $T$ is a Lorentz transformation. □

PROPOSITION 14. *A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transformation if and only if*

(1) *the image of the canonical basis $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ is Lorentz orthonormal.*
(2) *$T$ is linear.*

PROOF. $\Rightarrow$ Suppose $T$ is a Lorentz transformation.

$$\langle T(\mathbf{e}_i), T(\mathbf{e}_j) \rangle_L = \langle \mathbf{e}_i, \mathbf{e}_j \rangle \text{ since } T \text{ is a Lorentz transformation}$$
$$= J_{i,j} \text{ by Lemma 25, which implies } T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n) \text{ is Lorentz orthonormal.}$$

Suppose $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{e}_i \in \mathbb{R}^n$, and $T(\mathbf{x}) = \sum_{i=1}^n b_i T(\mathbf{e}_i)$. By orthnormality of the $\mathbf{e}_i$ and the $T(\mathbf{e}_i)$,

$$-b_1 = \langle T(\mathbf{x}), T(\mathbf{e}_1) \rangle_L \qquad = \langle \mathbf{x}, \mathbf{e}_1 \rangle_L = -x_1$$
$$b_i = \langle T(\mathbf{x}), T(\mathbf{e}_i) \rangle_L \qquad = \langle \mathbf{x}, \mathbf{e}_i \rangle_L = x_i \text{ for } 2 \leq i \leq n$$

Thus $b_i = x_i$ for $i = 1, \ldots, n$ so $T$ is linear.
$\Leftarrow$: The converse is Lemma 26. □

COROLLARY 8. *A Lorentz transformation is bijective.*

PROOF. This is because a Lorentz transformation is linear and maps a basis to a basis. □

COROLLARY 9. *A Lorentz transformation corresponds to an element of $O(1,n)$, and every element of $O(1,n)$ gives rise to a Lorentz transformation. A positive Lorentz transformation corresponds to an element of $O^+(1,n)$, and every element of $O^+(1,n)$ gives rise to a positive Lorentz transformation.*

PROOF. Proposition 14 says that all Lorentz tranformations are linear, and Proposition 13 says that all linear Lorentz tranformations are given by matrices in $O(1,n)$. The statement about positive Lorentz transformations follows from their definition. □

Let's characterise elements of $O^+(1,n)$.

LEMMA 27. *If $A \in O(1,n)$ then $A^T \in O(1,n)$.*

PROOF. $A \in O(1,n) \Rightarrow A^T JA = J \Rightarrow JA^T JA = I \Rightarrow (JA^T J)^{-1} = A \Rightarrow A(JA^T J) = I \Rightarrow AJA^T J = I \Rightarrow AJA^T = J \Rightarrow A^T \in O(1,n)$. □

PROPOSITION 15. *For $A = (a_{i,j}) \in O(1,n)$, we have $A \in O^+(1,n) \iff a_{1,1} > 0$.*

PROOF. Let $A$ have columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$.
$\Rightarrow$: If $A \in O^+(1,n)$, then $T(\mathbf{e}_1) = \mathbf{a}_1 = (a_{1,1}, a_{2,1}, \ldots, a_{n,1}) \in \mathcal{H}^n$, so $a_{1,1} > 0$.
$\Leftarrow$: Denote the first column of $A^T$ by $\mathbf{a}_1'$. By Lemma 27, $A^T \in O(1,n)$, and so by definition of $O(1,n)$, $\mathbf{a}'$ is in $\mathcal{H}^n$ or in $-\mathcal{H}^n$. Since $a_{1,1} > 0$, we have $\mathbf{a}' \in \mathcal{H}^n$. The first element of the vector $A\mathbf{x}$ is $\langle \mathbf{a}_1', \mathbf{x} \rangle = \beta \langle \mathbf{a}_1', \mathbf{x}/\beta \rangle$, where this is the usual inner product, and $\beta = -i||\mathbf{x}||_L > 0$. Since $\mathbf{x}/\beta \in \mathcal{H}^n$, by Lemma 22 $\langle \mathbf{a}_1', \mathbf{x}/\beta \rangle$ is positive, so $A\mathbf{x}$ is also positive, so we are done. □

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lecture 16
Hyperbolic isometries

## 27. Lecture 16: $\mathrm{Isom}(\mathcal{H}^n, d_{\mathcal{H}^n}) \cong O^+(1,n)$

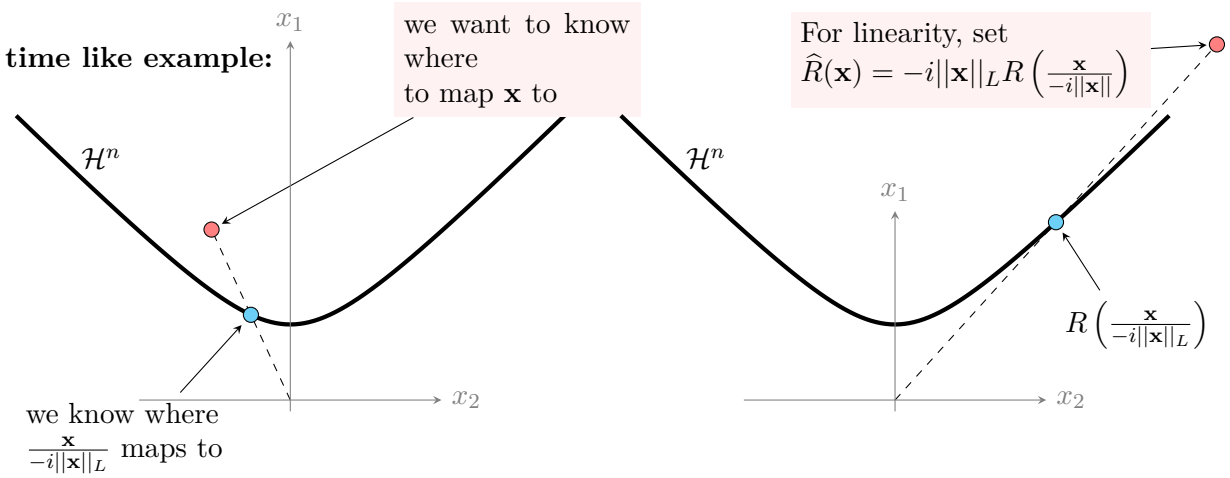
We have seen how positive Lorentz transformations restrict to isometries of $\mathcal{H}^n$. Now we show all Hyperbolic isometries can be obtained in this way.

LEMMA 28. *There is a basis of $\mathbb{R}^{n+1}$ consisting of time-like vectors, which may be taken to be elements of $\mathcal{H}^n$.*

PROOF. Take the basis $\mathbf{e}_1$ together with $e_i + \sqrt{2}\mathbf{e}_1$ for $i = 2, \dots n+1$. □

PROPOSITION 16. *Let $R \in \mathrm{Isom}(\mathcal{H}^n, d_{\mathcal{H}^n})$. Then there is a unique $A \in O^+(1,n)$ with $R = T_A|_{\mathcal{H}^n}$, where $T_A$ is the linear map on $\mathbf{R}^{n+1}$ given by $\mathbf{x} \mapsto A\mathbf{x}$.*

PROOF. Given $R \in \mathrm{Isom}(\mathcal{H}^n, d_{\mathcal{H}^n})$, we will define a Lorentz transformation $\widehat{R}$ with $\widehat{R}|_{\mathcal{H}^n} = R$. For a time-like vector, we can use radial extension, as in Proposition 11 for the sphere:




Positive time-like definition: For any $\mathbf{x} \in \mathbf{R}^{n+1}$, if $\mathbf{x}$ is a positive time like vector, then $-\langle \mathbf{x}, \mathbf{x} \rangle_L > 0$, so it has a postive real square root, equal to $-i||\mathbf{x}||_L$, and so $\mathbf{x}/(-i||\mathbf{x}||_L) \in \mathbb{R}^{n+1}$, and is positive. By bilinearity,

$$\left\langle \frac{\mathbf{x}}{-i||\mathbf{x}||_L}, \frac{\mathbf{x}}{-i||\mathbf{x}||_L} \right\rangle_L = \frac{\langle \mathbf{x}, \mathbf{x} \rangle_L}{-||\mathbf{x}||_L^2} = -1$$

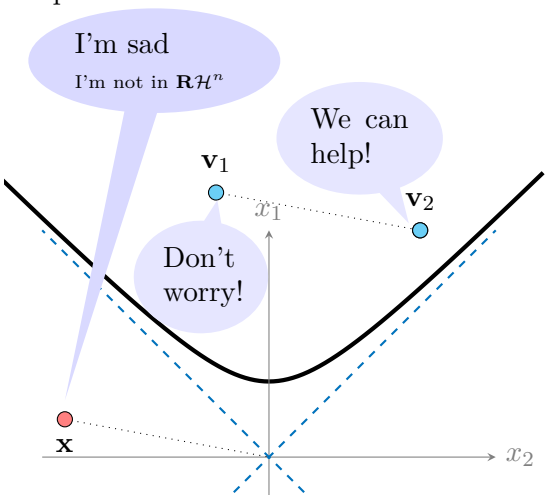
So $\mathbf{x}' := \mathbf{x}/(-i||\mathbf{x}||_L) \in \mathcal{H}^n$, and we can define $\widehat{R}(\mathbf{x})$ in terms of $R(\mathbf{x}')$ by:

$$\widehat{R}(\mathbf{x}) = -i||\mathbf{x}||_L R\left(\frac{\mathbf{x}}{-i||\mathbf{x}||_L}\right) \text{ if } \mathbf{x} \text{ is postive time like} \tag{7}$$

Since $R$ maps $\mathcal{H}^n$ to $\mathcal{H}^n$ and $-i||\mathbf{x}||_L \in \mathbb{R}_{\geq 0}$, this is positive time-like.
Exactly the same argument works for negative time like vectors, when $-i||\mathbf{x}||_L$ is now negative.
Non time-like definition: Suppose $\mathbf{x}$ is not time like. By Lemma 28, there is a basis of time-like vectors $\mathbf{v}_1, \dots, \mathbf{v}_{n+1}$, contained in $\mathcal{H}^n$ and $\alpha_1, \dots, \alpha_{n+1} \in \mathbb{R}$ with $\mathbf{x} = \alpha_1\mathbf{v}_1 + \cdots \alpha_n\mathbf{v}_{n+1}$, so we can set

$$\widehat{R}(\mathbf{x}) = \alpha_1\widehat{R}(\mathbf{v}_1) + \cdots + \alpha_{n+1}\widehat{R}(\mathbf{v}_{n+1}).$$

As long as $\mathbf{v}_1, \dots, \mathbf{v}_{n+1}$ is chosen to be a fixed basis, then the $\alpha_i$ are uniquely determnined, so this is well defined.




Since $R$ is an isometry on $\mathcal{H}^n$, it preserves $cosh^{-1}\langle \mathbf{x}, \mathbf{y} \rangle_L$, and so also preserves the Lorentz inner product of vectors in $\mathcal{H}^n$. We want to show $\widehat{R}$ preserves the inner product of all vectors in $\mathbb{R}^{n+1}$.

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If $\mathbf{x}$ and $\mathbf{y}$ are both time-like, checking $\left\langle \widehat{R}\mathbf{x}, \widehat{R}\mathbf{y} \right\rangle_L = \langle \mathbf{x}, \mathbf{y} \rangle_L$ is very similar to the method in Proposition 11, but we need to replace the Euclidean inner product with the Lorentz inner product. Details are left as an exercise.

Suppose neither $\mathbf{x}$ nor $\mathbf{y}$ is time-like, and $\mathbf{x} = \sum \alpha_i \mathbf{v}_i$ and $\mathbf{y} = \sum \beta_i \mathbf{v}_i$. Then using bilinearity, and that $\widehat{R}|_{\mathcal{H}^n} = R$ preserves the inner product on $\mathcal{H}^n$, we have

$$\begin{aligned}\left\langle \widehat{R}(\mathbf{x}), \widehat{R}(\mathbf{y}) \right\rangle_L &= \left\langle \sum_i \alpha_i \widehat{R}(\mathbf{v}_i), \sum_j \beta_j \widehat{R}(\mathbf{v}_j) \right\rangle_L = \sum_i \sum_j \alpha_i \beta_j \left\langle \widehat{R}(\mathbf{v}_i), \widehat{R}(\mathbf{v}_j) \right\rangle_L \\ &= \sum_i \sum_j \alpha_i \beta_j \langle R(\mathbf{v}_i), R(\mathbf{v}_j) \rangle_L = \sum_i \sum_j \alpha_i \beta_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle_L = \langle \mathbf{x}, \mathbf{y} \rangle_L .\end{aligned}$$

We need to check that the inner product is preserved if $\mathbf{x}$ is time like and $\mathbf{y}$ is not time-like. Suppose $\mathbf{x}$ is time like, and $\mathbf{y} = \sum_{i=1}^{n+1} a_i \mathbf{v}_i$. Then using linearity, and that $R$ preserves the Lorentz inner product on $\mathcal{H}^n$, we have

$$\begin{aligned}\left\langle \widehat{R}(\mathbf{x}), \widehat{R}(\mathbf{y}) \right\rangle_L &= \left\langle -i||\mathbf{x}||_L R\left(\frac{\mathbf{x}}{-i||\mathbf{x}||_L}\right), \sum_i \alpha_i R(\mathbf{v}_i) \right\rangle_L = \sum_i -i||\mathbf{x}||_L \alpha_i \left\langle R\left(\frac{\mathbf{x}}{-i||\mathbf{x}||_L}\right), R(\mathbf{v}_i) \right\rangle_L \\ &= \sum_i -i||\mathbf{x}||_L \alpha_i \left\langle \frac{\mathbf{x}}{-i||\mathbf{x}||_L}, \mathbf{v}_i \right\rangle_L = \left\langle \frac{-i||\mathbf{x}||_L \mathbf{x}}{-i||\mathbf{x}||_L}, \sum_i \alpha_i \mathbf{v}_i \right\rangle_L = \langle \mathbf{x}, \mathbf{y} \rangle_L .\end{aligned}$$

So we have shown that the Lorentz inner product is preserved for all vectors in $\mathbb{R}^{n+1}$, By Corollary 9 $\widehat{R}$ is a linear map with corresponding matrix $A \in O(1, n)$, and $\widehat{R} = T_A$.

By construction, $\widehat{R}$ maps positive time-like vectors to positive time-like vectors, so by Corollary 9 $\widehat{R}$ is a positive Lorentz transformation, so $A \in O^+(1, n)$.

By construction in Equation 7, we have $\widehat{R}(\mathbf{x}) = R(\mathbf{x})$ for $\mathbf{x} \in \mathcal{H}^n$, since in that case $-i||\mathbf{x}||_L = 1$. So $T_A|_{\mathcal{H}^n} = R$ as required.

Uniqueness follows since there is a basis of vectors for $\mathbb{R}^{n+1}$ contained in $\mathcal{H}^n$, so since $R$ is given on these values, and a linear map is determined by its action on a basis, $\widehat{R}$, and thus $A$, is unique. □

Note, I gave an alternative proof of above in class, see next page

COROLLARY 10. *There is a group isomorphism* $\mathrm{Isom}(\mathcal{H}^n) \cong O^+(1, n)$

PROOF. The maps of this isomorphism are given by

$$\begin{aligned}\phi : O^+(1, n) &\to \mathrm{Isom}(\mathcal{H}^n) \\ A &\to T_A|_{\mathcal{H}^n}\end{aligned}$$

$$\begin{aligned}\psi : \mathrm{Isom}(\mathcal{H}^n) &\to O^+(1, n) \\ R &\to A \text{ such that } T_A = \widehat{R}\end{aligned}$$

The existance of these maps comes from Corollary 9 and Proposition 16. To check that these maps are inverse to each other, and that $\phi$ is a group isomorphism is left as an exercise. The proof is similar to the proof of Corollary 5, the result for the sphere. A key point is to take a basis of $\mathbf{R}^n$ consisting of elements of $\mathcal{H}^n$, since linear maps are determined by the images of a basis. □

## 28. Lecture 16/17

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PROOF. Alternative Proof of Proposition 16

Woffle: In the Tuesday lecture, I decided to switch to an alternative proof, closer to the one given in [RS, 3.11]. The proof given above defines $\widehat{R}$ on all of $\mathbb{R}^{n+1}$, and shows it is a Lorentz transformation, and so is linear by Corollary 9. I.e., we extend first, then show it is linear.
An alternative method is to first show the map is linear on $\mathcal{H}^n$, and then extend, just by using the same matrix that works on $\mathcal{H}^n$ to define $\widehat{R}$ on all of $\mathbb{R}^{n+1}$. I.e., show it's linear first, then extend.
Both proofs are equally valid, so you can choose your favourite.

For this version of the proof, take the basis for $\mathbb{R}^{n+1}$ given in Lemma 28: $\mathbf{v}_1 = \mathbf{e}_1$ and for $2 \leq i \leq n+1$:

$$\mathbf{v}_i = \mathbf{e}_i + \sqrt{2}\mathbf{e}_1$$

Motivation: My goal is to show that $R$ is a linear map, which means that if $\mathbf{x} = \sum a_i \mathbf{v}_i$ then $R(\mathbf{x}) = \sum a_i R(\mathbf{v}_i)$, but this can't be possible unless the $R(\mathbf{v}_i)$ also form a basis for $\mathbb{R}^{n+1}$. So I want to show that $R(\mathbf{v}_i)$ is also a basis – how to do this? I know that a set of Lorentz orthonormal vectors is a basis, but these vectors are not Lorentz orthonormal, so I look for a different set which is Lorentz orthonomal.

Claim: Define a set of vectors

$$\mathcal{B} = \{R(\mathbf{v}_1) = R(\mathbf{e}_1)\} \cup \{R(\mathbf{v}_i) - \sqrt{2}R(\mathbf{e}_1) = R(\mathbf{e}_i + \sqrt{2}\mathbf{e}_1) - \sqrt{2}R(\mathbf{e}_1) \ : \ i = 2, \ldots, n+1\}$$

We claim that $\mathcal{B}$ is a Lorentz orthnormal set in $\mathbb{R}^n$.

Motivation: I would like to take $R(\mathbf{e}_i)$, but $R(\mathbf{e}_i)$ is not defined. But it was defined, and if $R$ was linear, then it would be equal to this, so let's just take this, and hope it all works out – it will turn out that it does. The reason we take this form is that $R$ is defined for all the terms it's applied to here. $R$ is not aprioi defined for general elements of $\mathbb{R}^{n+1}$ [Analogy (ignore this sentence if you want): in computing you might define a function for integers, and, if the computer program checks the type of input, it will break if you try and apply it to a decimal number that's not an integer, but it could well be the function can be extended, but you have to do some work to get to the extension. Another example: you might define a function on $(-1, 1)$ by $f(x) = \lim_{n\to\infty}(2\sin^{-1}(x)/\pi)^n)$. Apriori, this function is only defined on $(-1, 1)$, but if you compute it, you find out that you can extend it to all of $\mathbb{R}$ just by defining $\widehat{f}(x) = 0$ on $\mathbb{R}$.]

We can check the claim by just taking the Lorentz inner products and using that this is preserved by $R$. For $2 \leq i, j \leq n+1$, we have the following, where the computation works by using bilinarity of the Lorentz inner product, for the first equality, the fact that $R$ preserves the Lorentz inner product for the second equality, bilinearity again for the third equality, and the definition of $\mathbf{v}_i$ for the fourth equality.

$$\begin{aligned}
\left\langle R(\mathbf{v}_i) - \sqrt{2}R(\mathbf{e}_1), R(\mathbf{v}_j) - \sqrt{2}R(\mathbf{e}_1) \right\rangle_L &= \langle R(\mathbf{v}_i), R(\mathbf{v}_j)\rangle_L - \sqrt{2}\langle R(\mathbf{e}_1), R(\mathbf{v}_j)\rangle_L \\
&\quad - \sqrt{2}\langle R(\mathbf{e}_1), R(\mathbf{v}_i)\rangle_L + 2\langle R(\mathbf{e}_1), R(\mathbf{e}_1)\rangle_L \\
&= \langle \mathbf{v}_i, \mathbf{v}_j\rangle_L - \sqrt{2}\langle \mathbf{e}_1, \mathbf{v}_j\rangle_L \\
&\quad - \sqrt{2}\langle \mathbf{e}_1, \mathbf{v}_i\rangle_L + 2\langle \mathbf{e}_1, \mathbf{e}_1\rangle_L \\
&= \left\langle \mathbf{v}_i - \sqrt{2}\mathbf{e}_1, \mathbf{v}_j - \sqrt{2}\mathbf{e}_1 \right\rangle_L \\
&= \langle \mathbf{e}_i, \mathbf{e}_j\rangle_L = J_{i,j}.
\end{aligned}$$

(Details for $i = 1$ given in class; left as an exercise here.) By Lemma 25, this means that $\mathcal{B}$ is a Lorentz orthonormal basis. By Lemma 25 this means that $\mathcal{B}$ is a basis for $\mathbb{R}^{n+1}$, which means that the set $R(\mathbf{v}_i)$, for $i = 1, \ldots, n+1$ is a also basis for $\mathbb{R}^{n+1}$, which means that there are some $\alpha_i$ such that for any

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$\mathbf{x} \in \mathcal{H}^n$,

$$R(\mathbf{x}) = \sum \alpha_i R(\mathbf{v}_i)$$
$$(\star) \qquad = (a_1 + \sqrt{2}(\alpha_2 + \alpha_3 + \cdots + \alpha_{n+1}))R(\mathbf{e}_1) + \sum_{i=2}^{n+1} \alpha_i (R(\mathbf{v}_i) - \sqrt{2}R(\mathbf{e}_1))$$

On the other hand, since the vectors $\mathbf{v}_i$ form a basis, we have that for some $\beta_i$,

$$\mathbf{x} = \sum \beta_i \mathbf{v}_i$$
$$(\dagger) \qquad = (\beta_1 + \sqrt{2}(\beta_2 + \cdots + \beta_{n+1}))\mathbf{e}_1 + \sum_{i=2}^{n+1} \beta_i (\mathbf{v}_i - \sqrt{2}\mathbf{e}_i).$$

In order to show that $R$ is a linear map on $\mathcal{H}^n$, we want to show that $\beta_i = \alpha_i$.
Because $\mathcal{B}$ is Lorentz orthonomal, taking the inner product of them with $(\star)$ we have

$$(\bigstar) \qquad \begin{aligned} \langle R(\mathbf{x}), R(\mathbf{e}_1)\rangle_L &= -(\alpha_1 + \sqrt{2}(\alpha_2 + \alpha_3 + \cdots + \alpha_{n+1})) \\ \left\langle R(\mathbf{x}), R(\mathbf{e}_i + \sqrt{2}\mathbf{e}_1) - \sqrt{2}R(\mathbf{e}_1)\right\rangle_L &= \alpha_i \text{ for } i = 2, \ldots, n+1 \end{aligned}$$

Since $\mathbf{e}_i$ are Lorentz orthonomal, taking the inner products with $(\dagger)$ we have

$$(\ddagger) \qquad \begin{aligned} \langle \mathbf{x}, \mathbf{e}_1\rangle_L &= -(\beta_1 + \sqrt{2}(\beta_2 + \cdots + \beta_{n+1})) \\ \langle \mathbf{x}, \mathbf{e}_i\rangle_L &= \beta_i \text{ for } i = 2, \ldots, n+1 \end{aligned}$$

Since $R$ preserves the Lorentz inner product on $\mathcal{H}^n$, we have

$$\begin{aligned} \langle R(\mathbf{x}), R(\mathbf{e}_1)\rangle_L &= \langle \mathbf{x}, \mathbf{e}_1\rangle_L \\ \left\langle R(\mathbf{x}), R(\mathbf{e}_i + \sqrt{2}\mathbf{e}_1) - \sqrt{2}R(\mathbf{e}_1)\right\rangle_L &= \left\langle R(\mathbf{x}), R(\mathbf{e}_i + \sqrt{2}\mathbf{e}_1)\right\rangle_L - \sqrt{2}\langle R(\mathbf{x}), R(\mathbf{e}_1)\rangle_L \text{ (bilinearity)} \\ &= \left\langle \mathbf{x}, \mathbf{e}_i + \sqrt{2}\mathbf{e}_1\right\rangle_L - \sqrt{2}\langle \mathbf{x}, \mathbf{e}_1\rangle_L \text{ (preservation of } \langle\cdot\rangle_L) \\ &= \langle \mathbf{x}, \mathbf{e}_i\rangle_L \text{ (bilinearity)} \end{aligned}$$

So the expressions in $(\bigstar)$ are equal to those in $(\ddagger)$, and so $\alpha_i = \beta_i$ for $i = 1, \ldots, n+1$.
So $R$ is a linear map on $\mathcal{H}^n$, which means that it can be described, with respect to the basis $\mathbf{v}_i$, by a matrix $A$, the columns of which are given by the $R(\mathbf{v}_i)$ written in terms of the $\mathbf{v}_i$.
Now we have a linear map, we can extend to a linear map defined on all of $\mathbb{R}^{n+1}$, given by the matrix $A$, now applied to all vectors. Call this map $\widehat{R}$. Since $\widehat{R}$, and extends $R$, we have that the elements of $\mathcal{B}$ are:

$$R(\mathbf{e}_i + \sqrt{2}\mathbf{e}_1) - \sqrt{2}R(\mathbf{e}_1) = \widehat{R}(\mathbf{e}_i + \sqrt{2}\mathbf{e}_1) - \sqrt{2}\widehat{R}(\mathbf{e}_1) = \widehat{R}(\mathbf{e}_i).$$

So we have shown that the Lorentz orthonomal basis $\mathbf{e}_i$ maps to another Lorentz orthonomal basis, $\mathcal{B}$.
So by Proposition 14, $\widehat{R}$ is a Lorentz transformation, and so given by a matrix in $O(1, n)$. Since $R$ maps $\mathcal{H}^n$ to $\mathcal{H}^n$, positive time like vectors map to positive time like vectors, so $A \in O^+(1, n)$.
So, we have extended $R$ to a positive Lorentz transformation $\widehat{R}$, given by an element of $O^+(1, n)$ as required. □

Moral: Orthonormal basis are cool because you can easily find coefficients of vectors in terms of these basis by taking inner procucts with the basis elements.

LEMMA 29. *If* $\mathbf{w}$ *is Lorentz orthongonal to a time-like vector, then* $\mathbf{w}$ *is space like. In other words: If* $\mathbf{x} \in \mathbb{R}^n$ *with* $\langle \mathbf{x}, \mathbf{x}\rangle_L < 0$*, and* $\mathbf{w} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *with* $\langle \mathbf{x}, \mathbf{w}\rangle_L = 0$*, then* $\langle \mathbf{w}, \mathbf{w}\rangle_L > 0$.

PROOF. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. Let $\mathbf{x}' = \mathbf{x} - x_1\mathbf{e}_1$ and $\mathbf{w}' = \mathbf{w} - w_1\mathbf{e}_1$. Since $\langle \mathbf{x}', \mathbf{x}'\rangle_L \geq 0$ and $\langle \mathbf{x}, \mathbf{x}\rangle_L = -x_1^2 + \langle \mathbf{x}', \mathbf{x}'\rangle_L$, we must have $x_1 \neq 0$. If $w_1 = 0$, then since $\mathbf{w} \neq 0$, we have $\langle \mathbf{w}, \mathbf{w}\rangle_L = \langle \mathbf{w}', \mathbf{w}'\rangle_L > 0$, and we are done. So we may assume $w_1 \neq 0$. The vector $x_1\mathbf{w} - w_1\mathbf{x}$ has zero $\mathbf{e}_1$ component. This vector can not be zero, or $x_1\mathbf{w} = w_1\mathbf{x}$, but then $0 > \langle \mathbf{x}, \mathbf{x}\rangle_L = \left\langle \mathbf{x}, \frac{x_1}{w_1}\mathbf{w}\right\rangle_L =$

$\frac{x_1}{w_1}\langle \mathbf{x}, \mathbf{w}\rangle_L = 0$, a contradiction. Now

$$0 \leq ||x_1\mathbf{w} - w_1\mathbf{x}||_L^2 = \langle x_1\mathbf{w} - w_1\mathbf{x}, x_1\mathbf{w} - w_1\mathbf{x}\rangle_L = x_1^2\langle \mathbf{w}, \mathbf{w}\rangle_L + w_1^2\langle \mathbf{x}, \mathbf{x}\rangle_L$$

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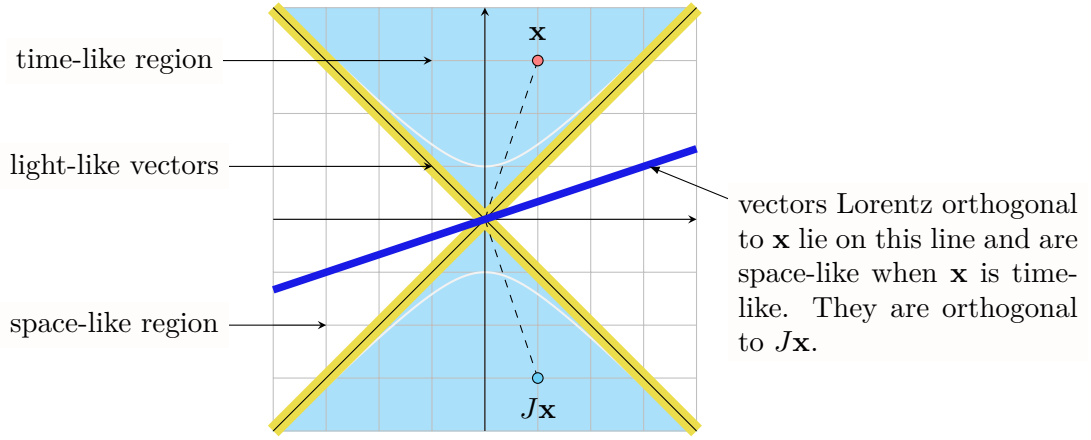
so

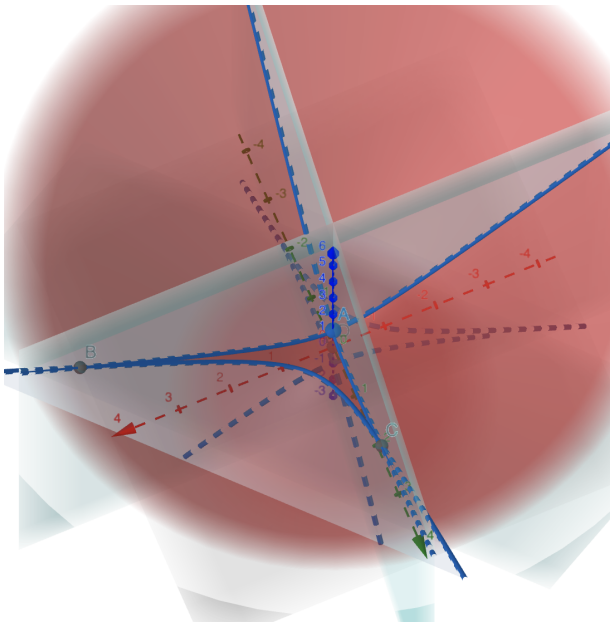
$$\langle \mathbf{w}, \mathbf{w}\rangle_L \geq -\frac{w_1^2}{x_1^2}\langle \mathbf{x}, \mathbf{x}\rangle_L > 0.$$

□

**Examples.** In $\mathcal{H}^1$, if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{w} = (w_1, w_2)$, then $\langle \mathbf{x}, \mathbf{w}, \rangle_L = 0$ means $x_1w_2 = x_2w_2$. For fixed $\mathbf{x}$ this gives an equation for a line through the origin on which $\mathbf{w}$ must lie.




For getting some intuition about hyperbolic space, play around with images in Geogebra, e.g.:



from `https://www.geogebra.org/3d/np9aseqx`

## 29. Lecture 18: Translation to nice coordinates

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LEMMA 30. *Given any linearly independent* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{H}^n$*, there is an element* $A \in O^+(1,n)$ *with*

$$A\mathbf{a} = (1,0,0,\ldots,0) = \mathbf{e}_1$$
$$A\mathbf{b} = (b_1, b_2, 0, \ldots, 0) \text{ with } b_1 > 0$$
$$A\mathbf{c} = (c_1, c_2, c_3, 0, \ldots, 0) \text{ with } c_1 > 0$$

PROOF. Let $\mathbf{z}_1 = \mathbf{a}, \mathbf{z}_2 = \mathbf{b}, \mathbf{z}_3 = \mathbf{c}, \mathbf{z}_4, \ldots, \mathbf{z}_{n+1}$ be a basis for $\mathcal{H}^n$, which is possible since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent, and from a result in linear algebra, can be extended to a basis.
We will construct a Lorentz orthonormal basis of $\mathbb{R}^{n+1}$, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n+1}$. Set

$$\mathbf{v}_1 = \mathbf{a}.$$

Since $||\mathbf{a}|| = i$, we have $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle_L = -1$, so this is a good first element of a Lorentz orthonomal basis. We proceed by induction, constructing each $\mathbf{v}_i$ to be in the span of $\mathbf{z}_1, \ldots, \mathbf{z}_i$. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_k$ has been constructed and is a Lorentz orthonomal set, then we set

$$\mathbf{w}_{k+1} = \mathbf{z}_{k+1} + \langle \mathbf{z}_{k+1}, \mathbf{v}_1 \rangle_L \mathbf{v}_1 - \langle \mathbf{z}_{k+1}, \mathbf{v}_2 \rangle_L \mathbf{v}_2 - \langle \mathbf{z}_{k+1}, \mathbf{v}_3 \rangle_L \mathbf{v}_3 - \cdots - \langle \mathbf{z}_{k+1}, \mathbf{v}_k \rangle_L \mathbf{v}_k.$$

Then $\mathbf{w}_{k+1}$ is orthogonal to $\mathbf{v}_1, \ldots, \mathbf{v}_k$, and so has positive norm by Lemma 29, so we can set

$$\mathbf{v}_{k+1} = \frac{\mathbf{w}_{k+1}}{||\mathbf{w}_{k+1}||_L}.$$

In this way, a Lorentz orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n+1}$ is constructed. With respect to this basis,

$$\begin{array}{lll}
\mathbf{a} & = \mathbf{v}_1 & = (\quad 1, \quad 0, \quad 0, \quad 0, \quad \ldots, 0) \\
\mathbf{b} & = -\langle \mathbf{v}_1, \mathbf{b} \rangle_L \mathbf{v}_1 + ||\mathbf{w}_2||_L \mathbf{v}_2 & = (\quad -\langle \mathbf{v}_1, \mathbf{b} \rangle_L, \quad ||\mathbf{w}_2||_L, \quad 0, \quad 0, \quad \ldots, 0) \\
\mathbf{c} & = -\langle \mathbf{v}_1, \mathbf{c} \rangle_L \mathbf{v}_1 + \langle \mathbf{v}_2, \mathbf{z} \rangle_L \mathbf{v}_2 + ||\mathbf{w}_3||_L \mathbf{v}_3 & = (\quad -\langle \mathbf{v}_1, \mathbf{c} \rangle_L, \quad \langle \mathbf{v}_2, \mathbf{c} \rangle_L, \quad ||\mathbf{w}_3||_L, \quad 0, \quad \ldots, 0),
\end{array}$$

which have the required form. The change of basis matrix $A^{-1}$ has Lorentz orthonormal columns $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$, and $\mathbf{v}_1 = \mathbf{a}$ is positive, so $A \in O^+(1,n)$, so we are done. □

These coordinates are used in `https://www.geogebra.org/3d/np9aseqx` previous page.
In the case $n = 2$, we have:

LEMMA 31. *There is a parameterization*

$$f : \mathbb{R}^2 \to \mathcal{H}^2$$
$$f : (t, \theta) \mapsto (\cosh(t), \cos(\theta)\sinh(t), \sin(\theta)\sinh(t)), \quad t \in [0, \infty), \theta \in [0, 2\pi),$$

*where* $\mathbb{R}^2$ *is given with polar coordinates, and* $\mathcal{H}^2$ *with the Cartesian coordinates of* $\mathbb{R}^3$*.*

PROOF. This is the inverse of the projection from $\mathbb{R}^3$ given by $(x_1, x_2, x_3) \to (x_2, x_3)$. This is a bijection between $\mathcal{H}^2$ and $\mathbb{R}^2$, since for any $(x_2, x_3)$ there is a unique solution to

$$(\star) \qquad -x_1^2 + x_2^2 + x_3^2 = -1 \text{ and } x_1 > 0.$$

In polar coordinates,

$$(x_2, x_3) = (r\cos(\theta), r\sin(\theta)),$$

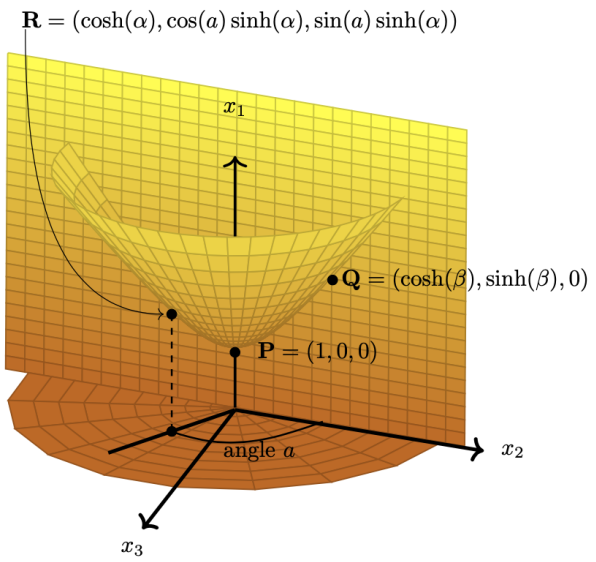
so $(\star)$ becomes

$$-x_1^2 + r^2 = -1, \quad x_1 > 0,$$

the equation for $\mathcal{H}^1$, parametrised by

$$(x_1, r) = (\cosh(t), \sinh(t)),$$

from which the result follows by substitution. □



## 30. Lecture 19: Hyperbolic lines

We need to prove the triangle inequality, and discuss lines and triangles. It will be useful to introduce some terminology about hyperbolic lines.

DEFINITION 33. *A 2-dimensional sub-vector space $V$ of $\mathbb{R}^{n+1}$ is called a Lorentz plane if it contains a timelike vector. A hyperbolic line is the intersection of $\mathcal{H}^n$ with any Lorentz plane.*

Hyperbolic line segments are the shortest distances between two points in the hyperbolic metric. (We do not have time to prove this.)
Examples are shown in the following Geogebra figures: line; triangle; parallel lines. [4]

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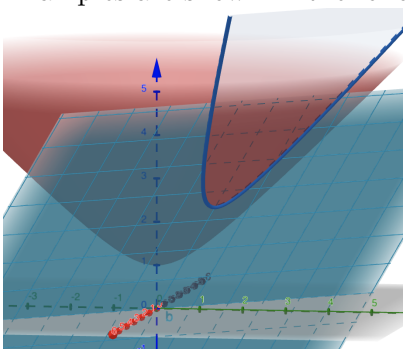
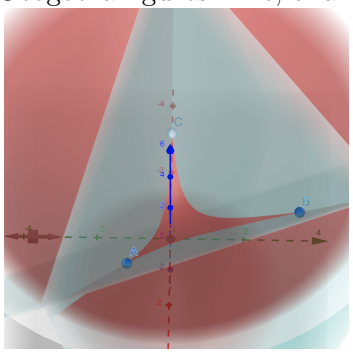
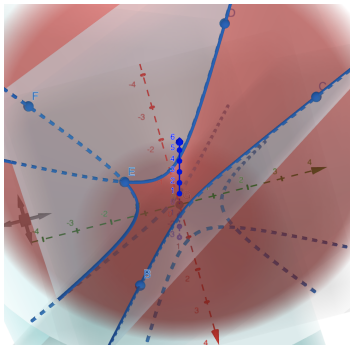
lecture 19
Hyperbolic lines

LEMMA 32. *For $\mathbf{P} \neq \mathbf{Q} \in \mathcal{H}^n$, there is a unique hyperbolic line $L$ containing $\mathbf{P}$ and $\mathbf{Q}$.*

PROOF. Let $V$ be the subvector space of $\mathbb{R}^{n+1}$ spanned by $\mathbf{P}$ and $\mathbf{Q}$, and set $L = \mathcal{H}^n \cap V$. □

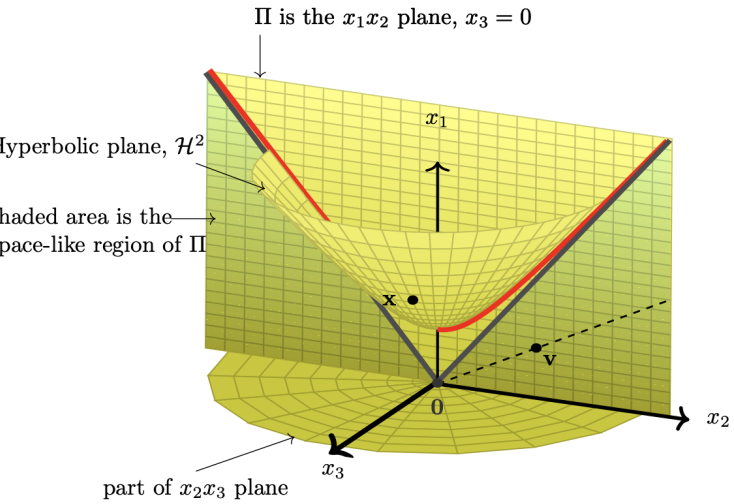
### 30.1. Hyperbolic lines in $\mathcal{H}^2$.

We will show that Euclid's parallel postulate does not hold in $\mathcal{H}^2$.

DEFINITION 34. *A one dimensional subvector space $V = \mathbb{R}\mathbf{v}$ of $\mathbb{R}^n$ is called time-like, space-like or light-like depending on whether $\mathbf{v}$ is time-like, space-like or light-like.*

DEFINITION 35. *Let $L_1 = \mathcal{H}^2 \cap \Pi_1$ and $L_2 = \mathcal{H}^2 \cap \Pi_2$ be two distinct lines in $\mathcal{H}^2$, for two dimensional subvector spaces $\Pi_1$ and $\Pi_2$. Let $\mathbf{v} \in \Pi_1 \cap \Pi_2 \setminus \{\mathbf{0}\}$, and $V = \mathbb{R}\mathbf{v} = \Pi_1 \cap \Pi_2$.*

1. *$V$ is time-like: $L_1$ and $L_2$ intersect at a point $\mathbf{x}$, with $\{\mathbf{x}\} = V \cap \mathcal{H}^2$.*
2. *$V$ is space-like: $L_1$ and $L_2$ are parallel and diverge.*
3. *$V$ is light-like: $L_1$ and $L_2$ are ultraparallel.*




THEOREM 9. *Let $\mathbf{x} \in \mathcal{H}^2$, and let $L$ be a line in $\mathcal{H}^2$. Then there are infinitely many lines in $\mathcal{H}^2$ which pass through $\mathbf{x}$ and do not intersect $L$.*

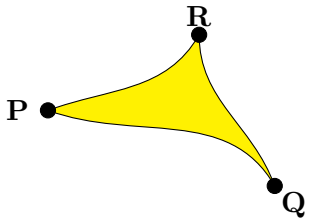
PROOF. Lines and incedence of lines is unchanged by a hyperbolic isometry. Let $\mathbf{P}$ and $\mathbf{Q}$ be two points on $L$ and $\mathbf{R} = \mathbf{x}$. Then by Lemma 30 and Lemma 31, we may assume $L = \mathcal{H} \cap \{(x_1, x_2, x_3) : x_3 = 0\}$, as shown on the left.[a] Now pick any vector $\mathbf{v} = (v_1, v_2, 0) \neq \mathbf{0}$, and let $\Pi_2 = \text{span}\{\mathbf{x}, \mathbf{v}\}$, and $L_2 = \mathcal{H}^2 \cap \Pi_2$. Then $L \cap L_2 = \mathbb{R}\mathbf{v} \cap \mathcal{H}^2$, which is the empty set unless $\mathbf{v}$ is time-like. We can take any space-like vector of the form $\mathbf{v} = (x_1, 1, 0)$ for $x_1 \in (-1, 1)$ to obtain infinitely many lines parallel to $L$ which pass through $\mathbf{x}$, and two ultra parallel lines, when $x_1 = \pm 1$. □

[a] see also `https://www.geogebra.org/3d/qxfcgn2u`

[4] `www.geogebra.org/3d/xk4cauzn` ; `www.geogebra.org/3d/ktnpt5ae` ; `www.geogebra.org/3d/bdtqpp8v`

## 31. Lecture 20: Hyperbolic triangles and angles

### 31.1. Hyperbolic triangles.




DEFINITION 36. *A hyperbolic triangle, denoted* $\Delta\mathbf{PQR}$*, consists of three distinct, non-collinear points,* $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ *on* $\mathcal{H}^n$*, and the finite hyperbolic line segments joining each pair of points, and the finite area enclosed by these lines.*
Note, collinear is used to mean points are contained in the same hyperbolic line.

**31.2. Hyperbolic angles: Motivation from spherical geometry.** If $\Delta\mathbf{PQR}$ is a spherical triangle in $S^2 \subset \mathbb{R}^3$, then the angle $a$ at $\mathbf{P}$ can be computed as follows. Recall that this is the dehedral angle between the planes containing the sides of the triangle containing $\mathbf{P}$. Let $\Pi_1$ be the plane containing $\mathbf{P}$ and $\mathbf{Q}$, and let $\Pi_2$ be the plane containing $\mathbf{P}$ and $\mathbf{R}$. Then

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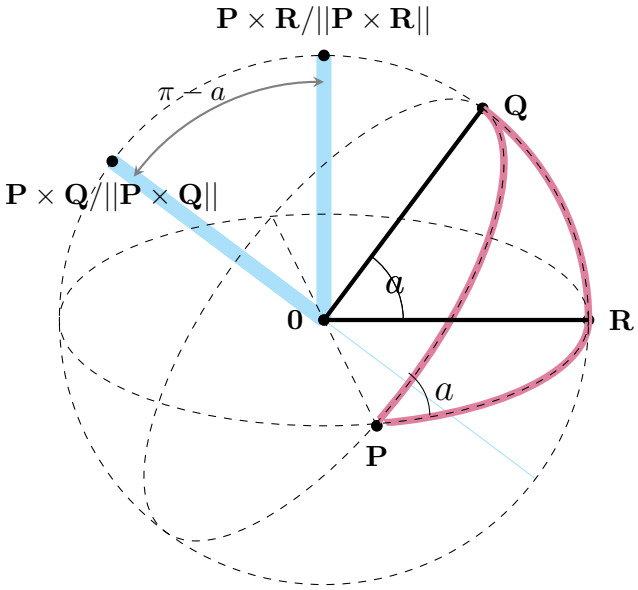
$$\Pi_1 = (\mathbb{R}(\mathbf{P} \times \mathbf{Q}))^\perp \text{ and } \Pi_2 = (\mathbb{R}(\mathbf{P} \times \mathbf{R}))^\perp.$$

If $\theta$ is the dihedral angle between two planes, with unit normals $\mathbf{N}_1$ and $\mathbf{N}_2$ is $\theta$, then we have $\cos(\theta) = \mathbf{N}_1 \cdot \mathbf{N}_2$ (the usual Euclidean inner product). So for $\Delta\mathbf{PQR}$ we have

$$\text{For a spherical triangle:} \quad \cos(a) = \frac{\langle \mathbf{P} \times \mathbf{R}, \mathbf{P} \times \mathbf{Q} \rangle}{||\mathbf{P} \times \mathbf{R}||||\mathbf{P} \times \mathbf{Q}||}.$$




For simplicty, $\mathbf{Q}$ and $\mathbf{R}$ are drawn so that the spherical distance between them is $a$, the angle at $\mathbf{P}$, i.e., the triangle $\Delta\mathbf{PQR}$ is isosceles, which is not true in general

This all works very well for the sphere, but this is not a good definition for hyperbolic space because the Euclidean inner product is not preserved by the isometies of $\mathcal{H}^n$, which preserve the Lorentzian inner product. If we used this definition for angles in triangles in $\mathcal{H}^n$, the angles would change as the triangle moves around. The point is that our embedding of $\mathcal{H}^n$ in $\mathbb{R}^{n+1}$ is not isometric, and so we can not expect the angles in the hyperbolic embedding of $\mathcal{H}^n$ to be computable as though we had an isometric embedding in Euclidean space.
Note, even in Euclidean space, projections of angles in a plane are not preserved if the projection is not to a parallel plane. We will see more of this when we discuss projctive geometry.

DEFINITION 37. *Let* $J = J_3$ *as defined in Lecture 22. For* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$*, the Lorentz cross-product of* $\mathbf{x}, \mathbf{y}$ *is*

$$\mathbf{x} \times_L \mathbf{y} := J \cdot (\mathbf{x} \times \mathbf{y}) = \begin{pmatrix} -x_2y_3 + x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} = (J\mathbf{y}) \times (J\mathbf{x}) = \left| \begin{pmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right|$$

DEFINITION 38. *Let* $\Delta\mathbf{PQR}$ *be a hyperbolic triangle in* $\mathcal{H}^2 \subset \mathbb{R}^3$*, defined by non-collinear points* $\mathbf{P}, \mathbf{Q}, \mathbf{R} \in \mathcal{H}^2$*. Then we will define the hyperbolic angle* $a$ *at* $\mathbf{P}$ *to be* $a \in [0, \pi]$ *such that*

$$\textit{For a hyperbolic triangle:} \quad \cos(a) = \frac{\langle \mathbf{P} \times_L \mathbf{R}, \mathbf{P} \times_L \mathbf{Q} \rangle_L}{||\mathbf{P} \times_L \mathbf{R}||_L||\mathbf{P} \times_L \mathbf{Q}||_L}. \tag{8}$$

It is not immediately obvious that this definition is well defined, since we need the right hand side to be a real number in $[-1, 1]$. (A better definition is given in [R, §3.5, page 78], but we don't have time for that.) To show that it is, we will first show that it is invarient under Lorentz transformations. We will use the following results, which we don't have time to prove in class:

LEMMA 33. *(Binet–Cauchy identity.) For* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$,

$$\langle \mathbf{x} \times \mathbf{y}, \mathbf{z} \times \mathbf{w} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle \cdot \langle \mathbf{y}, \mathbf{w} \rangle - \langle \mathbf{x}, \mathbf{w} \rangle \cdot \langle \mathbf{y}, \mathbf{z} \rangle$$

The following is a version of this identity for the Lorentz inner product, where the difference with the usual formula is highlighted in red.

LEMMA 34. *(Binet–Cauchy identity for Lorentz inner product.) For* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, *we have*

$$\langle \mathbf{x} \times_L \mathbf{y}, \mathbf{z} \times_L \mathbf{w} \rangle_L = -\langle \mathbf{x}, \mathbf{z} \rangle_L \cdot \langle \mathbf{y}, \mathbf{w} \rangle_L + \langle \mathbf{x}, \mathbf{w} \rangle_L \cdot \langle \mathbf{y}, \mathbf{z} \rangle_L$$

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PROPOSITION 17. *The hyperbolic angle is invariant under hyperbolic isometries.*

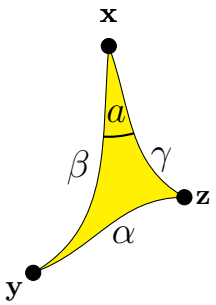
PROOF. By Corollary 10, a hyperbolic isometry is given by a Lorentz transformation, which preserves the Lorentz inner product. Since the right hand side of Equation 8 is given in terms of the Lorentz inner product, Lemma 34 implies the result. □

This result tells us that the expression on the right hand side of Equation 8 is invariant under Lorentz transformations, since by definition the Lorentz inner product is.

## 32. Proof of the triangle inequality

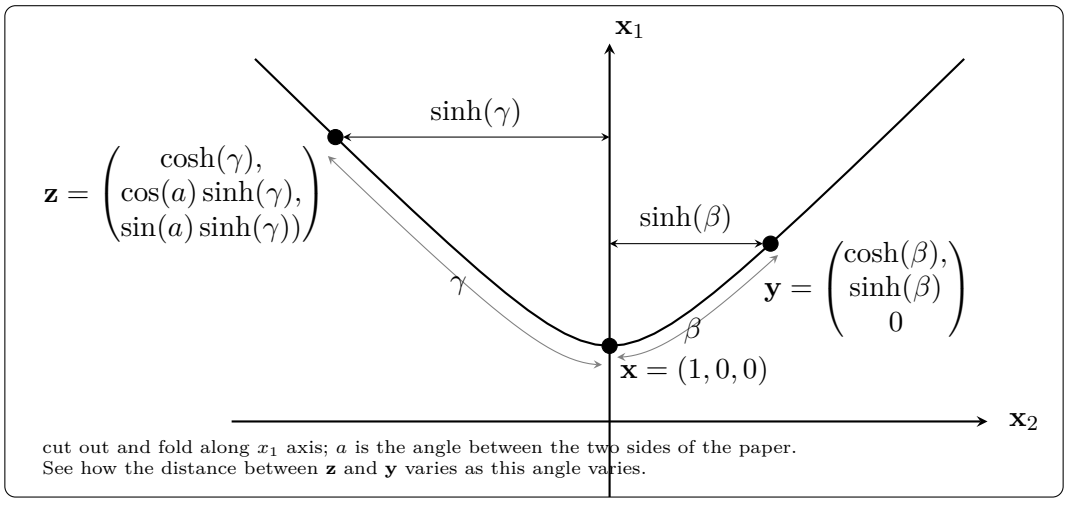
THEOREM 10. *(Main theorem of hyperbolic trigonometry) For distinct points* $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *in* $\mathcal{H}^n$, *let*

$$\alpha = d_{\mathcal{H}^n}(\mathbf{z}, \mathbf{y})$$
$$\beta = d_{\mathcal{H}^n}(\mathbf{x}, \mathbf{z})$$
$$\gamma = d_{\mathcal{H}^n}(\mathbf{x}, \mathbf{y})$$
$$a = \text{ the hyperbolic angle of } \Delta\mathbf{xyz} \text{ at } \mathbf{x}.$$




*Then*

$$\cosh\alpha = \cosh\beta \cdot \cosh\gamma - \sinh\beta \cdot \sinh\gamma \cdot \cos a$$



PROOF. By Lemma 30, we can assume these points have the form: $\mathbf{x} = (1, 0, 0, \ldots, 0)$, $\mathbf{y} = (y_1, y_2, 0, \ldots, 0)$, $\mathbf{z} = (z_1, z_2, z_3, \ldots, 0)$. Since the trailing zeros make no contribution to the inner product, without loss of generality we may take $n = 2$. Because of the parameterization of $\mathcal{H}^2$ in Lemma 31, we can find $\theta$, $t_1$ and $t_2$ such that

$$\begin{array}{rcccc} \mathbf{x} & = ( & 1, & 0, & 0) \\ \mathbf{y} & = ( & \cosh(t_1), & \sinh(t_1), & 0) \\ \mathbf{z} & = ( & \cosh(t_2), & \cos(\theta)\sinh(t_2), & \sin(\theta)\sinh(t_2)) \end{array}$$

In order to compute $\cos(a)$, note that for $(a_1, a_2, a_3) \in \mathbb{R}^3$,

$$\mathbf{x} \times_L (a_1, a_2, a_3) = \left| \begin{pmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & 0 \\ a_1 & a_2 & a_3 \end{pmatrix} \right| = (0, -a_3, a_2).$$

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So by the definition of the hyperbolic angle at $\mathbf{x}$,

$$\begin{aligned} \cos(a) &= \frac{\langle \mathbf{x} \times_L \mathbf{y}, \mathbf{x} \times_L \mathbf{z} \rangle_L}{||\mathbf{x} \times_L \mathbf{y}||_L ||\mathbf{x} \times_L \mathbf{z}||_L} \\ &= \frac{\langle (0, 0, \sinh(t_1)), (0, -\sin(\theta)\sinh(t_2), \cos(\theta)\sinh(t_2)) \rangle_L}{|\sinh(t_1)||\sinh(t_2)|} \\ &= \cos(\theta) \end{aligned}$$

Remark: Provided that $a$ and $\theta \in [0, \pi]$ (and with more work for the general case), we can conclude that

$$a = \theta.$$

Note, when $\mathbf{x} = (1, 0, 0)$, the hyperbolic angle is the same as the Euclidean angle, but this is not the case for other points on $\mathcal{H}^2$.
Now we can compute $\alpha, \beta, \gamma$ in terms of the coordinates of $\mathbf{x}, \mathbf{y}, \mathbf{z}$:

$$\begin{aligned} \cosh(\gamma) &= -\langle \mathbf{x}, \mathbf{y} \rangle_L = \cosh(t_1) \\ \cosh(\beta) &= -\langle \mathbf{x}, \mathbf{z} \rangle_L = \cosh(t_2) \\ \cosh(\alpha) &= -\langle \mathbf{z}, \mathbf{y} \rangle_L = \cosh(t_1)\cosh(t_2) - \cos(\theta)\sinh(t_1)\sinh(t_2) \end{aligned}$$

Combining these, and using $\theta = a$ gives the required result:

$$\cosh(\alpha) = \cosh(\gamma)\cosh(\beta) - \sinh\beta \cdot \sinh\gamma \cdot \cos a$$

□

## 33. Lecture 21: Triangle inequality in hyperbolic space

COROLLARY 11. *(The triangle inequality for the hyperbolic metric.) For* $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *in* $\mathcal{H}^n$*, we have*

$$d_{\mathcal{H}^n}(\mathbf{y}, \mathbf{z}) \leq d_{\mathcal{H}^n}(\mathbf{y}, \mathbf{x}) + d_{\mathcal{H}^n}(\mathbf{x}, \mathbf{z})$$

PROOF. Use notation of Theorem 10. Since $\cos(a) \geq -1$

$$\cosh(\alpha) \leq \cosh(\gamma)\cosh(\beta) + \sinh\beta \cdot \sinh\gamma = \cosh(\beta + \gamma).$$

Since cosh is increasing on $[0, \infty)$, this implies that $\alpha \leq \beta + \gamma$, the required result. □

COROLLARY 12. *The angles in a hyperbolic triangle are well defined.*

PROOF. We showed in Theorem 10 that the angle was well defined after a Lorentz translation of coordinates. Since we have seen that the angle is invariant under Lorentz translation, this implies that it is well defined. □

COROLLARY 13. *The hyperbolic metric is a metric.*

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PROOF. Recall that

$$d_{\mathcal{H}^n}(\mathbf{x}, \mathbf{y}) = \cosh^{-1}(-\langle \mathbf{x}, \mathbf{y} \rangle_L),$$

which we showed was well defined in Lemma 22.
Non degeneracy: If $d_{\mathcal{H}^n}(\mathbf{x}, \mathbf{y}) = 0$, then $-\langle \mathbf{x}, \mathbf{y} \rangle_L = 1$. In the Lemma 22, we showed that $-\langle \mathbf{x}, \mathbf{y} \rangle_L \geq 1$. Examining the proof, by Lemma 5, we see that there is equality if and only if $\hat{\mathbf{x}}$ is a multiple of $\hat{\mathbf{y}}$. But every vector in $\mathbb{R}^{n+1}$ intersects $\mathcal{H}^n$ in at most one point (see below), and so this implies $\mathbf{x} = \mathbf{y}$.
Symmetry: This is immediate from the fact that the Lorentz inner product is symmetric on $\mathbb{R}^{n+1}$.
Triangle inequality: This is Corollary 11.
Hence $d_{\mathcal{H}^n}$ satisfies the requirments to be a metric. □

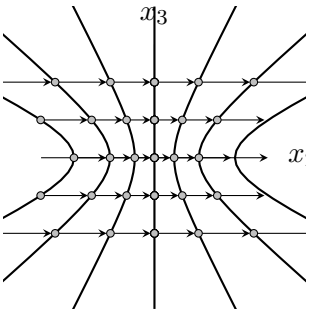
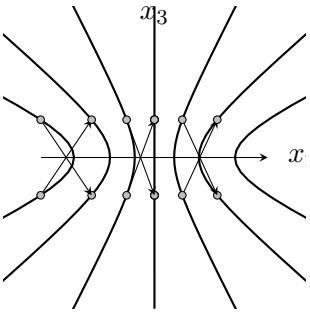
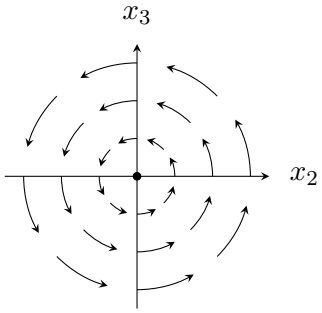
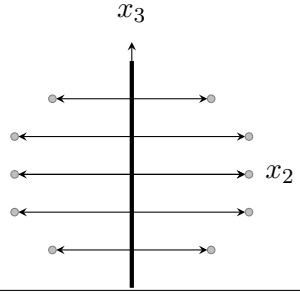
**Remark.** Note that we have equality in the triangle inequality if and only if $a = \pi$, which is if and only if $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are collinear (possibly up to reordering).

We proved the following in a class, but I am not sure it's written in the notes, so:
**Lemma.** *Every line through the origin in* $\mathbb{R}^{n+1}$ *intersects* $\mathcal{H}^n$ *in at most* 1 *point.*

PROOF. Suppose $\mathbf{x} = \lambda \mathbf{y}$ for some $\mathbf{x}, \mathbf{y} \in \mathcal{H}^n$.
Then by definition of $\mathcal{H}^n$, $\langle \mathbf{x}, \mathbf{x} \rangle_L = \langle \mathbf{y}, \mathbf{y} \rangle_L = -1$. But the Lorentz inner product is linear, so $\langle \mathbf{y}, \mathbf{y} \rangle = \lambda^2 \langle \mathbf{x}, \mathbf{x} \rangle_L$, so $\lambda = \pm 1$. But if $\mathbf{y} \neq \mathbf{x}$, then since the first component of $\mathbf{x}$ is positive, by definition of $\mathcal{H}^n$, the first component of $\mathbf{y}$ would be negative, but the by definition of $\mathcal{H}^n$, $\mathbf{y} \notin \mathcal{H}^n$. Hence $\mathbf{x} = \mathbf{y}$. □

## 34. Normal form for Lorentz transformations of $\mathcal{H}^2$

Almost every Lorentz transformation of $\mathcal{H}^2$ is given one of the following matrices, up to a Lorentz orthonormal change of basis, i.e., from the standard basis to another Lorentz orthormal basis.

| matrix | description | fixed points | eigen values | picture – projection to $x_2, x_3$ plane |
|---|---|---|---|---|
| $\begin{pmatrix} \cosh(\beta) & \sinh(\beta) & 0 \\ \sinh(\beta) & \cosh(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | Lorentz translation (by $\beta$ in the $x_2$ direction); orientation preserving (direct); hyperbolic | none (in $\mathcal{H}^2$) | $1, \exp(\pm\beta)$ |   |
| $\begin{pmatrix} \cosh(\beta) & \sinh(\beta) & 0 \\ \sinh(\beta) & \cosh(\beta) & 0 \\ 0 & 0 & -1 \end{pmatrix}$ | Lorentz glide (in $x_2$ direction); orientation reversing (indirect / opposite) | none | $-1, \exp(\pm\beta)$ |   |
| $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$ | rotation (about the point $(1,0,0)$); orientation preserving (direct); elliptic | one point | $1, \exp(\pm i\theta)$ |   |
| $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | reflection (in the line $x_2 = 0$); orientation reversing (indirect) | a line | $1, 1, -1$ |   |

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Note, each of the above matrices has an eigen value $\pm 1$ with an eigen vector which is space-like or time-like. There are elements of $O^+(1,2)$ with light-like eigen vectors with eigen value 1, such as $\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & -1 \\ -1 & 1 & 1 \end{pmatrix}$. These correspond to "parabolic" transformations of $\mathcal{H}^2$, which preserve the orientation, and don't fix any points of $\mathcal{H}^2$, but whereas a Lorentz translation fixes a space-like vector when extended to $\mathbb{R}^3$, parabolic transformations fix a light-like vector.

We do not have time to prove this result. See [RS, 3.11(page 47) and 3.16(page 58) and B.3 page 189], and the RS erratum If there is time I will draw some pictures in class, or add some pictures in these notes, and include examples on an assignment.

You are expected to be able to

- Check all the above are elements of $O^+(1,2)$.
- given an element of $O^+(1,2)$, determine which kind of transformation it corresponds to, e.g., by considering fixed points and eigen values
- sketch images of simple shapes e.g., lines, points, triangle, under the action of transformations
- compute the composition of hyperbolic isometries
- find the matrix for a hyperbolic isometry given a description

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Note, regarding sketches: the transformations will look different with respect to different basis. E.g., a Lorentz rotation about a point $p$ other than $(1,0,0)$ does not preseve the preimages of circles in the projection to the $x_2x_3$ plane which are concentric about the projection of $p$. Reflections will map points to other points equidistance from a line, but only for the lines through $(1,0,0)$ will the projection of the reflection be a Euclidean reflection in $\mathbb{R}^2$. You should be able to compute images of points under simple hyperbolic isometries of $\mathcal{H}^2$.

**Example computation.** Where does the line $x_2 = 0$ map to under the translation by $\beta$ in $x_2$ direction? Points on the line $x_2 = 0$ have the form $(\cosh(\alpha), 0, \sinh(\alpha))$, so we have

$$\begin{pmatrix} \cosh(\beta) & \sinh(\beta) & 0 \\ \sinh(\beta) & \cosh(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh(\alpha) \\ 0 \\ \sinh(\alpha) \end{pmatrix} = \begin{pmatrix} \cosh(\alpha)\cosh(\beta) \\ \cosh(\alpha)\sinh(\beta) \\ \sinh(\alpha) \end{pmatrix}$$

We know that $\cosh(\alpha) = \sqrt{1+\sinh(\alpha)^2}$. So, if we only care about the projection to the $x_2x_3$ plane, writing $a = \sinh(\alpha)$ and $b = \sinh(\beta)$, we have

$$\begin{pmatrix} 0 \\ a \end{pmatrix} \mapsto \begin{pmatrix} b\sqrt{1+a^2} \\ a \end{pmatrix},$$

so the image of the line $x_2 = 0$ under translation by $\beta$ in the $x_2$ direction is the line given by

$$L_b : x_2^2 = b^2(1 + x_3^2)$$

This line can be parameterised by $(x_2, x_3) = (b\cosh(t), \sinh(t))$, which is used to plot the images in the table. Call this translation $T_B$, with matrix $B = B(\beta)$. It can be checked (exercise!) that $B(\beta)^n = B(n\beta)$. So the image $T_B(L_b) = L_{2b}$, and $T_B(T_B(L_b)) = L_{3b}$ and so on, which gives the sucessive images in the picture.

What is the distance from $\mathbf{x}$ to $T_B(\mathbf{x})$?

for $\mathbf{x} = (\cosh(\alpha), 0, \sinh(\alpha))$ and $T_B(\mathbf{x}) = (\cosh(\alpha)\cosh(\beta), \cosh(\alpha)\sinh(\beta), \sinh(\alpha))$, we have

$$\cosh(d(\mathbf{x}, T_b(\mathbf{x})) = \cosh(\alpha)^2\cosh(\beta) - \sinh(\alpha)^2,$$

which is only equal to $\cosh(\beta)$ when $\alpha = 0$, and so the larger $|\alpha|$, the further $\mathbf{x}$ is translated.

On the other hand, $T_B(\mathbf{x}) = (\cosh(\alpha)\cosh(2\beta), \cosh(\alpha)\sinh(2\beta), \sinh(\alpha))$, so

$$\cosh(d(T_B(\mathbf{x}), T_b^2(\mathbf{x})) = \cosh(\alpha)^2\cosh(\beta)\cosh(2\beta) - \cosh(\alpha)^2\sinh(\beta)\sinh(2\beta) - \sinh(\alpha)^2$$

$= \cosh(\alpha)^2\cosh(\beta) - \sinh(\alpha)^2$, so $\mathbf{x}$ and $T_B(\mathbf{x})$ are translated the same amount – as expected, since $T_B$ is an isometry, so we always have $d(\mathbf{x}, T(\mathbf{x})) = d(T(\mathbf{x}), T^2(\mathbf{x}))$.

## 35. Hyperbolic triangle angles

THEOREM 11. *If $\Delta = \Delta\mathbf{PQR}$ is a hyperbolic triangle, with angles and sides as in the following diagram, then*

$$\text{Area}(\Delta) = \pi - (a + b + c)$$

We do not have time to prove this theorem. See [RS, 3.14] for a proof. We will aim to prove the weeker result:

PROPOSITION 18. *For a hyperbolic triangle with non-zero area, and angles $a, b, c$,*

$$a + b + c < \pi$$

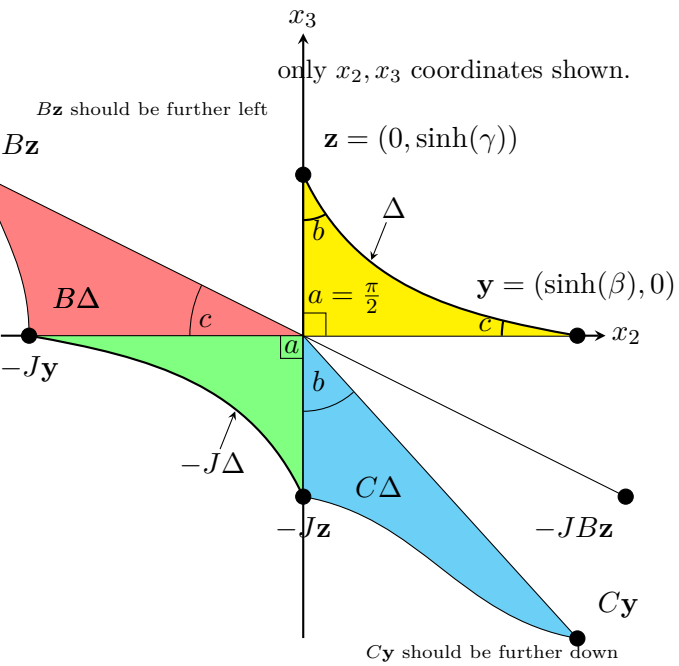
In fact, we will only prove an even weaker version. From this version it is a relatively simple step to prove Proposition 18, by dividing an arbitrary triangle into two right angle triangles. The proof is a hyperbolic version of the proof in the Euclidean case given in [RS, 1.16.2].

week 8 lecture 21 Hyperbolic triangle inequality

PROPOSITION 19. *For a hyperbolic triangle with non-zero area, and angles $\pi/2, b, c$,*

$$b + c < \pi/2$$

PROOF. By Lemmas 30, 31, and Proposition 17 we may assume the triangle, $\Delta$, has vertices $\mathbf{x} = (1, 0, 0)$, $\mathbf{y} = (\cosh(\beta), \sinh(\beta), 0)$ and $\mathbf{z} = ((\cosh(\gamma), 0, \sinh(\gamma))$, with angle $\pi/2$ at $\mathbf{x}$. The figure shows the projection to the $x_2x_3$ plane, i.e, a view from above.
By Theorem 10, the hyperbolic and Euclidean angles are the same for a vertex at $(1, 0, 0)$, and by Proposition 17 the hyperbolic angle is unchanged by application of a hyperbolic isometry. So, we apply the following isometries, to obtain the red, green, and blue triangles in the figure. One of these isometries is a rotation, and the others are hyperbolic translations.




| matrix $M = B, C,$ or $-J$ | $M\mathbf{x}$ | $M\mathbf{y}$ | $M\mathbf{z}$ |
|---|---|---|---|
| $B = \begin{pmatrix} c(\beta) & -s(\beta) & 0 \\ -s(\beta) & c(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} c(\beta) \\ -s(\beta) \\ 0 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} c(\beta)c(\gamma) \\ -s(\beta)c(\gamma) \\ s(\gamma) \end{pmatrix}$ |
| $-J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} c(\beta) \\ -s(\beta) \\ 0 \end{pmatrix}$ | $\begin{pmatrix} c(\gamma) \\ 0 \\ -s(\gamma) \end{pmatrix}$ |
| $C = \begin{pmatrix} c(\gamma) & 0 & -s(\gamma) \\ 0 & 1 & 0 \\ -s(\gamma) & 0 & c(\gamma) \end{pmatrix}$ | $\begin{pmatrix} c(\gamma) \\ 0 \\ -s(\gamma) \end{pmatrix}$ | $\begin{pmatrix} c(\beta)c(\gamma) \\ s(\beta) \\ -s(\gamma)c(\beta) \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ |

to save space, in this table $c = \cosh, s = \sinh$

In the diagram, we can determine whether $a + b + c < \pi$ by considering the relationship between $C\mathbf{y}$ and $-JB\mathbf{z}$. We have

$$\tan(b) = \frac{\sinh(\beta)}{\sinh(\gamma)\cosh(\beta)} \text{ and } \tan(\pi/2 - c) = \frac{\sinh(\beta)\cosh(\gamma)}{\sinh(\gamma)}$$

Since the triangle has non zero area, $\beta, \gamma > 0$. So we have

$$\frac{\tan(\pi/2 - c)}{\tan(b)} = \cosh(\beta)\cosh(\gamma) > 1$$

So $\tan(\pi/2 - c) > \tan(b)$, so, since tan is increasing on $[0, \pi/2)$, $\pi/2 - c > b$, and so $b + c < \pi/2$.

□

This illustrates that geometry in hyperbolic space does not follow the same rules as in Euclidean space. Unfortunately, we do not have time to investigate this further.

## 36. Lecture 22: Algebraic Projective Geometry: Overview

Algebraic Projective Geometry can be viewed as a solution about how to handle infinity when solving equations. E.g., in Euclidean space $\mathbb{R}^2$, we might colloquially say "two parallel lines meet at infinity". The projective plane makes this concept concrete. Note, the term "projection" here means a **radial** projection, not an orthogonal projection.
Goals of this section:

(1) Define $\mathbb{P}^n(k)$, the projective $n$ space of a field $k$
(2) Define Lines in $\mathbb{P}^n(k)$, show that all lines in $\mathbb{P}^2(\mathbb{R})$ intersect, so there are **no** parallel lines
(3) Explain important points: There are, a priori, no metric or angles in $\mathbb{P}^n(k)$ – these can be defined, just like $\mathbb{E}^n$ has no zero and is not a vector space, but can be given a zero, and the structure of a vector space. But a choice must be made, and different choices may give very different results.
(4) We don't have isometries of $\mathbb{P}^n(k)$ because we won't give a metric. We will define the group of projective transformations, $\mathrm{PGL}(n+1)$ of projective $n$ space, and show that transformations in this group preserve relationships such as incidence, map lines to lines, planes to planes, and preserve the cross ratio.
(5) Prove two classical results in projective geometry, Desargues Theorem and Pappus Theorem.
(6) Define axiomatic projective geometries, and state a theorem about how these are related to the algebraic construction.

week 8 lecture 22 Projective geometry

We will follow [RS, chapter 5] fairly closely for this section.

### 36.1. Algebraic Definition of Projective Space.

Definition 39. *Let $V$ be an $n+1$ dimensional vector space over a field $k$, so $V \cong k^{n+1}$. Let $V^* = V \setminus \{\mathbf{0}\}$. (This definition also works with $V = A^{n+1}$ for any division ring $A$. A division ring is a ring where $ax = b$ has a solution $x \in A$ for any non-zero $a, b \in A$. )*
*Define an equivalence relation on $V^*$, by defining for $\mathbf{x}, \mathbf{y} \in V$,*

$$\mathbf{x} \sim \mathbf{y} \iff k\mathbf{x} = k\mathbf{y}.$$

*Exercise: check that this is an equivalence relation, namely reflexive, transitive and symmetric.*
*The projective space of $V$ is the set of equivalence classes in $V^*$ under this relation:*

$$\mathbb{P}^n(k) := \mathbb{P}(V) := V^*/\sim$$

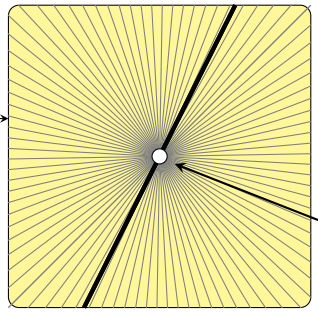
*For $\mathbf{x} = (x_1, x_2, \ldots, x_{n+1}) \in V^*$, we denote the equivalence class of $\mathbf{x}$ by*

$$[\mathbf{x}] = (x_1 : x_2 : \cdots : x_{n+1}) = \{\mathbf{v} \in V^* : \mathbf{v} \sim \mathbf{x}\}$$

*When $k = \mathbb{R}$, we write $\mathbb{P}^n = \mathbb{P}(\mathbb{R}^{n+1})$, and call this n-dimensional real projective space.*

Lemma 35. *There is a bijection between $\mathbb{P}(V)$ and the set of lines through the origin in $V$.*
*For $\mathbf{x}, \mathbf{y} \in V^*$, $\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = \lambda\mathbf{y}$ for some $\lambda \in k \setminus \{0\}$.*

Proof. Exercise. □



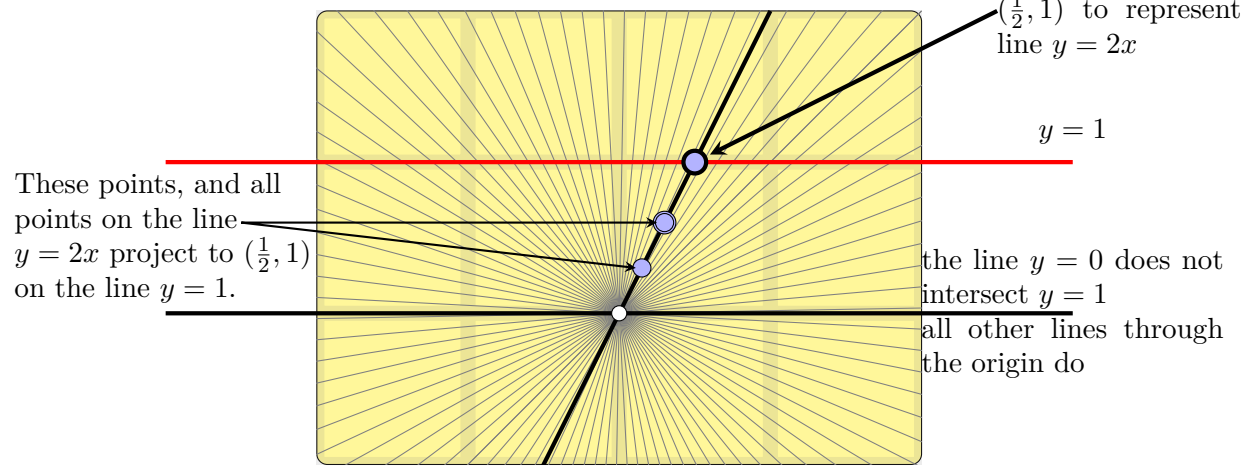
picture of $\mathbb{R}^2$ imagined as a square

Each line is an equivalence class of points in $\mathbb{P}^2$. The thick black line is the class
$[(1,2)] = (1:2) = \{(\lambda, 2\lambda) : \lambda \in \mathbb{R}\}$

**0** removed

## 37. The projective line $\mathbb{P}^1(\mathbb{R})$

**37.1. Equivalence class representatives for the projective line.** Consider the red line in this figure, given by $y = 1$. Then (nearly) every equivalence class of lines in $\mathbb{P}^2$ interests this line exactly once. E.g., the line $[(1, 2)]$ intersects at $(\frac{1}{2}, 1)$.

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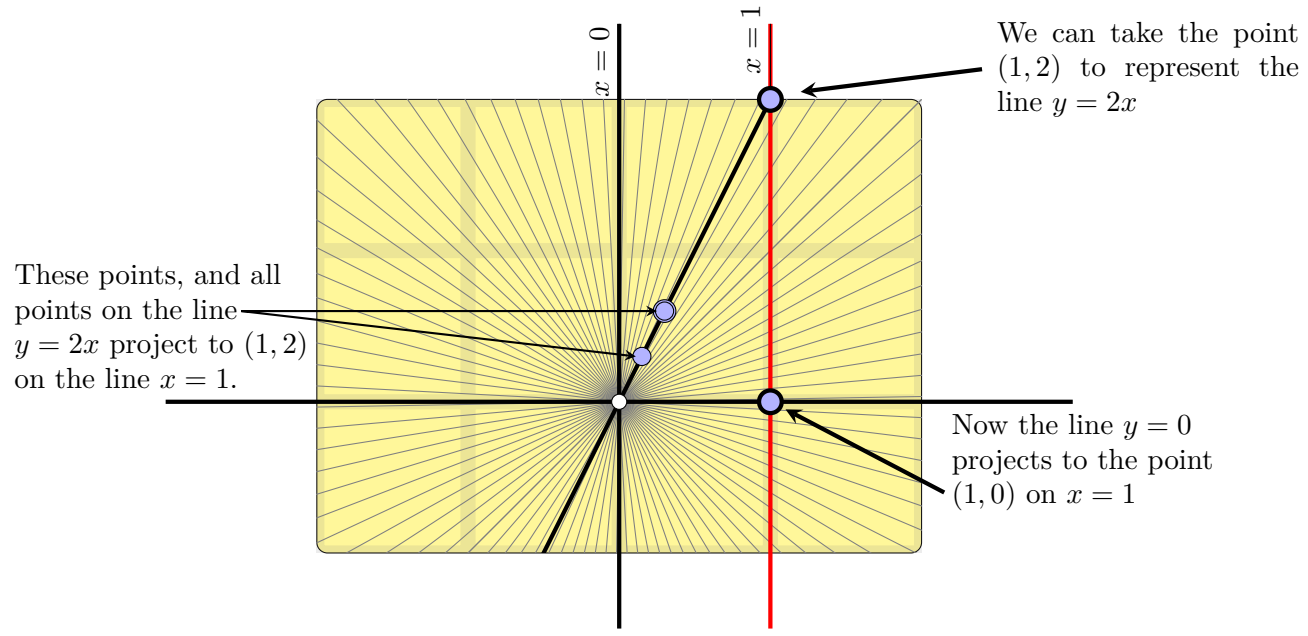
If you imagine the centre as a light source, and the line $y = 1$ as a screen, points on the same line through the origin project to the same line, so they are projectively equivalent because you can't tell them apart after projection.
The red line, which can be identified with $\mathbb{R}$, contains points for each line through the origin, except $y = 0$. So, we take a symbol called "$\infty$", and then we have a bijective map:

$$\begin{aligned} \phi_{y=1} : \mathbb{R} \cup \{\infty\} &\to \mathbb{P}^1 \\ t &\mapsto \left(\frac{1}{t} : 1\right) \text{ for } t \in \mathbb{R} \\ \infty &\mapsto (1 : 0) \end{aligned}$$

We could take a different line to project to, e.g., $x = 1$,



Now we get a map:

$$\begin{aligned}\phi_{x=1} : \mathbb{R} \cup \{\infty\} &\to \mathbb{P}^1 \\ t &\mapsto (1:t) \text{ for } t \in \mathbb{R} \\ \infty &\mapsto (0:1)\end{aligned}$$

These two maps give different projections, and so different representatives of the equivalence classes, e.g., for the first, the representative of the points on the line $y = 2x$ is $(1/2, 1)$, and for the second map, the respresentative point is $(1, 2)$. In either case, points which are equivalent under $\sim$ project to the same point.
The inverses, $\phi^{-1}_{y=1}$, or $\phi^{-1}_{x=1}$, which are injective, and surject to $\mathbb{P}^1 - \{\text{a point}\}$, can be viewed as completing $\mathbb{R}$ by adding a point at infinity. We have a choice about which point of $\mathbb{P}^1$ maps to $\infty$. By rotating the line of projection, any of the lines through the origin in $\mathbb{R}^2$ could be taken to be the point at infinity, so considered on $\mathbb{P}^1$, infinity is no longer a special point, it's just whatever point is missing when you try and make a bijection between $\mathbb{R}$ and $\mathbb{P}^1$. $\infty$ is still a special point on $\mathbb{R}$, just not on $\mathbb{P}^1$.

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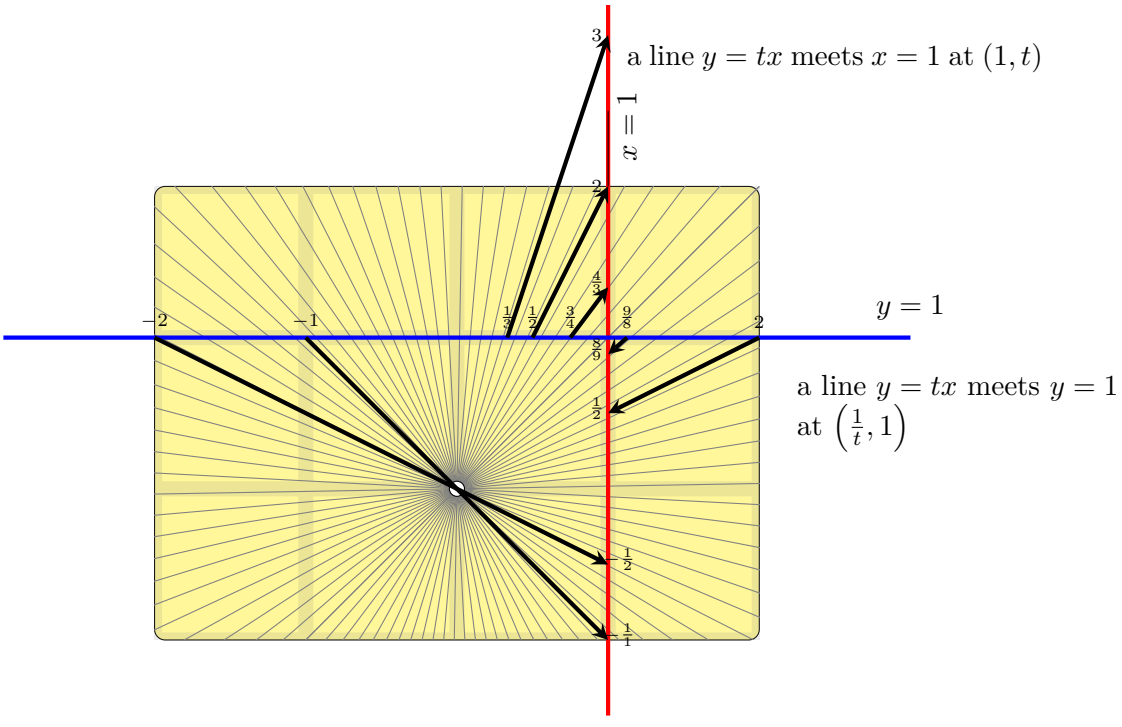
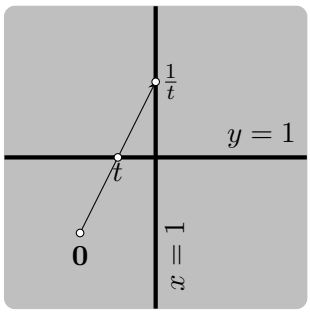
**Discussion point:** Is the projective line equal to $\mathbb{R} \cup \{\infty\}$, or is it equal to the set of lines though the origin in $\mathbb{R}^2$? Compare this question to the following: Are the numbers modulo 5 the set $\{0, 1, 2, 3, 4\}$, or are they the set of equivalence classes, $\{5\mathbb{Z}, 5\mathbb{Z} + 1, 5\mathbb{Z} + 2, 5\mathbb{Z} + 3, 5\mathbb{Z} + 4\}$? (where the equivalence relation is $a \equiv b \iff 5|(a - b)$). Or maybe numbers modulo 5 means all of $\mathbb{Z}$?

**37.2. Map of projective space.** Consider the two different (radial) projections to either $y = 1$, or $x = 1$. The two projections $\phi_{y=1}$ and $\phi_{x=1}$ can be considered as two different view points. How are they related? This figure illustrates the map from the projection to $y = 1$ to the projection to $x = 1$:

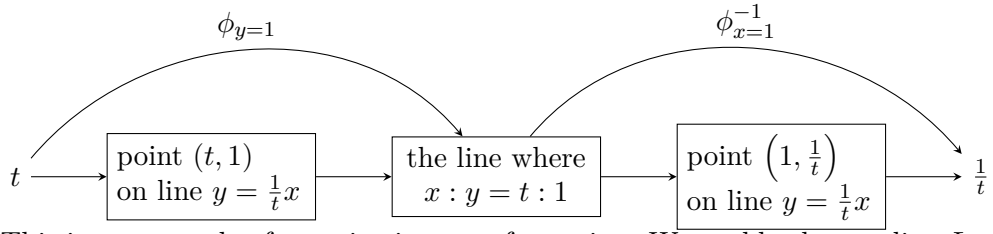


Let's represent a point $(a,1)$ on the line $y=1$ by $a \in \mathbb{R}$, and a point $(1,b)$ on $x=1$ by $b$. Then the map from $\{y=1\}$ to $\{x=1\}$ induced by radial projection from $\mathbf{0}$ is given by




$$\phi_{x=1}^{-1} \circ \phi_{y=1} : \overbrace{\mathbb{R} \cup \{\infty\}}^{\{y=1\}} \to \overbrace{\mathbb{R} \cup \{\infty\}}^{\{x=1\}}$$

$$t \to \begin{cases} \frac{1}{t} & \text{if } t \neq 0 \text{ or } \infty \\ 0 & \text{if } t = \infty \\ \infty & \text{if } t = 0 \end{cases}$$

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This is an example of a projective transformation. We could take any line $L$ not through the origin, and take $L \cup \{\infty\}$ as our representation of $\mathbb{P}^1$. For a general line, we map have to choose how to parametrise the line, so we know how to lable the intersection point.

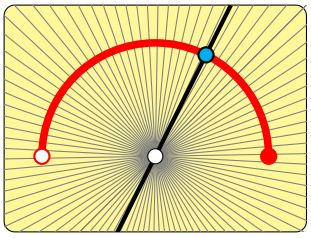
**Example:** For the line $x=2$, if we parameterise this by $t \to (2,t)$, then $\phi_{x=2}(t) = (2:t)$, and the relationship between the projections $\phi_{x=1}$ and $\phi_{x=2}$ is $\phi_{x=1}^{-1}\phi_{x=2}(t) = 2t$, i.e., things look twice as big when you move the "projection screen" twice as close.

We want to classify these kinds of transformations, so we need to work out what the quantities are that are preserved. The Euclidean distance is not preserved. E.g., $d(t,r) \neq d(\frac{1}{t},\frac{1}{r})$, where $d$ is the Euclidean distance.

**37.3. Projective space $\mathbb{P}^n$ for $n>1$.** Exactly the same kind of consideration allows us to map any point $\mathbf{x}$ in a plane $\Pi$ (basically a copy of $\mathbb{R}^2$) in $\mathbb{R}^3$, which does not pass through the origin, to a unique line through the origin in $\mathbb{R}^3$. The lines which will not be obtained in this way can be considered to correspond to "points at infinity" of the plane $\Pi$. These lines are those in the plane parallel to $\Pi$ which pass through the origin. So, in a sense, $\mathbb{P}^2$ is a completion of $\mathbb{R}^2$, with extra points added at infinity. Similar maps allow us to consider $\mathbb{P}^n$ as a completion of $\mathbb{R}^n$.

**37.4. Relationship between $\mathbb{P}^n$ and $S^n$.** An important model of projective space is given by the sphere $S^n$, with opposite points identified. This can also be considered as the half sphere, or half circle when $n=1$. An advantage of this choice of representatives of points in $\mathbb{P}^n$ is that all lines in $\mathbb{P}^n$ interesect a sphere centre at $\mathbf{0}$, whereas for a projection to a plane, some lines will not be included in the image.
As discussed in the section on spherical geometry, every line through the origin intersects the sphere twice, so, to take a representative of each line, we only need half the sphere or circle.

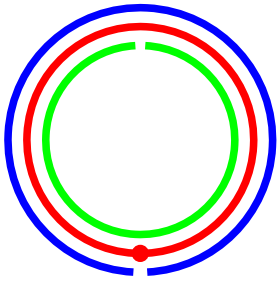
**37.5. Semi-circle identified with $\mathbb{P}^1$.**



Points on the semi-circle correspond to lines through $\mathbf{0}$. E.g., the point at $(\cos(\theta),\sin(\theta))$ corresponds to the line $y=\tan(\theta)x$

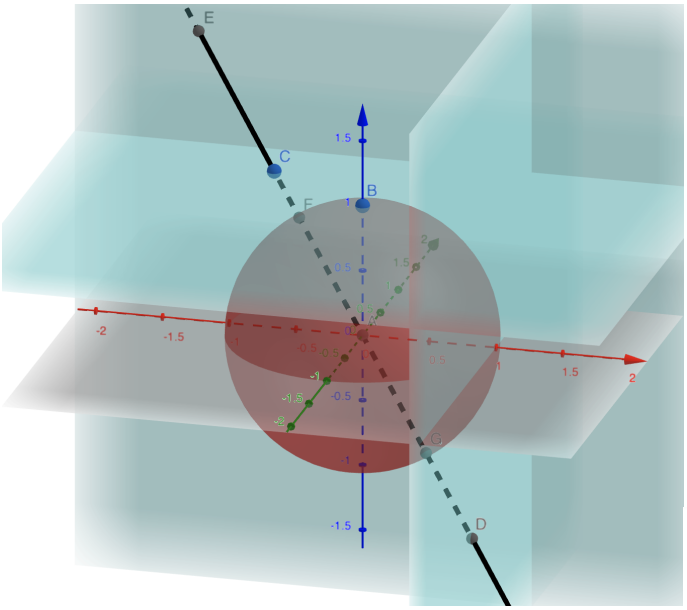
In the above picture, the red and white dots at the end of the semi circle correspond to the same line, $y=0$, so these points are identified. Topologically (in topology, you can bend and stretch but not tear a geometric object), $\mathbb{P}^1$ is a circle, which you can see after identifying these points, i.e., stretching round

and gluing together as in the next picture. We can also think of the $\{x = 1\}$ and $\{y = 1\}$ projections of $\mathbb{P}^1$ as real lines that together cover the whole of of $\mathbb{P}^1$, as in the following figure:



The red line represents all of $\mathbb{P}^1$.
The blue line represents the part of $\mathbb{P}^1$ that projects to $\{y = 1\}$
The green line represents the part of $\mathbb{P}^1$ that projects to $\{x = 1\}$
This is a schematic diagram, only topologically equivalent to $\mathbb{P}^1$.

**37.6. Projective plane $\mathbb{P}^2$.** The real projective plane is the set of lines through the origin in $\mathbb{R}^3$, but you can think of it as being represented by a hemisphere, or by projections to three different planes, e.g., the $X = 1$, $Y = 1$ and $Z = 1$ planes. (In algebraic geometry, it's quite common to write $X, Y, Z$ for the coordinates in $\mathbb{R}^2$, and $x, y$ for the coordinates in one of the projections.)

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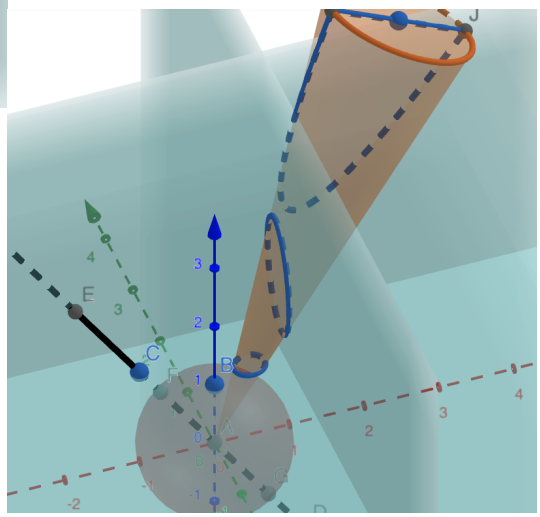
The black line in $\mathbb{R}^3$ in the figure on the left intersects the

| | | |
|---|---|---|
| sphere | at | $F$ and $G$ |
| $\{X = 1\}$ plane | at | D |
| $\{Y = 1\}$ plane | at | E |
| $\{Z = 1\}$ plane | at | C |

See `https://www.geogebra.org/3d/tprpwpwq` for a better picture. The "G" in this screen shot is hard to see.




In the figure on the right, the cone (shown truncated, because of contruction in geogebra, and to make it clear it's a cone), intersects the planes in either an ellipse of parabola. (Hyperbola and circles also possible with different planes; planes to project to can be any plane through the origin.) On the $Z = 1$ plane we have an ellipse, with an equation

$$r_1(x - c_1)^2 + r_2(y - c_2)^2 = 1$$

for some $r_1, r_2, c_1, c_2$. The cone through this ellipse has equation $r_1(X - c_1Z)^2 + r_2(Y - c_2Z)^2 = Z^2$. The same cone, on the $y = 1$ plane, becomes $r_1(x - c_1z)^2 + r_2(1 - c_2z)^2 = z^2$, which depending on the parameters, may be a different kind of conic. So, this shows the conics are all just different projection of the same object. The next page shows an example of different projections of a cubic curve.

**37.7. Algebra on $\mathbb{P}^n$.** [This page is non-examinable; motivation for the interested reader]
There are many applications of projective geometry, which has been studied since ancient times, and by many mathematicians, for example, Brook Taylor of Taylor series fame also wrote a treatise on "Linear Perspective" (1715). Understanding projective geometry is fundemantal to understanding modern methods of solutions to algebraic equations, i.e., polynomial equations in several variables; Diophantine equations; Fermat's last theorem, etc. This is the beginnings of algebraic geometry.
Supposing we take $\mathbb{P}^1$ as the set of lines through $\mathbf{0}$ in $\mathbb{R}^2$, taking coordinates $x, y$ for $\mathbb{R}^2$. Algebraic functions like $f(x,y) = xy + 1$ don't make sense on $\mathbb{P}^1$, because this function is not constant on lines through the origin.
However, a function like

$$(\bigstar) \qquad f(x,y) = \frac{x^2+y^2}{3x^2+xy}$$

does make sense, since for every point $(x,y)$ on a line $y = \lambda x$, for a fixed $\lambda$, this function has the same value. Also a function from $\mathbb{P}^1$ to $\mathbb{P}^1$, such as

$$f((x:y)) = (x^2+y^2 : 3x^2+xy)$$

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is well defined, because both components have the same degree, so

$$f((x:y)) = f((\lambda x : \lambda y))$$

for all $\lambda \in \mathbb{R}$.
Suppose we consider our map $\phi_{x=1}$, which induces an inclusion $\mathbb{R} \subset \mathbb{P}^1$, then the map $f$ restricted to this copy of $\mathbb{R}$, and considered as a map to $\mathbb{R}$, becomes

$$f_{x=1}(y) = \frac{1+y^2}{3+y}.$$

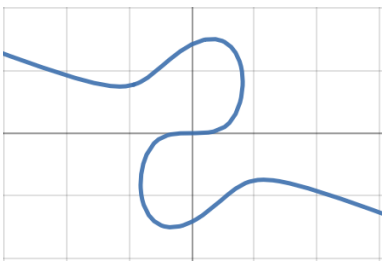
If we restrict to $y = 1$, we get

$$f_{y=1}(x) = \frac{x^2+1}{3x^2+x}.$$

This is equivalent to a change of coordinates $x = \frac{1}{y}$, but in projective space, we can view both cases at once, by writing points as ratios, $(x:y)$, and giving the function in the form $(\bigstar)$, which turns out to be very useful. Similar algebra can be done on $\mathbb{P}^n$ for any positive integer $n$. With this language, "solutions at infinity" can be precisely described.
**Example:** The following two (elliptic) curves are both just different projections of the same projective curve in $\mathbb{P}^2$ to different planes. Changing view points makes the equation easier to solve.

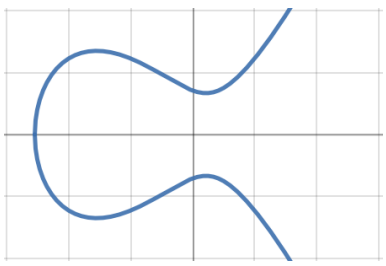
$$X^3 + 2X^2Z - XZ^2 + Z^3 = 2ZY^2$$

Projection to the $Y = 1$ plane;
substitute $z = Z/Y, x = X/Y$



$$x^3 + 2x^2z - xz^2 + z^3 = 2z$$

Projection to the $Z = 1$ plane;
substitute $x = X/Z, y = Y/Z$



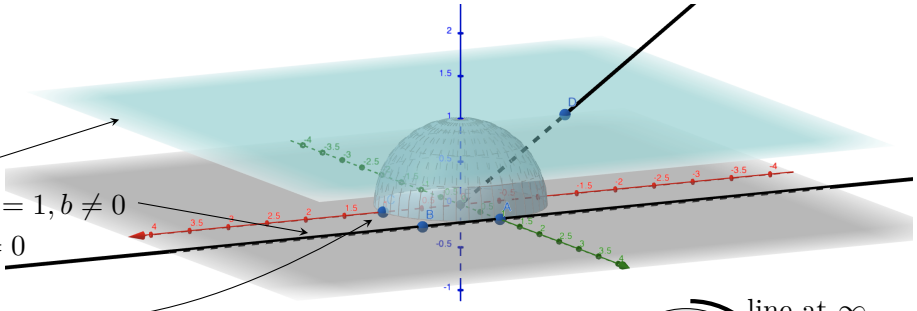
$$x^3 + 2x^2 - x + 1 = 2y^2$$

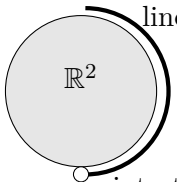
## 38. Topology of the projective plane $\mathbb{P}^2(\mathbb{R})$

This section is non-examinable, just here to give more intuition about the projective plane.
Every point in $\mathbb{P}^2$ can be represented by a point on $S^2$, but this includes each point twice. So we only need the upper hemisphere $\mathbb{Z} \geq 0$. And we only need one half of the $S^1 = S^2 \cap \{z = 0\}$.
Here is a Geogebra picture `https://www.geogebra.org/3d/katc2xrq`

Every point $(a : b : c)$ in $\mathbb{P}^2$ is represented by unique point on either

$$\begin{cases} \text{the plane} & Z = 1, \text{ if } c \neq 0 \\ \text{the line} & Z = 0, Y = 1 \text{ if } c = 1, b \neq 0 \\ \text{the point} & (1, 0, 0) \text{ if } b = c = 0 \end{cases}$$






$$\mathbb{P}^2 \cong \overbrace{\{Z = 1\}}^{\mathbb{R}^2} \cup \overbrace{\{Z = 0, Y = 1\}}^{\mathbb{R}} \cup \overbrace{\{(1, 0, 0)\}}^{\text{point}}$$

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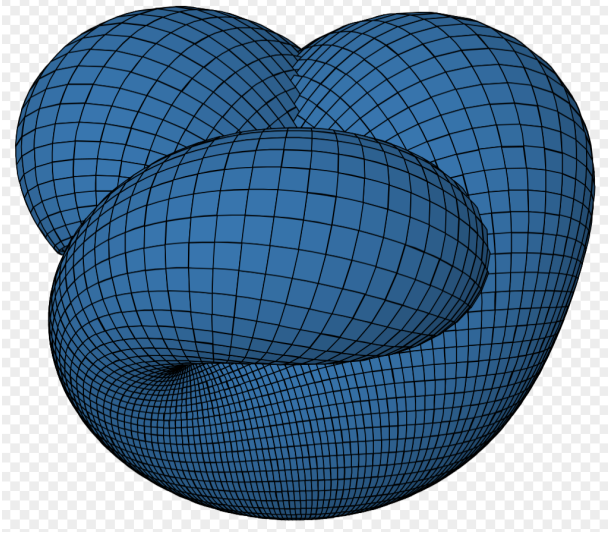
If you take the hemisphere, and distort it to "glue together" points on opposite side of the circular boundary on $Z = 0$, which correspond to the same line in $\mathbb{R}^3$, you get a closed surface. This surface only can be viewed in $\mathbb{R}^3$ if it crosses itself. One model is called "Boy's surface":



This is a picture from Wikipedia, by Ggleizer `https://commons.wikimedia.org/wiki/File:BoysSurfaceKusnerBry svg`. You can find lots of videos about this, e.g., at `https://www.youtube.com/watch?v=uiq-EcQz_uU`
This is not necessarily the most useful way to think about $\mathbb{P}^2$ for the purposes of algebraic computations.
You can also think of $\mathbb{P}^n$ as $\mathbb{R}^n$ with a point added at ininfty for every radial direction in $\mathbb{R}^n$ – same point added for both $\mathbf{v}$ and $-\mathbf{v}$.

## 39. Lecture 23: Lines and linear subspaces in projective geometry

For a vector space $V$, over a field $k$, there is a projection map from $V^* = V \setminus \{\mathbf{0}\}$ to $\mathbb{P}(V)$ given by

$$\pi : V^* \to \mathbb{P}(V)$$

$$\mathbf{v} \mapsto k\mathbf{v}$$

DEFINITION 40. *If $V$ is a vector space $V$, with a $k+1$ dimensional vector subspace $W$, then the $k$-dimensional projective linear subspace $\mathbb{P}(W)$ of $\mathbb{P}(V)$ is the image under $\pi$ of $W$ in $\mathbb{P}(V)$, that is*

$$\mathbb{P}(W) := \{\mathbf{x} \in \mathbb{P}(V) : \mathbf{x} \subset W\} = \pi(W)$$

**Warning:** the dimension of $\mathbb{P}(W)$ is one less than the dimension of $W$.

DEFINITION 41. *Let $W$ be a subvector space of $V$.*

$$\mathbb{P}(W) \text{ is a } \begin{cases} \text{point} & \text{if } \dim(W) = 1 \\ \text{line} & \text{if } \dim(W) = 2 \\ \text{plane} & \text{if } \dim(W) = 3 \end{cases}$$

## 40. The Fano plane

In this section, we discuss the smallest possible projective plane.

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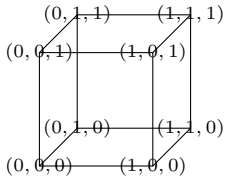
Let $V$ be the vector space $(\mathbb{F}_2)^3$ over the finite field of two elements, $\mathbb{F}_2 = \{0, 1\}$. What are the points and lines in $\mathbb{P}(V)$?

The elements of $V$ are $(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)$, which can be pictured as points at the vertices of a cube:

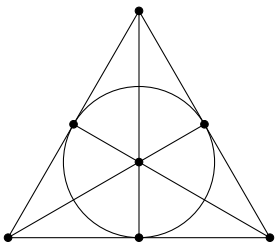



The points of $\mathbb{P}(V)$ are given by the one dimensional subspaces of $V$, which have the form $\mathbb{F}_2\mathbf{v} = \{0, \mathbf{v}\}$ for non zero $\mathbf{v}$. So there are 7 of these.

Lines are given by the two dimensional subspaces. Each of these has the form $\{0, \mathbf{v}, \mathbf{w}, \mathbf{v}+\mathbf{w}\}$ for some $\mathbf{v}, \mathbf{w} \in V$. For example, the plane $P = \{(0,0,0), (0,1,0), (1,0,1), (1,1,1)\}$. So, there are $\binom{7}{2}$ ways to choose two points $\mathbf{w}$ and $\mathbf{v}$, but any three non-zero points in $P$ span $P$, so this would count the planes 3 times, so the number of planes is $\frac{1}{3}\binom{7}{2} = 7$.

It's not a coincidence that this is the same as the number of points, since each plane can also be defined by the vector it is orthogonal to, e.g., $P$ above is orthogonal to $(1,0,1)$.

We can draw a diagram showing the points and lines as follows. Exercise: add labels for the points, with a vector in $V$ which generates each point in $\mathbb{P}(V)$. This is called the **Fano plane**.



**40.1. Further details on the Fano plane.** Since students may not be familiar with working over finite fields, here are a few more details.

The finite field with 2 elements can be thought of as $\{0, 1\}$, where multiplication and addition are given modulo 2, e.g., $1 + 1 = 0$.

**Lemma:** In

$$(\mathbf{F}_2)^3 = \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 = \{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$$

any one dimensional vector subspace is spanned by one element, and only contains one non-zero element.
**Proof**: this is because by definition, any one dimensional vector subspace is all the possible elements given by linear combination of $\mathbf{v}$ for some non-zero vector $\mathbf{v}$. All the vectors except $(0,0,0)$ are non-zero, so we have 7 non-zero vectors.
There are only two possible multiples of $\mathbf{v}$, that is, using $\langle\mathbf{v}\rangle$ to denote all vectors in the space spanned by $\mathbf{v}$, we have

$$\langle\mathbf{v}\rangle = \{\lambda\mathbf{v} : \lambda \in \mathbb{F}_2\} = \{\lambda\mathbf{v} : \lambda \in \{0,1\}\} = \{\mathbf{0},\mathbf{v}\}.$$

□

So, this means that the one dimensional vector subspaces (lines through the origin in $(\mathbf{F}_2)^3$ are

$$\{(0,0,0),(1,0,0)\}$$
$$\{(0,0,0),(0,1,0)\}$$
$$\{(0,0,0),(0,0,1)\}$$
$$\{(0,0,0),(1,1,0)\}$$
$$\{(0,0,0),(1,0,1)\}$$
$$\{(0,0,0),(0,1,1)\}$$
$$\{(0,0,0),(0,1,1)\}$$

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Since by definition points in $\mathbb{P}^2(\mathbb{F}_2)$ are lines in $(\mathbb{F}_2)^3$, there are seven points in $\mathbb{P}^2(\mathbb{F}_2)$, given by:

$$(0:0:1),(0:1:0),(0:1:1),(1:0:0),(1:0:1),(1:1:0),(1:1:1).$$

A line in $\mathbb{P}^2(\mathbb{F}_2)$ is a plane in $(\mathbb{F}_2)^3$.
A plane is by definition a two dimensional subvector space. A subvector space is a space closed under addition and scalar multiplication. The dimension is the number of elements in a basis. So, if we take two linearly independent vectors, which just means non-zero and different in $(\mathbb{F}_2)^3$, and take their span, we get a subspace of $(\mathbb{F}_2)^3$. So, since there are 7 vectors, we get 7 choose $2 = \frac{7!}{2!5!} = 21$ choices. These 21 choices sometimes give the same subvector space, because we can choose a different basis. If $\mathbf{v},\mathbf{w}$ are linearly independent vectors in $(\mathbb{F}_2)^3$, then their span is:

$$\langle\mathbf{v},\mathbf{w}\rangle = \{\mathbf{0},\mathbf{v},\mathbf{w},\mathbf{v}+\mathbf{w}\}$$

Unfortunately the span notation is the same as the inner product notation, so we have to deduce from context that here it means the span.
Now since we are working modulo 2, we have $(\mathbf{v}+\mathbf{w})+\mathbf{w} = \mathbf{v}$, so $\mathbf{v}$ is in the span of $\mathbf{v}+\mathbf{w}$ and $\mathbf{w}$, so $\mathbf{v}+\mathbf{w}$ is another choice of basis. We have

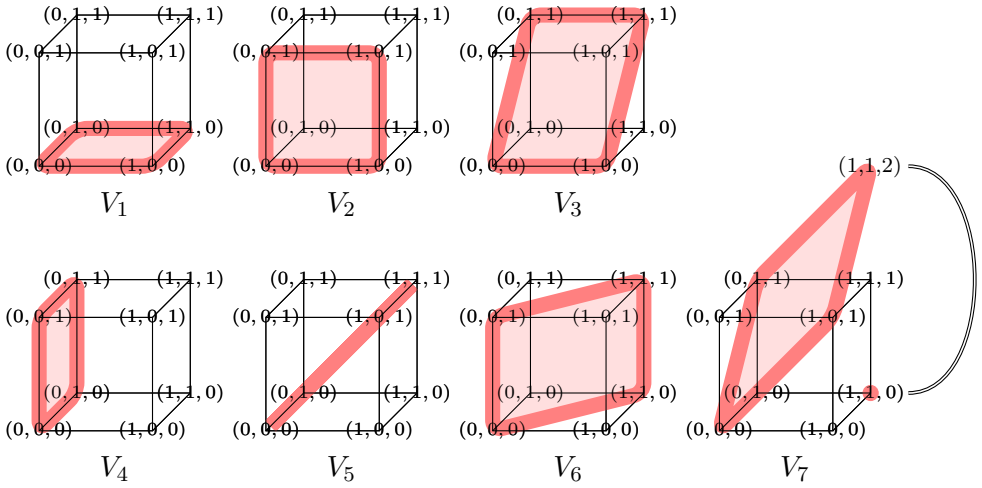
$$\langle\mathbf{v},\mathbf{w}\rangle = \langle\mathbf{v},\mathbf{w}+\mathbf{v}\rangle = \langle\mathbf{v}+\mathbf{w},\mathbf{w}\rangle$$

This means we have three possible choices of basis for each 2-dimensional subvector space of $(\mathbb{F}_2)^3$. These subspaces are:

$$\begin{array}{lllllllll}
V_1 = \langle(1,0,0),(1,1,0)\rangle & = & \langle(1,0,0),(0,1,0)\rangle & = & \langle(1,1,0),(1,1,0)\rangle & = & \{(0,0,0),(1,0,0),(1,1,0),(0,1,0)\} \\
V_2 = \langle(1,0,0),(0,0,1)\rangle & = & \langle(1,0,0),(1,0,1)\rangle & = & \langle(1,0,1),(0,0,1)\rangle & = & \{(0,0,0),(1,0,0),(1,0,1),(1,0,1)\} \\
V_3 = \langle(1,0,0),(1,1,1)\rangle & = & \langle(1,0,0),(0,1,1)\rangle & = & \langle(1,1,1),(0,1,1)\rangle & = & \{(0,0,0),(0,1,1),(1,1,1),(1,0,0)\} \\
V_4 = \langle(0,1,0),(0,1,1)\rangle & = & \langle(0,0,1),(0,1,0)\rangle & = & \langle(0,0,1),(0,1,1)\rangle & = & \{(0,0,0),(0,0,1),(0,1,1),(0,1,0)\} \\
V_5 = \langle(0,1,0),(1,0,1)\rangle & = & \langle(0,1,0),(1,1,1)\rangle & = & \langle(1,1,1),(1,0,1)\rangle & = & \{(0,0,0),(1,0,1),(1,1,1),(0,1,0)\} \\
V_6 = \langle(0,0,1),(1,1,0)\rangle & = & \langle(1,1,1),(0,0,1)\rangle & = & \langle(1,1,1),(1,1,0)\rangle & = & \{(0,0,0),(1,1,0),(0,0,1),(1,1,1)\} \\
V_7 = \langle(1,0,1),(0,1,1)\rangle & = & \langle(1,0,1),(1,1,0)\rangle & = & \langle(1,1,0),(0,1,1)\rangle & = & \{(0,0,0),(1,0,1),(0,1,1),(1,1,0)\}
\end{array}$$

These planes in $\mathbb{F}_2^3$ can be depicted as follows. This picture is really a picture in $\mathbb{R}^3$. Actually it's really a projection of a picture in $\mathbb{R}^3$ to a picture in $\mathbb{R}^2$, since your screen or paper is flat. However, the relationships between points in the planes can be seen, e.g., you can locate spanning vectors etc, and check the relationships between them. I have drawn the case $V_7$ slightly differently, so it looks like a

plane, using the fact that $(1,1,2) = (1,1,0)$ in $\mathbb{F}_2^3$. Note that there are other planes in $\mathbb{F}_2^3$, but these are all the planes that pass through the origin; these are all the 2-dimensional vector subspaces.




These can also be considered as the planes orthgonal to a given vector. We have

$$V_1 = (\mathbb{F}_2(0,0,1))^\perp$$
$$V_2 = (\mathbb{F}_2(0,1,0))^\perp$$
$$V_3 = (\mathbb{F}_2(0,1,1))^\perp$$
$$V_4 = (\mathbb{F}_2(1,0,0))^\perp$$
$$V_5 = (\mathbb{F}_2(1,0,1))^\perp$$
$$V_6 = (\mathbb{F}_2(1,1,0))^\perp$$
$$V_7 = (\mathbb{F}_2(1,1,1))^\perp$$

where $\mathbb{F}_2\mathbf{v} = \{\mathbf{0}, \mathbf{v}\}$, and $V^\perp = \{\mathbf{x} \in \mathbb{F}_2^3 : \langle \mathbf{x}, \mathbf{v}\rangle = 0 \; \forall \mathbf{v} \in V\}$, so $(\mathbb{F}_2\mathbf{v})^\perp = \{\mathbf{x} \in \mathbb{F}_2^3 : \langle \mathbf{v}, \mathbf{x}\rangle = 0\}$.

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Definition 42. *For a vector space $V$ and a subset $\Sigma \subset \mathbb{P}(V)$, the projective cone of $\Sigma$ is given by*

$$\widetilde{\Sigma} := \bigcup_{\mathbf{v} \in \Sigma} \mathbf{v}$$

*The linear span or span of $\Sigma$ is is the smallest projective linear subspace of $\mathbb{P}(V)$ containing $\Sigma$,*

$$\langle \Sigma \rangle := \mathbb{P}(span(\widetilde{\Sigma})),$$

*For a point $P \in \mathbb{P}(V)$, $\widetilde{P}$ is only defined up to direction, not magnitude, and given by:*

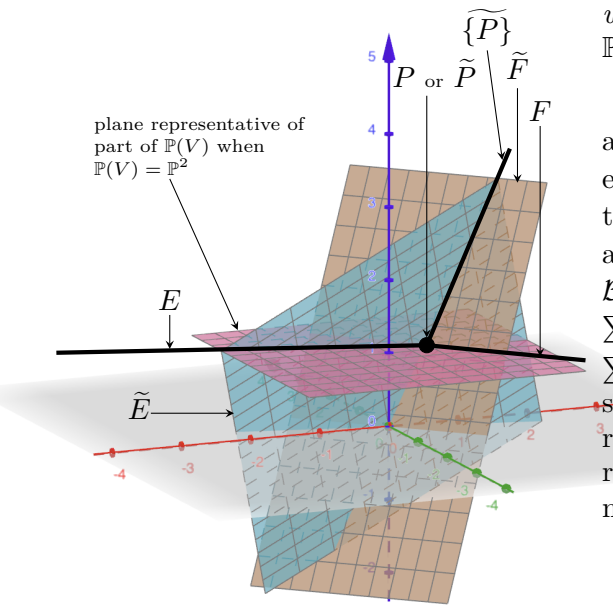
$$\widetilde{P} = \text{any choice of non zero } \mathbf{v} \in P$$

*For $s+1$ points $P_0, P_1, \ldots, P_s$ in $\mathbb{P}(V)$, the dimension of $\langle \{P_0, P_1, \ldots, P_s\} \rangle$ is at most $s$, (by definition of projective dimension).*
*We say $P_0, P_1, \ldots, P_s$ are linearly independent if $\dim \langle \{P_0, P_1, \ldots, P_s\} \rangle = s$.*

**Exercise:** $\pi(\widetilde{\Sigma}) = \Sigma$. Does $\widetilde{\pi(\mathbf{x})} = \mathbf{x}$? Does $\widetilde{P} = \widetilde{\{P\}}$?
**Quiz:** What is the maximum size of a linearly independent set of points in $\mathbb{P}^2$?
**Quiz:** True or false: If $E$ and $F$ are projective linear subspaces of $\mathbb{P}(V)$, then so is $E \cup F$?

Definition 43. *If $P$ and $Q$ are two distinct points in $\mathbb{P}(V)$, then their span $\langle P, Q\rangle$ is the unique projective line containing $P$ and $Q$.*

**Exercise:** $P_0, P_1, \ldots, P_s$ in $\mathbb{P}(V)$ are linearly independent if and only if $\widetilde{P_0}, \ldots, \widetilde{P_s}$ are linearly independent in $V$.




THEOREM 12. *For projective linear subspaces $E$ and $F$ of $\mathbb{P}(V)$,*

$$\dim(E \cap F) = \dim E + \dim F - \dim \langle E, F \rangle$$

*with the convention $\dim \varnothing = -1$, and $\langle E, F \rangle$ is the linear subspace of $\mathbb{P}(V)$ spanned by $E$ and $F$.*

PROOF. Let $n = \dim E$ and $m = \dim F$. Then $\dim \widetilde{E} = n + 1$ and $\dim \widetilde{F} = m + 1$. We also have $\widetilde{E \cap F} = \widetilde{E} \cap \widetilde{F}$ (proof: exercise). Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be a basis for $\widetilde{E} \cap \widetilde{F}$. Then we can extend this to a basis $\mathcal{B}_E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_{n+1}\}$ of $\widetilde{E}$, and a basis $\mathcal{B}_F = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_{m+1}\}$ of $\widetilde{F}$. The vectors in $\mathcal{B} := \{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_{m+1}\}$ clearly span $\widetilde{E} \cup \widetilde{F}$. Suppose $\sum_{i=1}^{n+1} a_i \mathbf{v}_i + \sum_{i=k+1}^{m+1} b_i \mathbf{v}_i = \mathbf{0}$, then $\sum_{i=1}^{n+1} a_i \mathbf{v}_i = -\sum_{i=k+1}^{m+1} b_i \mathbf{w}_i$, so since $\sum_{i=1}^{n+1} a_i \mathbf{v}_i \in \widetilde{E}$ and $\sum_{i=k+1}^{m+1} b_i \mathbf{w}_i \in \widetilde{F}$, we have $\sum_{i=k+1}^{n+1} b_i \mathbf{w}_i \in \widetilde{E} \cap \widetilde{F}$, and so can be written in terms of the basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$, but since this gives a relationship between elemtents of the basis $\mathcal{B}_F$, this must be the trivial relationship, since $\mathcal{B}_F$ is a basis, so all the coefficients must be zero. So $\mathcal{B}$ must be a basis for $\operatorname{span}(\widetilde{E}, \widetilde{F})$, and so

$$\dim(\operatorname{span}(\widetilde{E}, \widetilde{F})) = \overbrace{(n+1)}^{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}} + \overbrace{(m+1) - (k+1)}^{\mathbf{w}_{k+1}, \ldots, \mathbf{w}_{m+1}}$$

$$\Rightarrow \dim \langle E, F \rangle + 1 = (\dim E + 1) + (\dim F + 1) - (\dim E \cap F + 1)$$

and so the result follows. □

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**Quiz:** What is the corresponding result if $E$ and $F$ are replaced by vector subspaces $U$ and $V$ of $\mathbb{R}^n$?
**Example:** If $E = L_1$ and $F = L_2$ are lines in $\mathbb{P}^2$, then they have projective dimension 1. In general, their intersection is a point $P = L_1 \cap L_2$ , which has projective dimension 0. The union spans all of $\mathbb{R}^3$, which has dimension 2, so, provided $E$ and $F$ are distinct lines, we get the expected result, as shown in the above figure.

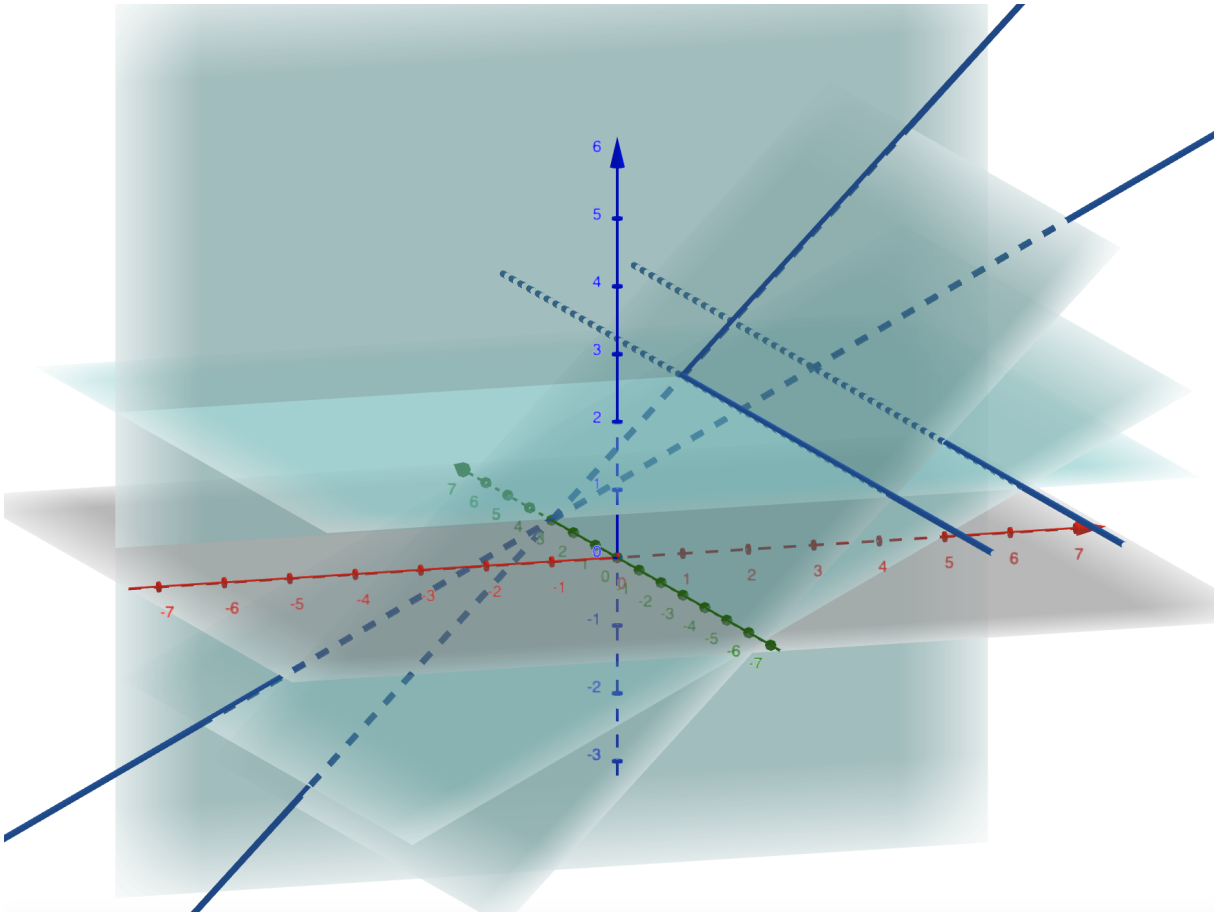
$$\overbrace{\dim(L_1 \cap L_2)}^{0} = \overbrace{\dim L_1}^{1} + \overbrace{\dim L_2}^{1} - \overbrace{\dim \langle L_1, L_2 \rangle}^{2}$$

In the figure,(`https://www.geogebra.org/3d/vry2xuh5`) $E$ is a line in $\mathbb{P}^2$, so it's drawn as a line in a plane representing $\mathbb{P}^2$, for psychological reasons. This is similar to how when we work with numbers modulo 5, we write things like 3, when really we mean $\{5\lambda + 3 : \lambda \in \mathbb{Z}\}$

THEOREM 13. *Any two distinct lines $L_1$ and $L_2$ in $\mathbb{P}^2$ intersect in a point.*

PROOF. The lines $L_1$ and $L_2$ are projections of planes through the origin, $\widetilde{L_1}$ and $\widetilde{L_2}$ in $\mathbb{R}^3$, to $\mathbb{P}^2$. Since $\widetilde{L_1}$ and $\widetilde{L_2}$ have dimension 2, and are distinct, their intersection must have dimension 1, by Theorem 12, since the dimension of the span of $\widetilde{L_1}$ and $\widetilde{L_2}$ is 3, since it can't be greater, since it's in $\mathbb{R}^3$, and can't be less, since $L_1 \neq L_2$. So the dimension of $L_1 \cap L_2$ is zero, and so $L_1 \cap L_2$ is a point. □

Theorem 13 shows that lines always meet in $\mathbb{P}^2$. This can also be seen as a conseqence of the theorem we proved that spherical lines always intersect, since $\mathbb{P}^2$ can be considered to be the sphere $S^2$ with antipodal points identified. We can also consider parallel lines in $\mathbb{R}^2$ considered as a subset of $\mathbb{P}^2$ to meet in the "line at infinity", which can be seen by projecting $\mathbb{P}^2$ to a different plane: (`https://www.geogebra.org/3d/gnkzhywe`)



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In this figure, the lines $x = 1$ and $x = 2$ are parallel, drawn on a copy of $\mathbb{R}^2$ which is the plane $Z = 1$ in $\mathbb{R}^3$ in this figure. $x, y$ are the coordinates on this plane. These lines are (radial) projections of the planes $X = Z$ and $X = 2Z$ in $\mathbb{R}^3$, where $\mathbb{R}^3$ has coordinates $X, Y, Z$. These planes project to $x = z$ and $x = 2z$ on the plane $Y = 1$, where they meet at $(x, y) = (0, 0)$.
Similarly, lines $y = mx + c$ for fixed $m$, and varying $c$, are all parallel in the plane $Z = 1$. These lines are projections of $Y = mX + cZ$ in $\mathbb{R}^3$. If we project to $X = 1$, we get lines $y = m + cz$, which all meet at $(y, z) = (m, 0)$. You may also wish to consider these lines as projections to the sphere.

## 41. Lecture 24: Projective transformations and projective frames of reference

DEFINITION 44. *The projective general linear group of a vector space $V$ over a field $k$ is the group of all invertible linear maps from $T : V \to V$ upto scalar multiplication, that is, we consider $T_1 \sim T_2$ if there is some non-zero element $\lambda \in k$ with $T_2 = \lambda T_1$.*
*We write $PGL(n) = PGL(\mathbb{R}^n)$. In this case, the elements of $PGL(n)$ are invertible matrices up to scalar muliplication, that is*

$$A \sim B \iff A = \lambda B \text{ for some } \lambda \in \mathbb{R}^*$$

Example, in $PGL(2)$, we have

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

DEFINITION 45. *If $A$ is an invertible $(n+1) \times (n+1)$ matrix, then the map*

$$T_A : \mathbb{P}^n \to \mathbb{P}^n$$

$$[\mathbf{v}] \mapsto [A\mathbf{v}]$$

*is called a projective transformation, or a projectivity or a projective linear map.*
*More generally, for any invertable linear map $T$ between any vector spaces $V$ and $W$, the induced map $T : [\mathbf{v}] \mapsto [T(\mathbf{v})]$ is called a projective linear map. I.e., the definition is basis independent, and doesn't whether the field is the reals.*

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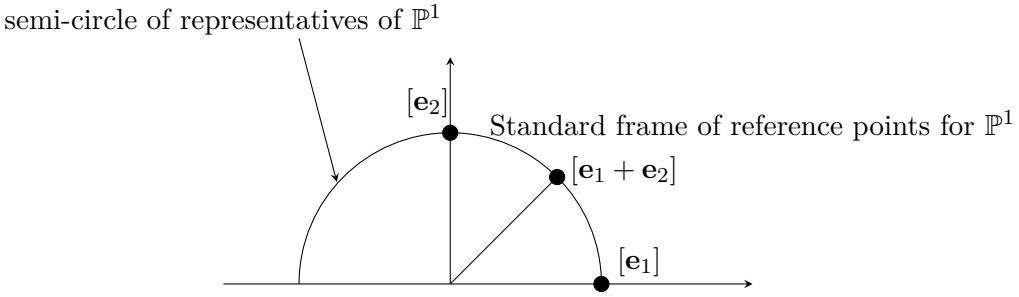
$T_A$ is well defined, since for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1}$, if $\mathbf{x} \sim \mathbf{y}$, that is, if $[\mathbf{x}] = [\mathbf{y}]$, then $\mathbf{x} = \lambda\mathbf{y}$ for some $\lambda \in \mathbb{R}$, so $A\mathbf{x} = A\lambda\mathbf{y} = \lambda A\mathbf{y}$, so $[A\mathbf{x}] = [A\mathbf{y}]$.
Also, if $A \sim B$, then $T_A = T_B$, since if $A = \lambda B$, then $[A\mathbf{x}] = [\lambda A\mathbf{x}] = [B\mathbf{x}]$. So, projective linear maps of $\mathbb{P}^n$ correspond to elements of $PGL(n+1)$.

DEFINITION 46. *A projective frame of reference for $\mathbb{P}^n$ is an ordered set of $n+2$ points, $P_0, P_1, \ldots, P_{n+1} \in \mathbb{P}^n$, any $n+1$ of which are linearly independent (and so span $\mathbb{P}^n$)*

DEFINITION 47. *The standard frame of reference for $\mathbb{P}^n$ is given by $[\mathbf{e}_1], \ldots, [\mathbf{e}_{n+1}]$ together with $[\mathbf{e}_1 + \cdots + \mathbf{e}_{n+1}]$, where $\mathbf{e}_i$ are the standard basis for $\mathbb{R}^{n+1}$.*




**Example:** Any three distinct points $P_1, P_2, P_3$ in $\mathbb{P}^1$ is a frame of reference for $\mathbb{P}^1$, since being distinct means that $\widetilde{P_1}$, $\widetilde{P_2}$, $\widetilde{P_3}$ are pairwise linearly independent.
**Motivation:** Why do we need $n+2$ points? This is because $n+2$ points is enough to determine a projecive linear map of $\mathbb{P}^n$, but $n+1$ is not. This might be suprising, since the images of $n$ points is enough to determine a linear map on $\mathbb{R}^n$.
**Example:** There are infinitely many projective linear maps $\mathbb{P}^1 \to \mathbb{P}^1$ with

$$(1:0) \mapsto (1:0)$$
$$(0:1) \mapsto (0:1)$$

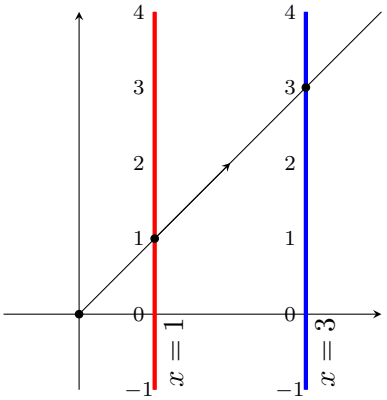
namely, any diagonal map, of the form

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

However, these maps are not all the same map, since we have

$$A : (1:0) \mapsto (1:0)$$
$$A : (0:1) \mapsto (0:1)$$
$$A : (1:1) \mapsto (\alpha:\beta)$$

If we fix the image of $(1:1)$, then we uniquely determine the projective linear map.




**Example:** The radial projection, with centre **0**, from the line $x = 1$ to the line $x = 3$ takes a point with $y = 1$ to a point with $y = 3$. We can paramaterise points on these lines by their $y$-coordinates. If we complete these lines to form copies of $\mathbb{P}^1$, with point with parameter $y$ maping to $(y:1)$, then the map is given by $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{P}^1 \to \mathbb{P}^1$

**Example:** The map $t \mapsto \frac{1}{t}$ from $\mathbb{R}$ to $\mathbb{R}$ given in Lecture 22 can be considered as a map on $\mathbb{P}^1$. On $\mathbb{R} \subset \mathbb{P}^1$, given by

$$\left[\begin{pmatrix} t \\ 1 \end{pmatrix}\right] \mapsto \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ t \end{pmatrix}\right],$$

extended to a map on $\mathbb{P}^1$ by allowing the map to apply to $(1:0)$. This map has the effect of switching 0 and $\infty$, thought of as the points $(0:1)$ and $(1:0)$.

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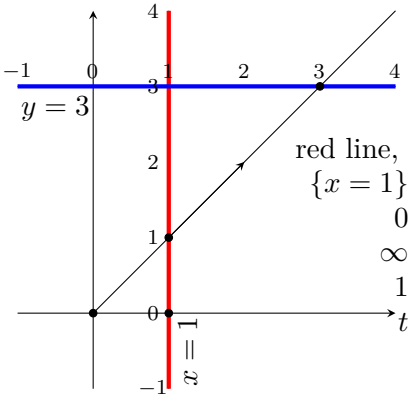
Any map $\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} : \mathbb{P}^1 \to \mathbb{P}^1$, for $\alpha, \beta \neq 0$ also switches $(0:1)$ and $(1:0)$, and corresponds to $t \mapsto \frac{\alpha}{\beta t}$.

By specifying the image of $(1:1)$, the map is uniquely determined.

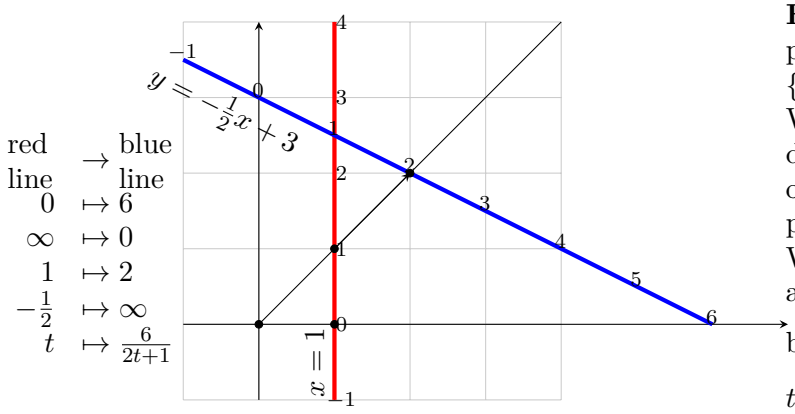
The diagram on the right shows a projection from $\{x = 1\}$ to $\{y = 3\}$, which corresponds to the map $\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} : \mathbb{P}^1 \to \mathbb{P}^1$ sending $(1:1) \mapsto (3:1)$




**Example:** In the figure on the right, we project from the line $\{x = 1\}$ to the line $\{y = -\frac{1}{2}x + 3\}$.

We parameterise $\{x = 1\}$ by the $y$ coordinate, and $\{y = -\frac{1}{2}x + 3\}$ by the $x$ coordinate, though we could choose different parameterisations.

With respect to this parameterisation, as a map from $\mathbb{P}^1$ to $\mathbb{P}^1$, this map is given by the matrix $\begin{pmatrix} 0 & 6 \\ 2 & 1 \end{pmatrix}$, and corresponds to $t \mapsto \frac{6}{2t+1}$ as a map on $\mathbb{R}$




The maps in these examples have been written as maps from $\mathbb{P}^1$ to $\mathbb{P}^1$, but drawn as maps from $\mathbb{R} \to \mathbb{R}$. To extend from $\mathbb{R}$ to $\mathbb{P}^1$, we should embed each copy of $\mathbb{R}$ in a copy of $\mathbb{P}^1$, which can be acheived by replacing $\mathbb{R}^2$ in these figures by $\mathbb{P}^2$. Then these are called projective perspectives, which we will come to next lecture.

THEOREM 14. *There is a bijection between projective transformations of $\mathbb{P}^n$ and projective frames of reference of $\mathbb{P}^n$, defined as follows:*

$$\begin{aligned}\phi : PGL(n+1) &\to \{\textit{projective frames of reference}\}\\ T &\mapsto \{T([\mathbf{e}_1]), T([\mathbf{e}_2]), \ldots, T([\mathbf{e}_{n+1}]), T([\mathbf{e}_1 + \cdots + \mathbf{e}_{n+1}])\}\end{aligned}$$

PROOF. We need to show that this map is well defined, injective and surjective.
Well defined: The image of the standard projective frame of reference under $T$ must also be a projective frame of reference, because $T$ is invertible, so maps any basis of $\mathbb{R}^{n+1}$ to another basis.
Surjectivity: Suppose we are given a projective frame of reference, $P_0, P_1, \ldots, P_{n+1}$. By definition, any choice of $n+1$ of these these are linearly independent in $\mathbb{P}^n$, which means that a choice of lifts, $\mathbf{a}_0 = \widetilde{P_0}, \mathbf{a}_1 = \widetilde{P_1}, \ldots \mathbf{a}_n = \widetilde{P_n}$ is linearly independent in $\mathbb{R}^{n+1}$, and so for any nonzero $\alpha_0, \ldots, \alpha_n$, the matrix $A$ with columns $\alpha_i \mathbf{a}_i$, is invertible.

$$A = \begin{pmatrix} | & | & \ldots & | \\ \alpha_0\mathbf{a}_0 & \alpha_1\mathbf{a}_1 & \ldots & \alpha_n\mathbf{a}_n \\ | & | & \ldots & | \end{pmatrix}$$

We know that $A\mathbf{e}_i$ is the $i$th column of $A$, and so for $i = 0, \ldots, n$,

$$[T_A(\mathbf{e}_i)] = [A\mathbf{e}_i] = [\alpha_i\mathbf{a}_i] = [\mathbf{a}_i] = P_i$$

We want

$$\left[T_A\left(\sum_{i=1}^{n} \mathbf{e}_i\right)\right] = P_{n+1}.$$

Let $\mathbf{a}_{n+1} = \widetilde{P_{n+1}}$, so $P_{n+1} = [\mathbf{a}_{n+1}]$. We have $[T_A(\sum_{i=1}^n \mathbf{e}_i)] = [A(\sum_{i=1}^n \mathbf{e}_i)] = [\sum_{i=1}^n \alpha_i\mathbf{a}_i]$, so it's sufficient to solve $\sum_{i=1}^n \alpha_i\mathbf{a}_i = \mathbf{a}_{n+1}$, i.e., $A(\alpha_0, \alpha_1, \ldots, \alpha_n)^T = \mathbf{a}_{n+1}$. Since $A$ is invertible, set

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = A^{-1}\mathbf{a}_{n+1}.$$

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Since $\mathbf{a}_0, \ldots, \mathbf{a}_{n+1}$ is a projective frame of reference, all $\alpha_i$ in this solution must be non-zero, since otherwise there would be a set of $n$ vectors from $\mathbf{a}_0, \ldots, \mathbf{a}_n$ which was not linearly independent. Hence $A \in PGL(n+1)$. So $\phi(T_A) = \{P_0, \ldots, P_{n+1}\}$. Thus $\phi$ is surjective.
Injectivity:
Suppose that $\phi(T_A) = \phi(T_B)$ for matrices $A, B \in PGL(n)$. Then for $i = 0, \ldots, n$, we have

$$T_A(\mathbf{e}_i) = T_B(\mathbf{e}_i),$$

so the columns of $B$ are multiples of the columns of $A$, so for some diagonal matrix $\mathrm{Diag}(\beta_0, \ldots, \beta_n)$,

$$B = A \cdot \mathrm{Diag}(\beta_0, \ldots, \beta_n).$$

Since $\phi(T_A) = \phi(T_B)$, we have $[T_A(\sum \mathbf{e}_i)] = [T_B(\sum \mathbf{e}_i)]$ so

$$\left[A\sum \mathbf{e}_i\right] = \left[B\sum \mathbf{e}_i\right] = \left[A\sum \beta_i\mathbf{e}_i\right].$$

We know that $A$ is invertible, so we can't have $A\sum \mathbf{e}_i = \mathbf{0}$, since then we would have $\sum \mathbf{e}_i = A^{-1}\mathbf{0} = \mathbf{0}$, but we know that $\sum \mathbf{e}_i \neq \mathbf{0}$. So for some scalar $\lambda$,

$$\lambda A\sum \mathbf{e}_i = A\sum \beta_i\mathbf{e}_i$$

rearranging, $A\sum(\lambda - \beta_i)\mathbf{e}_i = \mathbf{0}$ and since $A$ is invertible, $\sum(\lambda - \beta_i)\mathbf{e}_i = \mathbf{0}$ which is only possible if $\lambda = \beta_i$ for each $i = 0, \ldots, n$, which implies $B = \lambda A$, and so $T_A$ and $T_B$ are equal as elements of $PGL(n)$. □

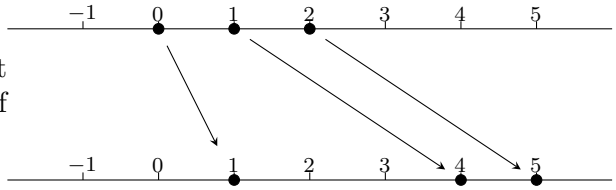
## 42. Lecture 25: Projectivities

COROLLARY 14. *Any three distinct points $P_1, P_2, P_3$ in $\mathbb{P}^1$ can be projected onto any other three distinct points $Q_1, Q_2, Q_3$. This is called* **three transitivity** *of the action of $PGL(2)$ on $\mathbb{P}^1$.*

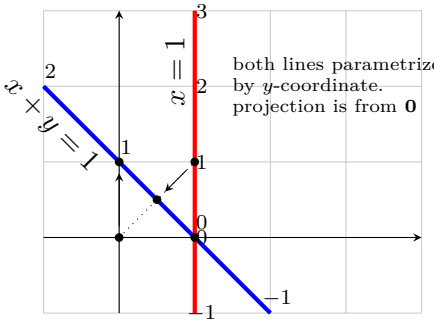
PROOF. By Theorem 14 there is some $A$ maping the standard frame of reference to $P_1, P_2, P_3$, and $B$ maps the standard frame of reference to $Q_1, Q_2, Q_3$, then $BA^{-1}$ maps $P_1, P_2, P_3$ to $Q_1, Q_2, Q_3$. □

**Example:**

**OK**, so how do I project between these set of points???




I am going to take $\mathbb{R}^1 \subset \mathbb{P}^1$ via $t \mapsto (1:t)$.
To project $[\mathbf{e}_1], [\mathbf{e}_2], [\mathbf{e}_1 + \mathbf{e}_2]$ (think "$0, \infty, 1$") to $(1:0), (1:1), (1:2)$, (think "0,1,2") we first take $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which gives $T_A([\mathbf{e}_1]) = (1:0)$, $T_A([\mathbf{e}_2]) = (1:1)$, but does not map $(1:1)$ to $(1:2)$.




| red line | → | blue line |
|---|---|---|
| 0 | ↦ | 0 |
| ∞ | ↦ | 1 |
| 1 | ↦ | $\frac{1}{2}$ |

Now take

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix},$$

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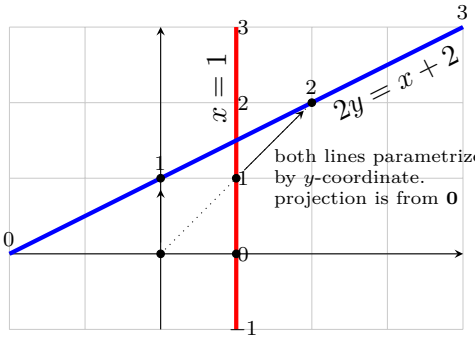
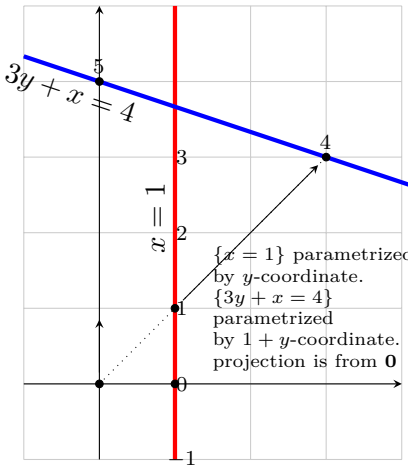
so the required element of $PGL(2)$ is

$$\begin{pmatrix} -1 & 2 \\ 0 & 2 \end{pmatrix}$$




| red line | → | blue line |
|---|---|---|
| 0 | ↦ | 0 |
| ∞ | ↦ | 1 |
| 1 | ↦ | 2 |




| red line | → | blue line |
|---|---|---|
| 0 | ↦ | −1 |
| ∞ | ↦ | 4 |
| 1 | ↦ | 5 |

To project $[\mathbf{e}_1], [\mathbf{e}_2], [\mathbf{e}_1 + \mathbf{e}_2]$ to $(1:1), (1:4), (1:5)$, we first take $A = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$, which gives $T_A([\mathbf{e}_1]) = (1:1)$, $T_A([\mathbf{e}_2]) = (1:4)$. Take the column multiplier:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix},$$

so the required element of $PGL(2)$ is $\begin{pmatrix} 1 & -4 \\ 1 & -16 \end{pmatrix}$

Note that we have ignored the determinant factor when computing the inverse of $A$, since elements of $PGL(2)$ are only defined up to multiplcation by non-zero scalar.

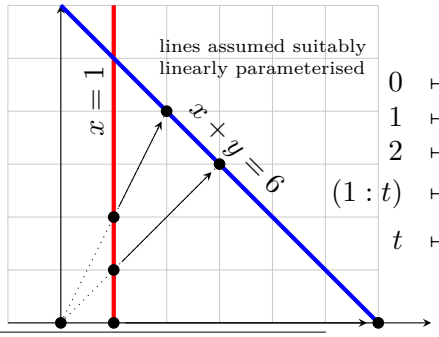
Now to map $(0:1), (1:1), (1:2)$ to $(1:1), (1:4), (1:5)$, we take

$$A = \begin{pmatrix} 1 & -4 \\ 1 & -16 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -4 \\ 1 & -16 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 14 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 7 \end{pmatrix}$$

Again, this is a computation in $PGL(2)$, so any scale factors of the whole matrix are ignored.
We can check the result, e.g.,

$$T_A((2:1)) = \left[\begin{pmatrix}1 & 1\\ 1 & 7\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}\right] = \left[\begin{pmatrix}3\\15\end{pmatrix}\right] = (1:5)$$

Note, although the element of $PGL(2)$ is unique, the picture is not unique, because we could choose different parameterisations. Finding a suitable geometric interpretation is a little more work.




**Example:** Find an element of $PGL(4)$ mapping the standard frame of reference of $\mathbb{P}^3$ to $(1:2:3:4), (0:1:0:1), (1:0:1:0), (1:1:1:0), (1:0:0:1)$
**Solution:** The required matrix must have the form

$$A = \begin{pmatrix}a & 0 & c & d\\ 2a & b & 0 & d\\ 3a & 0 & c & d\\ 4a & b & 0 & 0\end{pmatrix}$$

We need to solve $[A(1,1,1,1)^T] = (1:0:0:1)$, which we could solve by finding an inverse of a certain $4\times 4$ matrix, or by row reduction:

$$\left(\begin{array}{cccc|c}1 & 0 & 1 & 1 & 1\\ 2 & 1 & 0 & 1 & 0\\ 3 & 0 & 1 & 1 & 0\\ 4 & 1 & 0 & 0 & 1\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}-2 & 0 & 0 & 0 & 1\\ -2 & 0 & 0 & 1 & -1\\ 3 & 0 & 1 & 1 & 0\\ 4 & 1 & 0 & 0 & 1\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}-2 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & -2\\ 3 & 0 & 1 & 1 & 0\\ 0 & 1 & 0 & 0 & 3\end{array}\right) \Rightarrow \left(\begin{array}{cccc|c}1 & 0 & 0 & 0 & -\frac{1}{2}\\ 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 1 & 0 & \frac{7}{2}\\ 0 & 1 & 0 & 0 & 3\end{array}\right)$$

This has solution $(a:b:c:d) = (-\frac{1}{2}:3:\frac{7}{2}:-2) = (-1:6:7:-4)$. Hence the required matrix is

$$A = \begin{pmatrix}-1 & 0 & 7 & -4\\ -2 & 6 & 0 & -4\\ -3 & 0 & 7 & -4\\ -4 & 6 & 0 & 0\end{pmatrix}$$

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Any scalar multiple of this is also correct, as it will be equal to this matrix in $PGL(4)$.

---

**Example:** Considering $\mathbb{R}$ as a subset of $\mathbb{P}^1$, included as $\{(t:1) : t\in\mathbb{R}\} \subset \mathbb{P}^1$, we can ask how to extend a linear map on $\mathbb{R}$ to a projective linear map of $\mathbb{P}^1$. For example, $t \to t+a$ extends to:

$$(x:y) \to \left[\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\right] = \left[\begin{pmatrix}x+ay\\ y\end{pmatrix}\right] = (x+ay:y)$$

If $y=1$ and $x=t$, this becomes $(t:1)\to(t+a:1)$.
Projective linear maps of $\mathbb{P}^1$ become ratios of linear maps when restricted to $\mathbb{R}^1$.
**Example:** Consider $\mathbb{R}^2$ as a subset of $\mathbb{P}^2$, and the map

$$(x:y:z) \to \left[\begin{pmatrix}1 & 2 & 3\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\right] = (x+2y+3z : x+z : x+y)$$

The restriction to the copy of $\mathbb{R}^2$ given by setting $z=1$ is a map

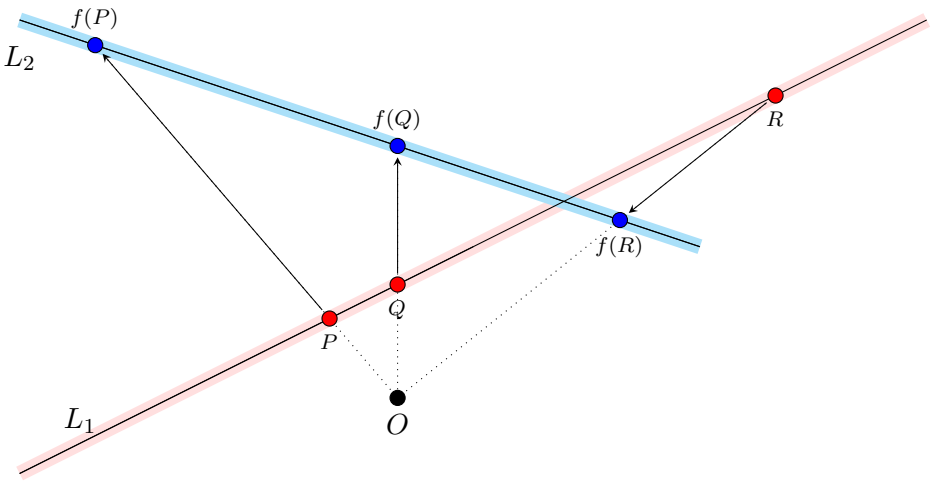
$$(x,y) \to \left(\frac{x+2y+3}{x+y}, \frac{x+1}{x+y}\right)$$

**Example:** Theorem 14 implies that we can project any quadrilateral on a plane in $\mathbb{R}^2$ to any other quadrilateral on some other plane in $\mathbb{R}^2$, provided they both are actual quadrilaterals, not triangles or lines or points, and that we can move the planes to suitable positions.

## 43. The cross ratio and projective perspectives

The projection maps we have been discussing are called "perspectivities".

DEFINITION 48. *A perpsectivity $f$ is a map between two distinct hyperplanes (linear subspaces of dimension $n-1$), $\Pi_1$ and $\Pi_2$ in $\mathbb{P}^n$, given by projection from a point $O \notin \Pi_1 \cup \Pi_2$. That is, if $P \in \Pi_1$, then $f(P) \in \Pi_2$ is the point such that $0, P, f(P)$ all lie on the same line.*




LEMMA 36. *Perspectivities are well defined.*

PROOF. We require in this definition that $f(P)$ exists and is unique. This follows from Theorem 12, with $E = L := \langle O, P\rangle$ and $F = \Pi_2$. We have $\dim(L) = 1$ and $\dim(\Pi_2) = n-1$. Since $O \notin \Pi_2$, we must have $\dim(\langle L, \Pi_2\rangle) = n$, so by the theorem,

$$\dim(L \cap \Pi_2) = 1 + (n-1) - n = 0$$

and so $L \cap \Pi_2$ is a single point, which we define to be $f(P)$. □

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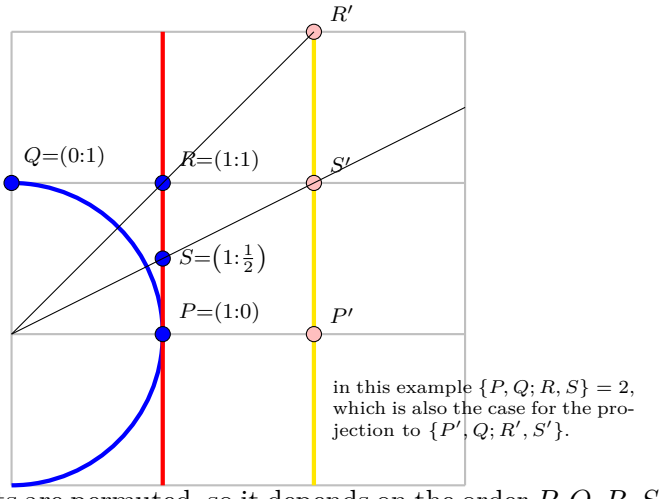
Corolary 14 shows that we can map any three distinct points on a projective line to any other three disctinct points, by a projective linear map. However, this is not true for four points on a line. We will show that the **cross ratio** of four points has to be the same for it to be possible to map a set of four points on a line to another set of four points on a line. So the cross ratio is an invariant of projective linear maps.

DEFINITION 49. *Let $P, Q, R, S$ be four distinct points on a line in $\mathbb{P}^n$. Choose an approrpiate basis such that $P = (1:0)$ and $Q = (0:1)$ (followed by $n-1$ zeros, which we can ignore). Since $\widetilde{P}$ and $\widetilde{Q}$ span the line that $R$ and $S$ lie on, and since $R, S \neq P$, there are $\lambda, \mu \in \mathbb{R}$ such that we can write with respect to this basis:*

$$R = (1:\lambda), \quad S = (1:\mu)$$

*Then the* **cross ratio** *is*

$$\{P, Q; R, S\} = \frac{\lambda}{\mu}$$




**Warning:** The cross ratio will change when the points are permuted, so it depends on the order $P, Q, R, S$ are given in.

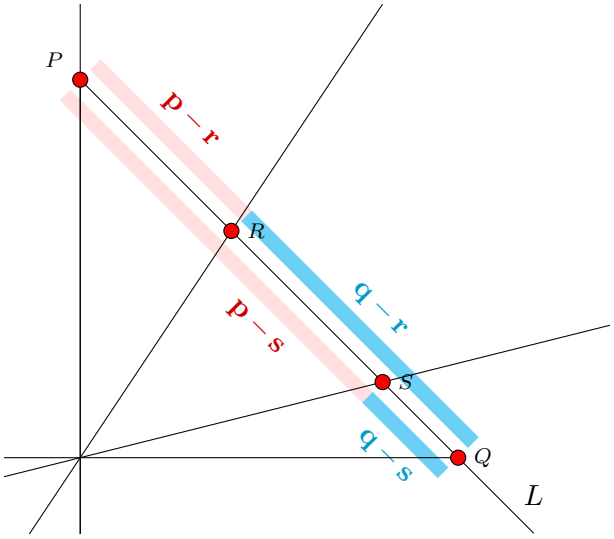
## 44. Lecture 26: The cross ratio and projective perspectives

**Note:** We could also map $P, Q, R$ to the standard projective frame of reference for $\mathbb{P}^1$, then the cross ratio is the image of $S$.
It is convenient not to have to make a change of basis in order to compute the cross ratio, so the following result is useful:

PROPOSITION 20. *If $P, Q, R, S$ are points in $\mathbb{P}^1$, then they can be considered as lines in $\mathbb{R}^2$. Let $L$ be any line in $\mathbb{R}^2$ not through the origin. Let $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ be the vector coordinates of the intersections of $P, Q, R, S$ with $L$. Then*

$$\{P, Q; R, S\} = \left(\frac{\mathbf{p}-\mathbf{r}}{\mathbf{p}-\mathbf{s}}\right)\left(\frac{\mathbf{q}-\mathbf{s}}{\mathbf{q}-\mathbf{r}}\right),$$

*where the ratios of vectors make sense, because they all in the direction of $L$, and we define $\frac{\lambda\mathbf{v}}{\mathbf{v}} = \lambda$.*




PROOF. Suppose $T$ is an invertible linear map. Then if $\mathbf{v}$ is a non-zero vector, and $\lambda$ a non-zero scalar, the by definition of linearity, $T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$, so

$$\frac{T(\lambda\mathbf{v})}{T(\mathbf{v})} = \frac{\lambda\mathbf{v}}{\mathbf{v}}$$

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and so when we apply a change of basis transformation $T$, with

$$T(P) = (1:0), \quad T(Q) = (0:1), \quad T(R) = (1:\lambda), \quad T(S) = (1:\mu)$$

By definition,

$$\{P, Q; R, S\} = \frac{\lambda}{\mu}.$$

Now for some non-zero scalars $a, b, c, d$, we have

$$T(\mathbf{p}) = (a, 0), \quad T(\mathbf{q}) = (0, b), \quad T(\mathbf{r}) = (c, c\lambda), \quad T(\mathbf{s}) = (d, d\mu)$$

So we have

$$\left(\frac{\mathbf{p}-\mathbf{r}}{\mathbf{p}-\mathbf{s}}\right)\left(\frac{\mathbf{q}-\mathbf{s}}{\mathbf{q}-\mathbf{r}}\right) = \left(\frac{(a,0)-(c,c\lambda)}{(a,0)-(d,d\mu)}\right)\left(\frac{(0,b)-(d,d\mu)}{(0,b)-(c,c\lambda)}\right) = \left(\frac{(c-a,c\lambda)}{(d-a,d\mu)}\right)\left(\frac{(d,d\mu-b)}{(c,c\lambda-b)}\right).$$

It is given that the points $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ all lie on a line, and so their images under $T$ also lie on a line, since $T$ is linear. So, the ratio of the vectors in this expression is given by either taking the ratio of the $x$ or $y$ components, since the result will be the same, since $\lambda(x, y) = (\lambda x, \lambda y)$, so we have

$$\left(\frac{(c-a,c\lambda)}{(d-a,d\mu)}\right)\left(\frac{(d,d\mu-b)}{(c,c\lambda-b)}\right) = \frac{c\lambda}{d\mu}\frac{d}{c} = \frac{\lambda}{\mu}.$$

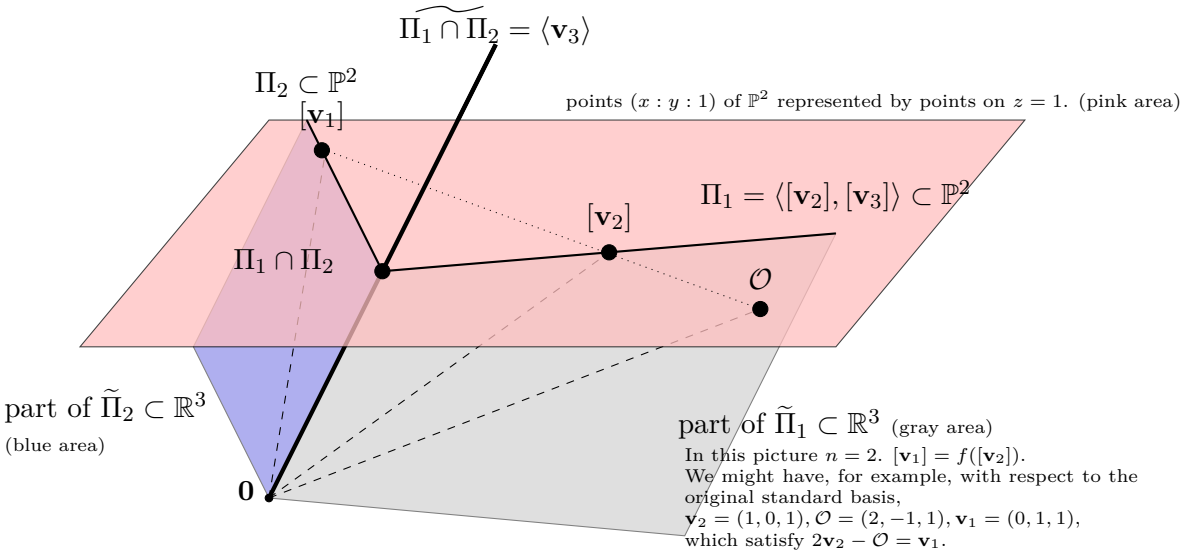
□

LEMMA 37. *For hyperplanes $\Pi_1, \Pi_2$ in $\mathbb{P}^n$, a perspectivity $f : \Pi_1 \to \Pi_2$ is a projective linear transformation.*

PROOF. We want to show that the perspectivity defined by projection from a point $\mathcal{O} \in \mathbb{P}^n$ not in $\Pi_1$ or $\Pi_2$ can be induced by a projective linear map (i.e., an invertible linear map, defined up to scalar multiplcation) from $\widetilde{\Pi_1}$ to $\widetilde{\Pi_2}$.
If $\Pi_1 = \Pi_2$, then $f$ is the identity, and the result is trivially true, so we may assume $\Pi_1 \neq \Pi_2$.
Since $\dim(\Pi_1) = \dim(\Pi_2) = n-1$, and $\Pi_1 \neq \Pi_2$, their span must be all of $\mathbb{P}^n$, and so by Theorem 12, $\dim(\Pi_1 \cap \Pi_2) = 2(n-1) - n = n-2$. Let $\mathbf{v}_3, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}$ be a basis of $\widetilde{\Pi_1 \cap \Pi_2}$.
Let $\mathbf{v}_2$ be a vector extending the basis for $\widetilde{\Pi_1 \cap \Pi_2}$ to a basis for $\widetilde{\Pi}_1$.
Let $\mathbf{v}_1$ be a vector such that $f([\mathbf{v}_2]) = [\mathbf{v}_1]$. By Lemma 36, $[\mathbf{v}_1]$ is well defined. Since $f([\mathbf{v}_2]) = [\mathbf{v}_1]$, this means that $\widetilde{\mathcal{O}}, \mathbf{v}_1, \mathbf{v}_2$ are non-zero vectors in the same plane, and so there is a linear relationship,

$$(\bigstar) \qquad \mathbf{v}_1 = \lambda \mathbf{v}_2 + \mu \widetilde{\mathcal{O}}$$

for some $\lambda, \mu \in \mathbb{R}$. (Because by definition, $[\mathbf{v}_1]$ is the point where the line through $\mathcal{O}$ and $[\mathbf{v}_2]$ meets $\Pi_2$. This line is the projectivisation of the plane spanned by $\mathbf{v}_1$ and $\widetilde{\mathcal{O}}$, and all points on this plane are given as a linear combination of $\mathbf{v}_1$ and $\widetilde{\mathcal{O}}$ by definition of the span.) By definition $[\mathbf{v}_1] \in \Pi_2$. Claim: $[\mathbf{v}_1] \notin \Pi_1$. This is because $[\mathbf{v}_2] \in \Pi_1$, and so if $[\mathbf{v}_1], [\mathbf{v}_2] \in \Pi_1$, any linear combination, in particular, $\mathcal{O}$, would be in $\Pi_1$, but by assumption $\mathcal{O} \notin \Pi_1$. Thus $[\mathbf{v}_1] \notin \Pi_1$, so $[\mathbf{v}_1] \in \Pi_2 \setminus \Pi_1$, and so $\mathbf{v}_1$ must extend the basis of $\widetilde{\Pi_1 \cap \Pi_2}$ to a basis for $\Pi_2$.

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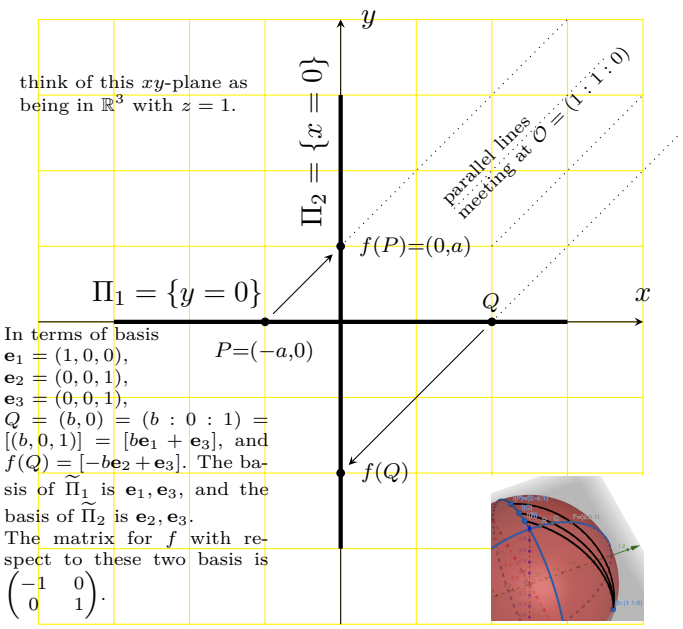
Since $\mathbf{v}_1 \notin \widetilde{\Pi}_1$, and $\widetilde{\Pi}_1$ has codimension 1 in $\mathbb{R}^{n+1}$, $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ is a basis for $\mathbb{R}^{n+1}$.
From now on, we write points in $\mathbb{R}^{n+1}$ and in $\mathbb{P}^n$ in terms of this basis, e.g., $(x_1, x_2, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} x_i \mathbf{v}_i$, and $(x_1 : x_2 : \cdots : x_{n+1}) = [\sum_{i=1}^{n+1} x_i \mathbf{v}_i]$.
Since all points in $\widetilde{\Pi}_1$ are in the span of $\mathbf{v}_2, \ldots, \mathbf{v}_{n+1}$, in terms of this basis, $\Pi_1 = \{x_1 = 0\} := \{(0 : x_2 : x_3 : \cdots : x_{n+1}) \in \mathbb{P}^n, x_i \in \mathbb{R}\}$. Similarly $\Pi_2 = \{x_2 = 0\}$.
In Equation $(\bigstar)$, $\mu \neq 0$, since otherwise $\mathbf{v}_2$ is in $\widetilde{\Pi}_2$, but $\mathbf{v}_2$ is chosen to extend the basis for $\widetilde{\Pi_1 \cap \Pi_2}$ to a basis for $\widetilde{\Pi}_1$, so can not be in $\widetilde{\Pi}_2$. Thus Equation $(\bigstar)$ can be rearanged to

$$\widetilde{\mathcal{O}} = \frac{1}{\mu}\mathbf{v}_1 - \frac{\lambda}{\mu}\mathbf{v}_2$$

We can scale to replace $\mathbf{v}_1$ and $\mathbf{v}_2$ by $\frac{1}{\mu}\mathbf{v}_1$ and $-\frac{\lambda}{\mu}\mathbf{v}_2$, so that $\widetilde{\mathcal{O}} = \mathbf{v}_1 + \mathbf{v}_2$, and the equations for $\Pi_1$ and $\Pi_2$ are unchanged. So now with respect to the basis $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$, $\mathcal{O} = (1 : 1 : 0 : \cdots : 0)$.




In this picture $n = 2$.
$\mathbf{v}_1 = (1, 0, 0), \mathbf{v}_2 = (0, 1, 0), \mathbf{v}_3 = (0, 0, 1)$
Consider this a picture of $\mathbb{R}^2$, with $\mathbb{R}^2 \subset \mathbb{P}^2$ via $(x, y) \mapsto (x : y : 1)$. So the origin $(0, 0)$ corresponds to the point $(0 : 0 : 1)$.
The hyperplanes $\Pi_1 = \{x = 0\} = \{(0 : r : t) : (r : t) \in \mathbb{P}^1$ and $\Pi_2 = \{y = 0\} = \{(r : 0 : t) : (r : t) \in \mathbb{P}^1$ are lines, given by the coordinate axis when restricted to $\mathbb{R}^2$.
However, the point $\mathcal{O} = (1 : 1 : 0)$ is not on this copy of $\mathbb{R}^2$, since $z = 0$, so it is a point at infinity (with respect to this choice of $\mathbb{R}^2$).
The point at $(1 : 1 : 0)$ can be intuitively considered to be infinitely far away in the $(1, 1)$ direction, since $(1 : 1 : 0)$ is "close" to $(1 : 1 : \epsilon) = (\frac{1}{\epsilon} : \frac{1}{\epsilon} : 1)$ for small $\epsilon$ – though remember that distance is only defined when we fix points in a copy of $\mathbb{R}$, and since $(1 : 1 : 0)$ is not in this plane, the distance to $(1 : 1 : 0)$ is not defined if we decide to define the distance from $(x : y : 1)$ to $(x' : y' : 1)$ to be the distance from $(x, y)$ to $(x', y')$. But this gives an intuitive idea of why we should consider $(1 : 1 : 0)$ to be in the $(1, 1)$ direction.
Intuitively, lines from a point at infinity are parallel, just as the rays from the sun are often considered to be parallel, since it is a long way off. (The geogebra drawing of a hemispherical representation of $\mathbb{P}^2$, bottom right, might make it clearer that lines intersect at $(1 : 1 : 0)$, but look parallel near $(0 : 0 : 1)$ . See `https://www.geogebra.org/3d/t5h4xem9`)
Algebraically, a line from $(a : b : 1)$ to $(1 : 1 : 0)$ is given by points of the form $(\lambda a + \mu : \lambda b + \mu : \lambda)$.
This line can be described by the equation $y = x + (b - a)z$, which is $y = x + (b - a)$ when $z = 1$, i.e., lines from $(1 : 1 : 0)$ to points with $z \neq 1$ are parallel to the direction $(1, 1)$. (Note that all points with $z = 0$ lie on the line $z = 0$, but this line will not appear in this picture, because this will be the "line at infinity".)

Now given a point $[\mathbf{x}] = (0 : x_2 : x_3 : \cdots : x_n) \in \Pi_1$, then the line through $\mathcal{O}$ and $[\mathbf{x}]$ consists of points of the form

$$(\lambda : \mu x_2 + \lambda : \mu x_3 : \cdots : \mu x_{n+1}).$$

In more details, this is because the line through $\mathcal{O}$ and $[\mathbf{x}]$ is the projection of the plane through $\widetilde{\mathcal{O}}$ and $\mathbf{x}$. The plane spanned by these points consists of all points of the form

$$\lambda(1, 1, 0, \ldots, 0) + \mu(0, x_2, x_3, \ldots, x_{n+1}) = (\lambda, \lambda + \mu x_2, \mu x_3, \mu x_4, \ldots, \mu x_{n+1}).$$

These vectors correspond to points in $\mathbb{P}^n$ provided $\lambda, \mu$ are not both zero.
This line intersects the hyperplane $\Pi_2$ when the second component is zero, i.e., $\lambda = -\mu x_1$, so

$$f([\mathbf{x}]) = (-x_1 : 0 : x_2 : \cdots : x_n).$$

week 9
lecture 26
cross ratio

Now taking a basis $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \ldots, \mathbf{v}_{n+1}$ for $\widetilde{\Pi}_1$ and $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \ldots, \mathbf{v}_{n+1}$ for $\widetilde{\Pi}_2$, the map $f$ is be described by the matrix

$$\begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & I_n \end{pmatrix},$$

which is clearly linear, and so $f$ is a projective linear transformation. □

PROPOSITION 21. *If $P_1, Q_1, R_1, S_1$ and $P_2, Q_2, R_2, S_2$ are two sets of distinct points lying on lines $L_1$ and $L_2$, then if there is a perspectivity $f$ taking the first set of points to the second, then we have equality of cross ratios:*

$$\{P_1, Q_1, R_1, S_1\} = \{P_2, Q_2, R_2, S_2\}$$

PROOF. In the proof of Proposition 20, we show that the cross ratio is invariant under projective linear transformations, and in Lemma 37, we show that perspectivity is a projective linear transformation. □

**Example:** Does there exist a projective linear map taking points $P_1, Q_1, R_1, S_1 = 0, 1, 2, 3$ to points $P_2, Q_2, R_2, S_2 = 0, 1, 7, 8$, where the point $t \in \mathbb{R}$ is interpreted as a point $(t : 1)$ in $\mathbb{P}^1$?
**Solution:** The cross ratios are

$$\{P_1, Q_1; R_1, S_1\} = \left(\frac{0-2}{0-3}\right)\left(\frac{1-3}{1-2}\right) = \frac{4}{3} \neq \{P_2, Q_2; R_2, S_2\} = \left(\frac{0-7}{0-8}\right)\left(\frac{1-8}{1-7}\right) = \frac{7 \times 7}{8 \times 6}$$

Since these are not equal, there is no such projective linear map.
**Example:** Does there exist a projective linear map taking points $P_1, Q_1, R_1, S_1 = 0, 1, 2, 5$ to points $P_2, Q_2, R_2, S_2 = 1, 4, 5, 6$, where the point $t \in \mathbb{R}$ is interpreted as a point $(t : 1)$ in $\mathbb{P}^1$?

**Solution:** The cross ratios are

$$\{P_1, Q_1; R_1, S_1\} = \left(\frac{0-2}{0-5}\right)\left(\frac{1-5}{1-2}\right) = \frac{8}{5} = \{P_2, Q_2; R_2, S_2\} = \left(\frac{1-5}{1-6}\right)\left(\frac{4-6}{4-5}\right) = \frac{8}{5}$$

So, there is such a map, and in fact this is the map given in Example 1 in Lecture 24, pages 84 and 85. Note that to compute the map, we only need to consider the images of the first three points. Given that the cross ratios are equal, the fourth point of one set will automatically get mapped to the fourth point of the other set.

**Remark (suggestion for further reading):** We do not have time to prove the converse of Proposition 21. We have also not shown that if a map between projective spaces preserves lines and the cross ratio, then it's a perspectivity, and thus given by an element of $PGL(n)$.

Note that the preservation of the cross ratio is a kind of generalisation of the fact that for 3 points $P, Q, R$ in a line in $\mathbb{R}^2$, the ratio of the distance between $PQ$ and $QR$ is preserved by isometries of $\mathbb{R}^2$, Proposition 2.

If there is time, we will prove the following, which will be useful in the proof of Desargues Theorem:

LEMMA 38. *The projection $\pi$ of $\mathbb{P}^3 \setminus \{M\}$ from a point $M$ in $\mathbb{P}^3 \setminus \mathbb{P}(V)$ to any plane $\mathbb{P}(V)$, where $V$ is any 3 dimensional subspace of $\mathbb{R}^4$ is given by $[\mathbf{v}] \to [T(\mathbf{v})]$ for some linear map $T$.*

PROOF. By applying a projective linear map, we may assume that $M = (0:0:0:1)$, and that $\mathbb{P}(V) = \{(x:y:z:0) : (x,y,z) \in \mathbb{R}^3\}$. Then $P = (x:y:z:w) \in \mathbb{P}^3$ lies on the line $\langle P, M\rangle = (\lambda x : \lambda y : \lambda z : \lambda w + \mu)$, which intersects $\mathbb{P}^2$ at $(x:y:z:0)$, by taking $\mu = -\lambda w$, and the map

$$(x:y:z:w) \mapsto (x:y:z:0)$$

is clearly a linear map. □

**Note:** This map is not well defined at the point $M$. The map is induced by a linear map on the underlying space, but it's not a projective linear map because it's not invertible.

week 9 lecture 26 cross ratio

**Note:** In the support class 9 notes, I go over how the cross ratio changes when the points $P, Q, R, S$ are permuted. For example, if we switch $P$ and $Q$, the cross ratio is inverted. You can see this by considering the equation and plugging in the values of $\mathbf{p}, \mathbf{s}, \mathbf{r}, \mathbf{s}$ in different orders. So for example, $\lambda$ and $1/\lambda$ can be cross ratios of the same set of points but in a different order.

E.g., by definition, $\{P = (1:0), Q = (0:1); R = (1:2), S = (1:1)\} = 2$. Suppose we permute $P, Q$. Let $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\{(0:1),(1:0);(1:1),(1:2)\} = \{T((0:1)), T((1:0)); T((1:2)), T((1:1))\} = \{(1:0),(0,1);(2:1),(1:1)\} = \{(1:0),(0,1);\left(1:\frac{1}{2}\right),(1:1)\} = \frac{1}{2}$.

However, $\lambda$ and $-\lambda$ are in general not the cross ratio of the same set of points in any order – though they might be in certain cases, since it turns out that $\lambda$ and $\frac{1}{1-\lambda}$ are the cross ratio of the same set of points in a different order, and $\frac{1}{1-\lambda} = -\lambda$ if $(1-\lambda)\lambda + 1 = 0$, that is, $\lambda^2 - \lambda + 1 = 0$, so $\lambda = 1 \pm \sqrt{-3}$, so at least this is possible for $\mathbb{P}^1(\mathbb{C})$.

## 45. Lecture 27: Desargues Theorem

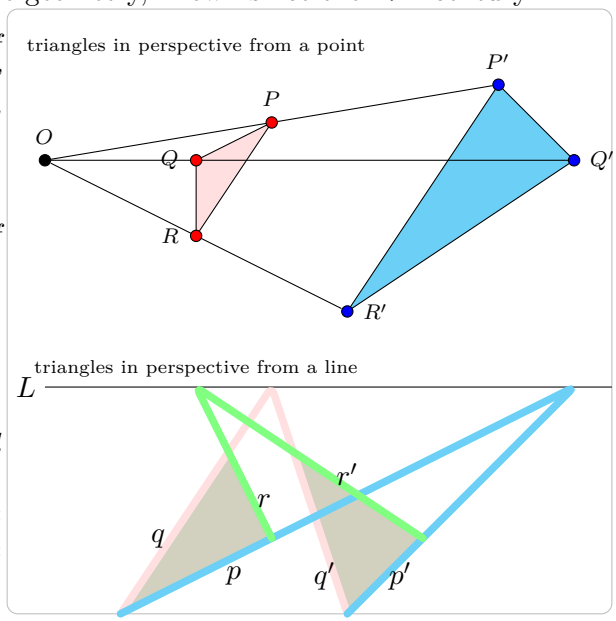
Desargues's theorem is a classical result in Projective geometry, known since the $17^{th}$ century.

DEFINITION 50. *A triangle $\Delta PQR$ in $\mathbb{P}^n$ is a set of three distinct points, $P, Q, R$ in $\mathbb{P}^n$, and the "sides" of the triangle, which are the three lines spanned by the three pairs of points.*

DEFINITION 51. *Two triangles $\Delta PQR$ and $\Delta P'Q'R'$ in $\mathbb{P}^n$ are said to be in perspective from a point $\mathcal{O}$ if the lines $\langle P, P'\rangle$, $\langle Q, Q'\rangle$ $\langle R, R'\rangle$ all intersect in a common point $\mathcal{O}$.*

The "dual" concept is 'perspective from a line' – see [RS 5.11] for more on the principal of duality.

DEFINITION 52. *Two triangles $\Delta PQR$ and $\Delta P'Q'R'$, with sides $p, q, r$ and $p', q', r'$ in $\mathbb{P}^n$ are said to be in perspective from a line $L$ if the points $p \cap p'$, $q \cap q'$, $r \cap r'$, all lie on a common line $L$.*




THEOREM 15. *(Desargues' Theorem) If $\Delta PQR$ and $\Delta P'Q'R'$ are two distinct triangles in $\mathbb{P}^n$ which are in perspective from a point, then they are also in perspective from a line.*

[We skipped the proof due to lack of time, and so you will not be asked to prove this result in the exam, but you should understand the statement of this result. We also had to skip the proof of Pappus' theorem.]

PROOF. Notation: Let $p = \langle Q, R\rangle, q = \langle P, R\rangle, r = \langle Q, P\rangle$, and $p' = \langle Q', R'\rangle, q' = \langle P', R'\rangle, r' = \langle Q', P'\rangle$. Let $\Pi_1$ be the plane spanned by $P, Q, R$ and $\Pi_2$ be the plane spanned by $P', Q', R'$. Let $O$ be the point from which $\Delta PQR$ and $\Delta P'Q'R'$ are in perspective.

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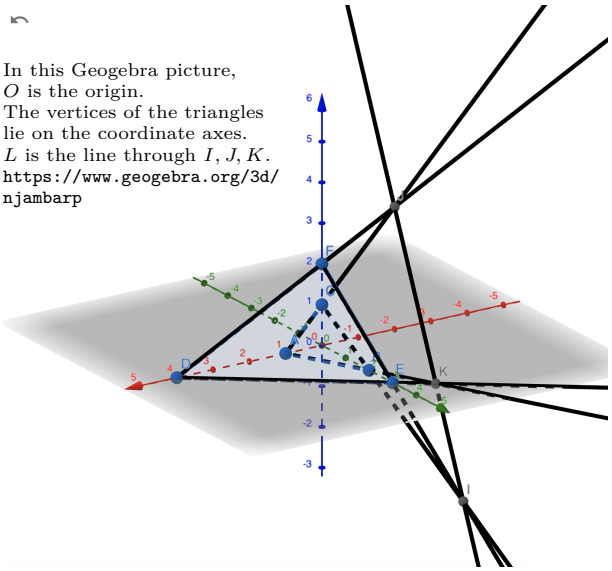
**Claim:** $p \cap p'$, $q \cap q'$, $r \cap r'$ are non empty: The plane spanned by $O, Q', R'$ also contains $Q$ and $R$, since these are on the lines $OQ'$ and $OR'$. So, the lines $p$ and $p'$ must lie in this plane, and so by Theorem 13 they intersect in a point, unless they are equal. Similarly for $q \cap q'$ and $r' \cap r'$.
Since $O, Q', R', Q$ and $R$ are all contained in a single plane, and $P$ is in the same plane as $OP$, the space spanned by $\Pi_1$ and $\Pi_2$ is at most 3 dimensional.
**Case 1:** Suppose that $\dim(\langle \Pi_1, \Pi_2\rangle) = 3$, so $\Pi_1 \neq \Pi_2$. Let $L = \Pi_1 \cap \Pi_2$. By Theorem 12 $\dim(L) = 1$. Since $p \subset \Pi_1$ and $p' \subset \Pi_2$, we must have $p \cap p' \subset L$, and similarly for $q \cap q'$ and $r \cap r'$, and so the result is true.

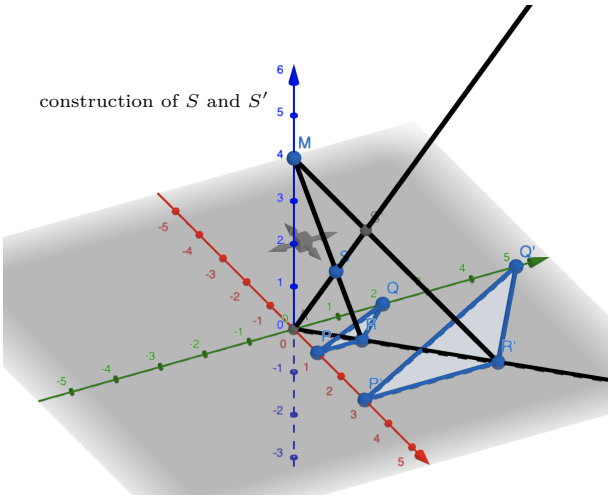
In this Geogebra picture,
$O$ is the origin.
The vertices of the triangles lie on the coordinate axes.
$L$ is the line through $I, J, K$.
https://www.geogebra.org/3d/njambarp

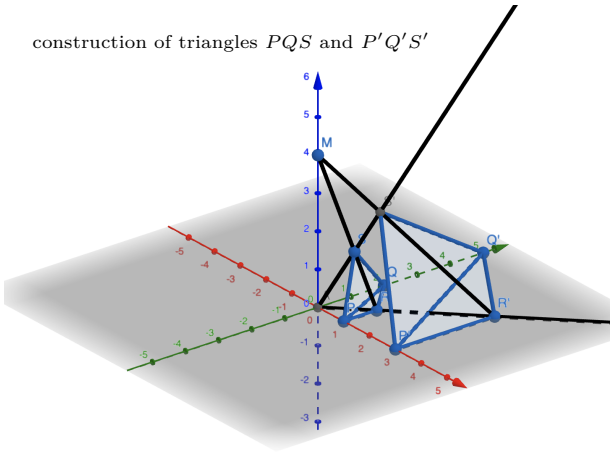


**Case 2:** Suppose that $\Pi_1 = \Pi_2$. By a change of basis we can identify $\Pi_1$ with $\mathbb{P}^2 = \{(x : y : z : 0)\} \subset \mathbb{P}^3$. We construct two triangles which are not coplanar, and which project to $\Delta PQR$ and $\Delta P'Q'R'$.

We may assume $R \neq R'$, since the triangles are distinct, so at least one point differs. Let $M$ be a point in $\mathbb{P}^3 \setminus \Pi_1$. Let $S$ be any point on the line $MR$ which is not in $\Pi_1$, and not equal to $M$. Since $R, R'O$ are on a line, $R, R', O, M, S$ are in a plane, so by Theorem 13 the lines $MR'$ and $OS$ intersect in a point, $S'$.

Now the triangles $\Delta PQS$ and $\Delta P'Q'S'$ can't lie in a plane, since $S$ and $S'$ are not in the same plane as $P, Q, P', Q'$.




By construction, the triangles $\Delta PQS$ and $\Delta PQS'$ project from $M$ to the triangles $\Delta PQR$ and $\Delta P'Q'R'$ in the plane $\Pi_1$. Let $\pi$ be the projection map from $M$ to $\Pi_1$.

By Case 1, the pairs of edges of $\Delta PQS$ and $\Delta P'Q'S'$ have intersection points all lying on a line, $L$. By Lemma 38, $\pi$ is described by a linear map, so the image of $L$ is a line (or if a point, it is contained in a line). Since $\pi$ is a projective linear map, it maps the lines spanned by edges of $\Delta PQS$ and $\Delta P'Q'S'$ to the lines spanned by edges of $\Delta PQR$ and $\Delta PQR'$. So the points $p \cap p'$, $q \cap q'$, $r \cap r'$ all lie on the line $\pi(L)$.

(See `https://www.geogebra.org/3d/thzujqpa`, `https://www.geogebra.org/3d/whpsdqxa`.) □

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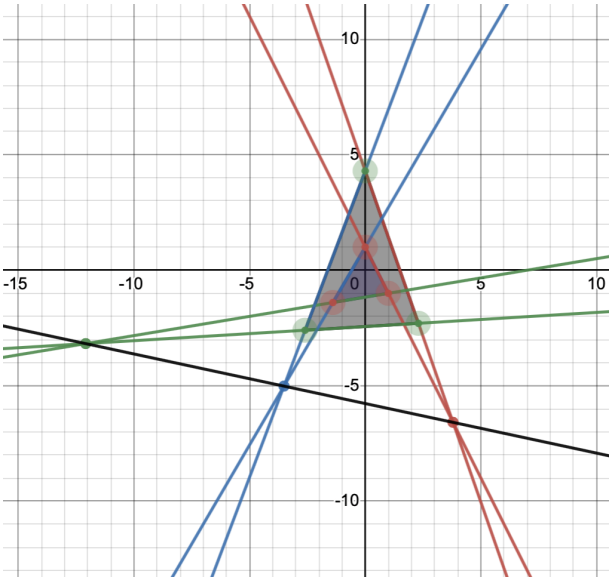
**Remark:** This result is true in $\mathbb{P}^n$. It is not always true in $\mathbb{R}^n$, since intersections of edges might be at infinity, or the line $L$ might end up being the line at infinity, which happens if pairs of sides of the two triangles are parallel. Working in projective space generally results in more uniform results without special exceptions. (Here "infinity" means any point in $\mathbb{P}^n \setminus \mathbb{R}^n$, where we can include $\mathbb{R}^n \subset \mathbb{P}^n$ as the subset with $x_{n+1} = 1$.)

Desmos version: `https://www.desmos.com/calculator/uq2okdn9ku`



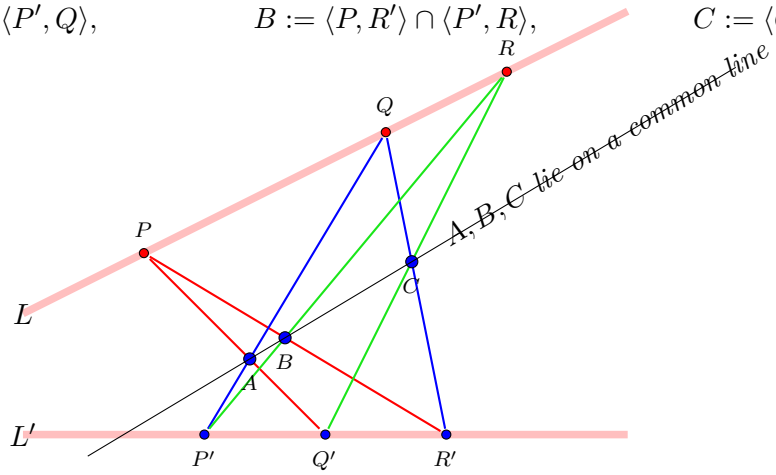
## 46. Pappus' Theorem

Pappus' Theorem has been known since the ancient Greeks. It is closely related to Desargues Theorem.

THEOREM 16 (Pappus' Theorem). *Let $L, L' \subset \mathbb{P}^2$ be distinct projective lines.*
*Let $P, Q, R \in L \setminus L'$, $P', Q', R' \in L' \setminus L$ be distinct points.*
*Then the intersection points*

$$A := \langle P, Q' \rangle \cap \langle P', Q \rangle, \qquad B := \langle P, R' \rangle \cap \langle P', R \rangle, \qquad C := \langle Q, R' \rangle \cap \langle Q', R \rangle$$

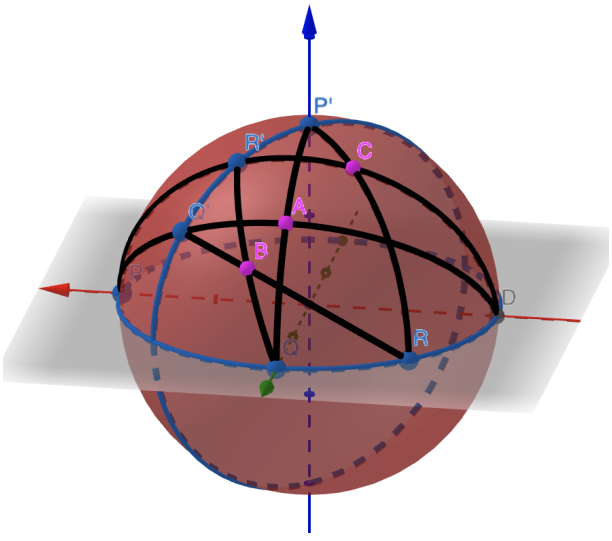
*are colinear.*




PROOF. Since $P, Q, P', Q'$ are distinct and on two distinct lines, these points must be a projective frame of reference, and so by Theorem 14 there is a projective transformation which takes these points to the standard frame of reference for $\mathbb{P}^2$.
Since a projective linear map preserves lines and intersection points, we can now assume

$$P = (1:0:0), \qquad Q = (0:1:0), \qquad P' = (0:0:1), \qquad Q' = (1:1:1).$$

The picture, from `https://www.geogebra.org/3d/zhwkackg` shows these points on the projective plane projected to a sphere. The line through $P, Q, R$ is the "line at infinity", where the sphere intersects the $z = 0$ plane.




Since $R \in \langle P, Q \rangle$, $R$, being a linear combination of $P$ and $Q$, has the form $R = (1 : \alpha : 0)$, where we may assume the first coordinate is 1, since it can not be zero, since $R \neq Q$. If the first coordinate is non-zero, by scaling we can set it equal to 1. Since $R'$ is on $\langle P', Q' \rangle$, $R'$ has the form $R' = (1 : 1 : \beta)$, where again, the first coordinates can not be zero, since $R' \neq P'$.

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Now compute the lines and intersection points:

$$\left.\begin{array}{ll} \langle P, Q' \rangle & = \{(\lambda + \mu : \mu : \mu)\} \\ \langle P', Q \rangle & = \{(0 : \lambda_2 : \mu_2)\} \end{array}\right\} A = (0:1:1)$$

$$\left.\begin{array}{ll} \langle Q, R' \rangle & = \{(\mu_2 : \mu_2 + \lambda_2 : \mu_2\beta)\} \\ \langle R, Q' \rangle & = \{(\mu + \lambda : \mu\alpha + \lambda : \lambda)\} \end{array}\right\} B = (1 : \alpha + \beta - \alpha\beta : \beta)$$

$$\left.\begin{array}{ll} \langle P, R' \rangle & = \{(\mu_2 + \lambda_2 : \mu_2 : \mu_2\beta)\} \\ \langle R, P' \rangle & = \{(\mu : \mu\alpha : \lambda)\} \end{array}\right\} C = (1 : \alpha : \alpha\beta)$$

Now check that $\alpha\widetilde{A} + \widetilde{C} = \widetilde{B}$, where $\widetilde{A} = (0, 1, 1), \widetilde{C} = (1, \alpha, \alpha\beta), \widetilde{B} = (\mu + \lambda, \mu\alpha + \lambda, \lambda)$, which shows that $A, B, C$ are on a line. □

## 47. Lecture 28: Axiomatic projective geometry

DEFINITION 53. *An axiomatic projective plane $(P, L, I)$ consists of a set $P$, the set of points, and a set $L$, the set of lines, and a relation $I \subset P \times L$, the incidence relation, i.e., how lines intersect.*
*For $p \in P$ and $\ell \in L$, we write*

$$p \in \ell \iff (p, \ell) \in I,$$

*and say that $p$ is contained in $\ell$.*
*Notation: for $\ell_1, \ell_2 \in L$,*

$$\ell_1 \cap \ell_2 = \{p \in P : p \in \ell_1 \text{ and } p \in \ell_2\}$$

*These sets must satisfy the following axioms:*
*(1) Every line contains at least distinct three points:*

$$\forall \ell \in L \exists \text{ distinct } x, y, z \in P, \text{ such that } (x, \ell), (y, \ell), (z, \ell) \in I$$

*i.e., $x, y, z \in \ell$.*
*(2) Every point is contained in at least three distinct lines:*

$$\forall x \in P \exists \text{ distinct } l, m, n \in L, \text{ such that } (x, l), (x, m), (x, n) \in I$$

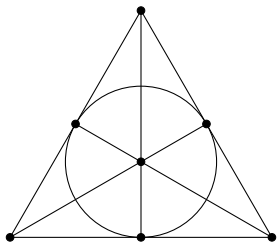
*i.e., $x \in \ell \cap m \cap n$*
*(3) Any two points span a unique line:*

$$\forall x \neq y \in P \exists! \ell \in L \text{ such that } (x, \ell), (y, \ell) \in I$$

*(4) Any two distinct lines intersect at a unique point:*

$$\forall l \neq m \in L \exists! x \in P \text{ such that } (x, l), (x, m) \in I$$

**Example** $\mathbb{P}^2$ is an axiomatic projective plane, where lines and points and intersections are as already defined. $\mathbb{P}^n$ is not a projective plane for $n \neq 2$. $\mathbb{R}^2$, with the usual Euclidean definitions of lines, points, intersections, is not an axomatic projective plane, since not all lines intersect.
$\mathbb{P}^2(k)$ is a projective plane for any field $k$, and even for $k$ replaced by any division ring. E.g., the Fano plane, $\mathbb{P}^2(\mathbb{F}_2)$ is the smallest axiomatic projective plane, and has 7 points and 7 lines.



week 10 lecture 28 Axiomatic projective geometry

PROPOSITION 22. *The projective plane $\mathbb{P}^2(\mathbb{F}_q)$ for a field $\mathbb{F}_q$ with $q$ elements has $q^2+q+1$ points. There are also $q^2+q+1$ lines, and each line contains $q+1$ points.*

PROOF. There are a total of $q^3 - 1$ points in $(\mathbb{F}_q^3)^*$. Each line in $(\mathbb{F}_q^3)^*$ contains $q-1$ points (all the points on the line in $\mathbb{F}_q^3$ except $\mathbf{0}$. Lines in $\mathbb{F}_q^3$ only intersect at $\mathbf{0}$, so there are

$$\frac{q^3-1}{q-1} = q^2 + q + 1$$

lines in $\mathbb{F}_q^3$, and so this is the number of points in $\mathbb{P}^2(\mathbb{F}_q)$. The number of lines is equal to the number of points because any line in $\mathbb{P}^2(\mathbb{F}_q)$ corresponds to a plane through the origin in $\mathbb{F}_q^3$, and any plane corresponds to the line which is orthogonal to the plane. Each plane contains $q^2 - 1$ non-zero points, and so there are $q + 1 = (q^2 - 1)/(q - 1)$ lines on each plane in $\mathbb{F}_q^3$, which after projectivising become $q + 1$ points on each line. □

## 48. Lecture 29: Finite Axiomatic Projective Geometries

PROPOSITION 23 (Proposition/Definition). *A finite axiomatic projective plane has $n^2+n+1$ points for some integer $n$, and has $n+1$ points on each line and $n+1$ lines through each point. Such a projective plane is called a projective plane of order $n$.*

PROOF. Not all points are on the same line, since by axiom 2, each point is contained in at least three lines, say $l_1$ and $l_2$ and $l_3$, and each line contains at least three points by axiom 1. The points $p, q, r$ on $l_1$ can not all lie on $l_2$, since by axiom 4, two distinct lines meet in a unique point, so at most one of these points is in $l_2$, similarly any line not equal to $l_1$ conatins at most one of $p, q, r$, and so there are at least two of these points not on the line. By switching $l_1$ and $l_2$, the same holds for $l_1$, so for any line, there is a point not on the line.
We know there are at least 3 lines, $l_1, l_2, l_3$. Suppose $l_1$ contains $N_1$ points. Let $p$ be a point not on $l_1$. Now consider all the lines through $p$. Each line intersects $l_1$ in a unique point, by axiom 4. Each point on $l_1$ is contained in exctly one of the lines through $p$ by axiom 3. So there is a bijection between lines through $p$ and points on $\ell$. Hence all points not on $l_1$ are contained in exactly $N_1$ lines.
Similarly, all points not on $l_2$ are contained in exactly $N_2$ lines, where $N_2$ is the number of points on $l_2$. Now let $p_0$ be the intersection of $l_1$ and $l_2$, and let $p_1$ and $p_2$ be points on $l_1$ and $l_2$ different from $p_0$. Since each line has three points, the line containing $l_1$ and $l_2$ must have third point $q$ not on $l_1$ or $l_2$. This point is contained in exactly $N_1$ lines and on exactly $N_2$ lines. Hence $N_1 = N_2$, and all points are contained in the same number of lines, and this number is equal to the number of points on each line.
Let $n = N_1 - 1$. So every line contains $n+1$ points, and every point is on $n+1$ lines.
Now take any point $p$. Every other point lies on exactly one of the $n+1$ lines through $p$, and each of these lines contains $n$ points not counting $p$, so this gives $(n+1)n$ points not equal to $p$, so together with $p$ there are $n^2+n+1$ points. □

**Example:** The Fano plane is a finite projective plane of order 2. $\mathbb{P}^2(\mathbb{F}_3)$ is a finite projective plane of order 3.
**Remark:** Desargues Theorem and Pappus' Theorem are entirely expressed in terms of incidence of points and lines, so they can be considered as statements about axiomatic projective planes. **But** they are not true for an arbitrary axiomatic projective plane. A projective plane is called "Desarguian" if Desargues Theorem holds.

DEFINITION 54. *We say two axiomatic projective planes $(P, L, I)$ and $(P', L', I')$ are isomorphic if there are bijective maps*

$$f_P : P \to P', \qquad f_L : L \to L', \qquad f_I : I \to I'$$

*such that for $p \in P, \ell \in L$*

$$p \in \ell \iff f_L(p) \in f_L(\ell)$$

THEOREM 17 (Hilbert's Theorem). *Let $(P, L, I)$ be an axiomatic projecive plane which satisfies Desargue's theorem. Then $(P, L, I)$ is isomorphic to the projecive plane of some division ring $k$.*
*If Pappus' Theorem is satisfied, then $k$ is a field.*

PROOF. See Projective Geometry from Foundations to Applications by Albrecht Beutelspacher and Ute Rosenbaum, page 117, 3.4.2. □

week 10 lecture 29 Finite projective planes

**Remark:** Wedderburn's little Theorem says that all finite divisions are fields, and so all finite Desarguian planes are projective planes of a finite field. However, there do exist non Desarguian planes.
**Conjecture:** Every finite projective plane has order $q$, where $q$ is a power of a prime.

There are finite projective fields of order $q$ for all powers of prime $q$, namely $\mathbb{P}^2(\mathbb{F}_q)$.

However, it's not true that every finite projective plane is isomorphic to $\mathbb{P}^2(\mathbb{F}_q)$ for some finite field $\mathbb{F}_q$ with $q = p^n$ elements for some prime $p$. There are 4 different finite projective planes of order 9. Just one of these is $\mathbb{P}^2(\mathbb{F}_9)$.

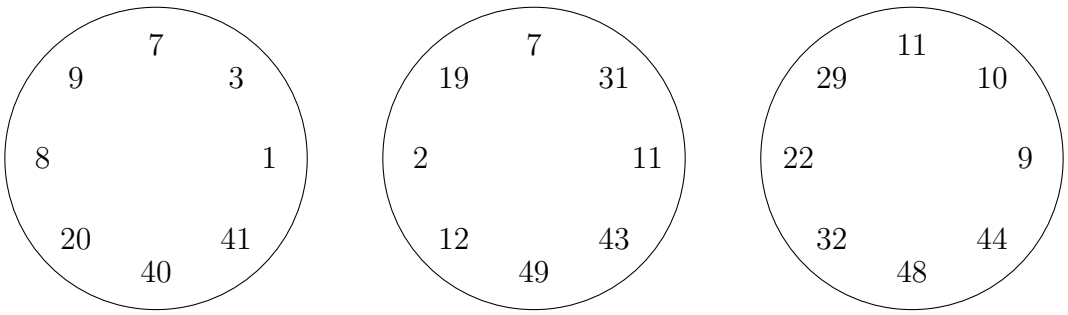
Not very much is known about this problem.

See "A001231: Number of non-isomorphic projective planes of order n." at `https://oeis.org/A001231`, for the list that is known; the number of projective planes of order $n = 2, \ldots, 10$:

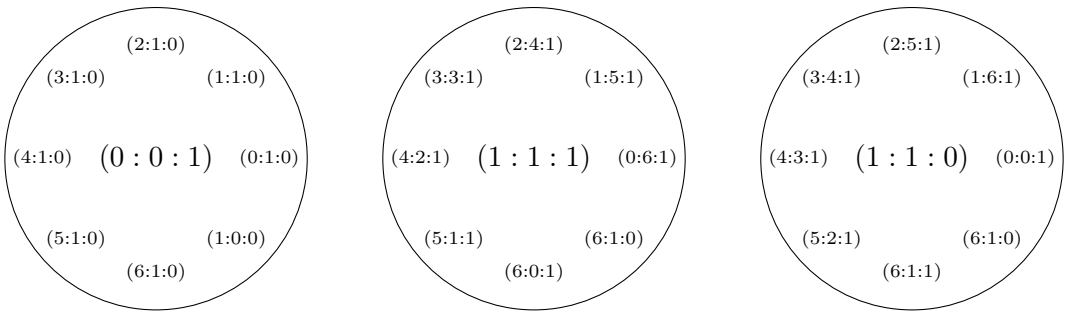
| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $n^2+n+1$ | 7 | 13 | 21 | 31 | (43) | 57 | 73 | 91 | (111) |
| number of non-isomorphic planes | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 4 | 0 |

I have put 43 and 111 in brackets, because there are no projective planes with this number of points. It is not currently known whether or not there is a projective plane of order 12.
The Bruck–Ryser Theorem says that for a finite projective plane of order $q$, with $q \equiv 1$ or $2 \mod 4$, $q$ is a sum of two squares.

**48.1. Example: The Axiomatic projective plane associated with "Dobble".** Dobble is a children's card game where each card has 8 pictures, and there are a total of 57 symbols. The game is sold with 55 cards.
Let's write numbers instead of pictures, then some cards might look like:




Every card has exactly one picture in common with one of the other cards.
There are various ways to play with these cards, but they all generally involve being the first to spot the unique picture (or number in this example) contained on a pair of cards.
This game is equivalent to a projective plane of order 8. $8^2 - 8 + 1 = 57$. Each card corresponds to a point, and each line corresponds to one of the symbols. This is isomorphic to $\mathbb{P}^2(\mathbb{F}_7)$. So, we can label each card with an element of $\mathbb{P}^2(\mathbb{F}_7)$. A line is given by a plane in $(\mathbb{F}_7)^3$, which can be specified by the vector it is orthogonl to. So, the lines any point is on correspond to all the orthogonal vectors, up to multiplcation by a non-zero element of $\mathbb{F}_7$.
So, rather than labeling the cards with pictures or numbers we could label them with points in $\mathbb{P}^2(\mathbb{F}_7)$. E.g., with the number of the card in the middle, and the number corresponding to a line it's on round the edge:

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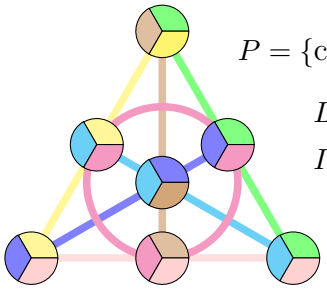
These cards all lie on the same line, since $(0,0,1) + (1,1,0) = (1,1,1)$. The line spanned by these points consists of the points

$$(0:0:1), (1:1:0), (1:1:1), (1:1:2), (1:1:3), (1:1:4), (1:1:5), (1:1:6)$$

As sold, the game only has 55 cards, so two are missing! If you have a set of this game, maybe you can work out what the missing cards are!

**48.2. Example: Dobble from the Fano plane.** We can make a seven symbol, seven card version of Dobble, by assigning each line a symbol. I'll use colours instead of symbols. The 7 cards are shown as the vertices of the graph. Cards have a match if they contain a region of the same colour.



$$P = \{\text{cards}\} = \left\{ \left\{ \begin{matrix}\text{green,}\\ \text{brown,}\\ \text{yellow}\end{matrix} \right\}, \left\{ \begin{matrix}\text{cyan,}\\ \text{magenta,}\\ \text{yellow}\end{matrix} \right\}, \left\{ \begin{matrix}\text{blue,}\\ \text{magenta,}\\ \text{green}\end{matrix} \right\}, \left\{ \begin{matrix}\text{cyan,}\\ \text{blue,}\\ \text{brown}\end{matrix} \right\}, \left\{ \begin{matrix}\text{yellow,}\\ \text{blue,}\\ \text{pink}\end{matrix} \right\}, \left\{ \begin{matrix}\text{magenta,}\\ \text{brown,}\\ \text{pink}\end{matrix} \right\}, \left\{ \begin{matrix}\text{cyan,}\\ \text{green,}\\ \text{pink}\end{matrix} \right\} \right\}$$

$$L = \{\text{Lines}\} = \{\text{green, brown, blue, magenta, pink, yellow, cyan}\}$$

$$I = \{(p, \ell) : p \in P, \ell \in L, \text{ if } \ell \in p\}$$

**48.3. Example: Junior Dobble.** A smaller version of the game has only 30 cards, 31 different symbols, and corresponds to a finite projective plane of order 6. This is isomorphic to $\mathbb{P}^2(\mathbb{F}_5)$. Each line contains on 6 points, each point is on 6 lines. In the game as sold, one card is missing.

The missing card is:
{dog, cat, camel, fish, ladybird, horse}.

Each line corresponds to a different animal.

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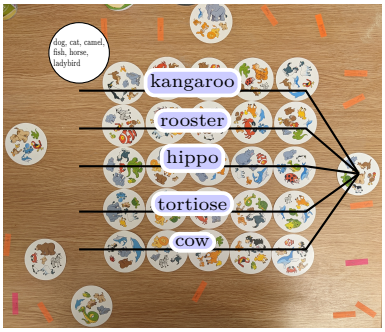
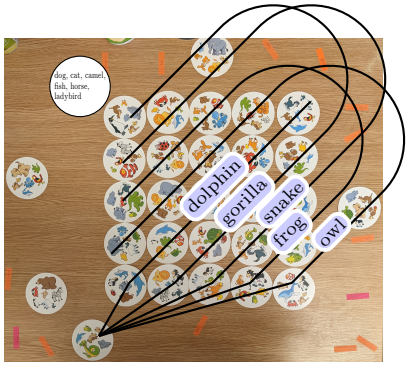
There are 31 different animal symbols, one for each line in the projective space $\mathbb{P}^2(\mathbb{F}_5)$.
With the cards as in the above layout, we have the 31 lines as follows. These lines are divided into sets of "parallel" lines, ie., sets of lines that have intersection at infinity. In this layout, I have chosen "dog" to be the line at infinity, so any card with "dog" on is a point on the line at infinity. For example, {Elephant, whale, tiger, dog, rabbit, } is one of the cards at infinity, and in the chosen layout, the lines corresponding to each of these animals correspond to the vertical lines. Each card on the "dog" line correponds to a different set of parallel lines, as in the following figures:

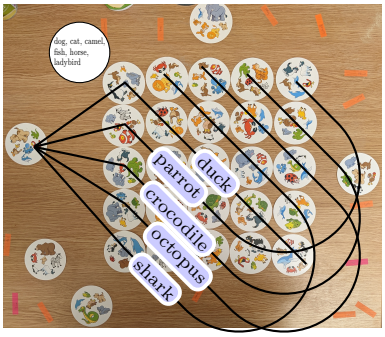
lines of the form $\{(\alpha s : t : s) : (t : s) \in \mathbb{P}^1\}$



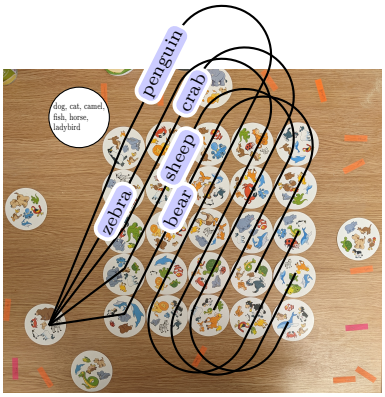

lines of the form $\{(t : \alpha s : s) : (t : s) \in \mathbb{P}^1\}$



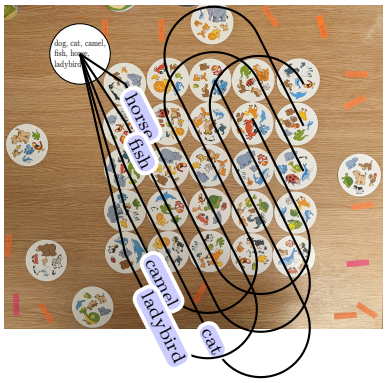

lines of the form $\{(t : t + \alpha s : s) : (t : s) \in \mathbb{P}^1\}$



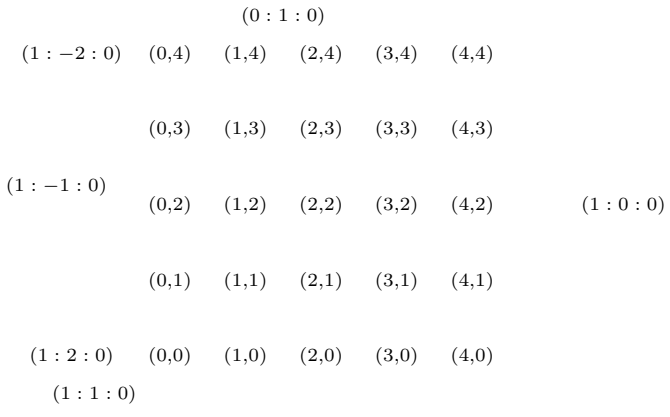

lines of the form $\{(t : \alpha s - t : s) : (t : s) \in \mathbb{P}^1\}$




lines of the form $\{(t : 2t + \alpha s : s) : (t : s) \in \mathbb{P}^1\}$




lines of the form $\{(t : \alpha s - 2t : s) : (t : s) \in \mathbb{P}^1\}$




interpret $(x, y)$ as $(x : y : 1)$; all computations mod 5

This shows the 6 families of "parallel" lines. Note that "parallel" here means not meeting in the copy of $\mathbb{F}_5^2 \subset \mathbb{P}^2(\mathbb{F}_5)$ given by $(x, y) \mapsto (x : y : 1)$. The last figure shows a possible isomorphism between this game of Dobble and elements of $\mathbb{P}^2(\mathbb{F}_5)$. The "cat" line (for example), consists of points $(x : y : z)$ with $y = 2z - 2x$, i.e., $(x, y, z) \cdot (2, 1, -2) = 0$ meaning "cat" is the line orthogonal to $(2 : 1 : -2) \equiv (4 : 2 : 1)$. In a similar way, each of the remaining animals can be identified with an element of $\mathbb{P}^2(\mathbb{F}_5)$. Other identifications can be made, but the identification is not arbitrary.

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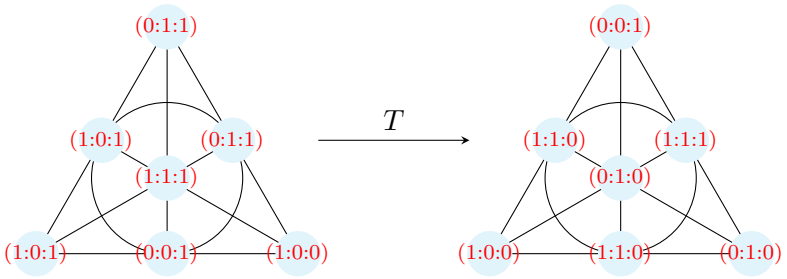
**Quiz:** Suppose you wanted a game where there were $N$ cards, and each card has exactly one symbol in common with another card. How many different symbols would be necessary to use?
**Solution:** If we call each card a point, and each symbol a line, and require the axioms of a projective plane to be satisfied, then if the projective plane has order $q$, we need $q^2 + q + 1 \geq N$. So for a game

with 55 cards, we need at least $q = 7$, since $q = 6$ gives only $6^2 + 6 + 1 = 43$ cards. (In fact there is no projecive plane of order 6.)

**Example** We have defined the concept of an isomorphism of projective planes. An isomorphim from a projective plane to itself is called an *automorphism*. Here is an example of an automorphism of the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$.




| $p$ | $T(p)$ |
|---|---|
| $(0:0:1)$ | $(0:1:1)$ |
| $(1:0:1)$ | $(1:1:0)$ |
| $(0:1:1)$ | $(1:1:1)$ |
| $(1:1:1)$ | $(0:1:0)$ |
| $(1:0:0)$ | $(1:0:1)$ |
| $(1:1:0)$ | $(0:0:1)$ |
| $(0:1:0)$ | $(1:0:0)$ |

This map corresponds to the projective linear map $T_A$ with matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

E.g., we can check that $A(1,1,1) = (0,1,0)$, so $T_A(1:1:1) = (0:1:0)$.

Note that elements of an axiomatic projective plane don't necessarily have to be given in terms of a finite field. E.g., in the example of Dobble from a Fano plane, the map $\{\text{cards}\} \to \mathbb{P}^2(\mathbb{F}_2)$ could be given by a map $T$ defined by:

| card | | | | | | | |
|---|---|---|---|---|---|---|---|
| $T(\text{card})$ | $(0:1:1)$ | $(1:0:1)$ | $(0:0:1)$ | $(1:0:1)$ | $(1:0:0)$ | $(1:1:1)$ | $(0:1:1)$ |

The map $T$ is not given by a linear map, because cards are not objects that you can apply linear maps to.

**48.4. Kirkman's Fifteen Schoolgirl Problem.** Dobble was apparently inspired by Kirkman's Schoolgirl Problem:

If there are 15 girls, who go for a walk in groups of three every day, how can you arrange so that each day of the week they all walk in a different group of three, and each girl walks in the same group as every other girl on exactly one day?

By making girls correspond to points and groups correspond to lines, this problem corresponds to a finite geometry. However, it is not an axiomatic projective plane, but one solution is given by identifying the girls with the points of the three dimensional projective space $\mathbb{P}^3(\mathbb{F}_2)$.

We don't have time to discuss this problem. This section has given a glimpse of a combinatorial view of projective geometry.

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## 49. MA243 Support Class 1, week 2, October 2022

(1) (Composition of isometries in $\mathbb{R}^2$)
   (a) Find the matrix corresponding to the reflection in the line $y = ax$
   (b) Find $A$ and $\mathbf{b}$ such that $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ is the reflection in the line $y = ax + b$
   (c) Let $T$ be the reflection in the line $x + y = 1$ and let $R$ be the reflection in the line $y = 1$. Write $T$, $R$ and the composition $T \circ R$ in the form $A\mathbf{x} + \mathbf{b}$.
   (d) What is the point fixed by the rotation $T \circ R$?
   (e) Is the composition of two reflections always a rotation?

(2) (isometries in $\mathbb{R}^2$) For the following questions, write the maps in the form $\mathbf{x} \to A\mathbf{x} + \mathbf{b}$. You might find it helpful to draw a diagram and find the center of rotation or line of reflection.
   (a) Find the reflection that maps $(0, 0)$ to $(2, 2)$
   (b) Find the rotation that maps

$$(0, 0) \mapsto (2, 2)$$
$$(1, -1) \mapsto (1, 1)$$

   (c) what are the eigen values and eigen vectors of the matrices involved in these transformations?
   (d) can you tell whether a matrix is a rotation or reflection matrix from its eigen values?
   (e) why do we talk about eigen values and eigen vectors of matrices, but not of isometries?

(3) (Composition of isometries in $\mathbb{R}^2$) Suppose you want to move your sofa on your own, and can only move one end at a time, rotating the sofa about the other end. Represent your sofa by the line $(0, 0)(1, 0)$ in $\mathbb{R}^2$. Describe how you can move your "sofa" to another line segment of length one by applying a sequence of such rotations. Is your sequence unique? You do not have to give matrices for this question, just describe a general procedure

(4) Show that if $(X, d_1)$ and $(Y, d_2)$ are isometric Euclidean spaces, then they are both isometric to $\mathbb{E}^n$ for some $n$.

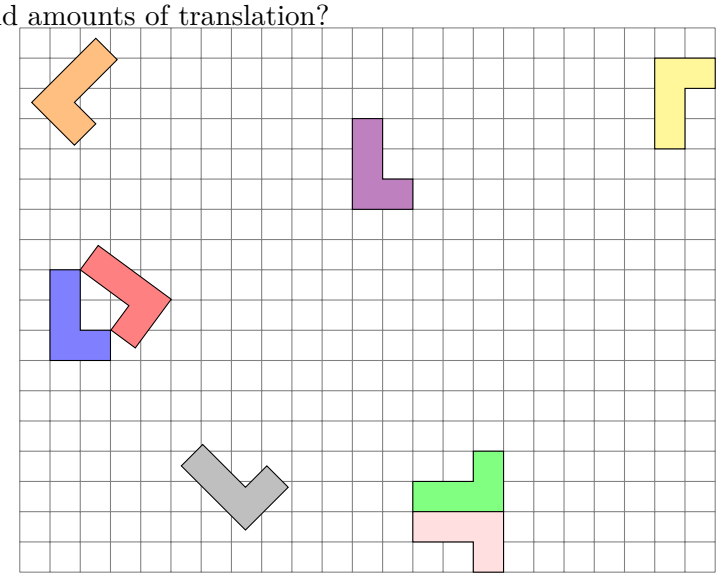
(5) Points $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are collinear if they are all on the same line $L$. Check you understand why this implies that $\mathbf{z} = \mathbf{x} + \lambda(\mathbf{y} - \mathbf{x})$ for some $\lambda \in \mathbb{R}$.

   Show that if $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are colinear, then one point is "in the middle" by which we mean $\mathbf{z} = \mathbf{x} + \lambda(\mathbf{y} - \mathbf{x})$ with $\lambda \in (0, 1)$.

(6) prove that the inverse of an isometry is an isometry

(7) give an example of a map on $\mathbb{R}^2$ which maps lines to lines, but which is not an isometry.

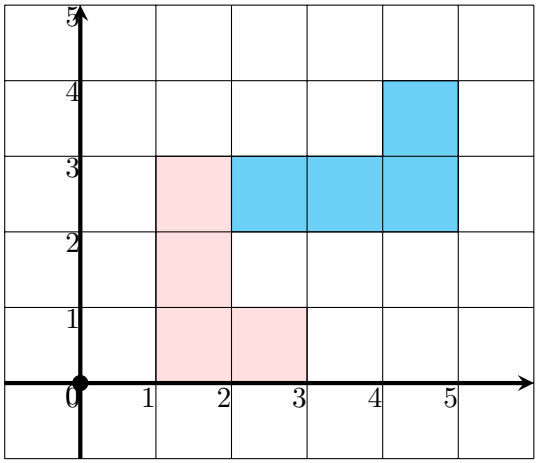
(8) discuss isometries of $\mathbb{R}^2$: translations, reflections, rotations, glide reflections. Describe which kind of isometry relates the following figures? Can you find centers of rotations, mirror lines, and amounts of translation?



Support class 1

Find equations for the isometries taking some of these to each other. To describe these in terms of linear maps, you will need to fix an origin, and basis. Try with **0** at lower left.

For vevox quiz in Lecture 2. This may be useful to go over a concrete example before the abstract example in Q(1).




What is the isometry $T$ that takes the pink L to the blue L?
Answer: a rotation 90° anti-clockwise about $(2, 3)$
To write this in the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$, use that you should know that rotation 90° about anti-clockwise the origin is given by

$$\mathbf{x} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}$$

One way to find an equation for $T$ is as follows:

$$\begin{array}{lllllll} \text{translate center of rotation to } \mathbf{0}: & R: & \mathbf{x} & \mapsto & \mathbf{y} & = & \mathbf{x} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ \text{rotate } 90° \text{ about } \mathbf{0}: & T: & \mathbf{y} & \mapsto & \mathbf{z} & = & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{y} \\ \text{translate } \mathbf{0} \text{ back to center of rotation}: & R^{-1}: & \mathbf{z} & \mapsto & & & \mathbf{z} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{array}$$

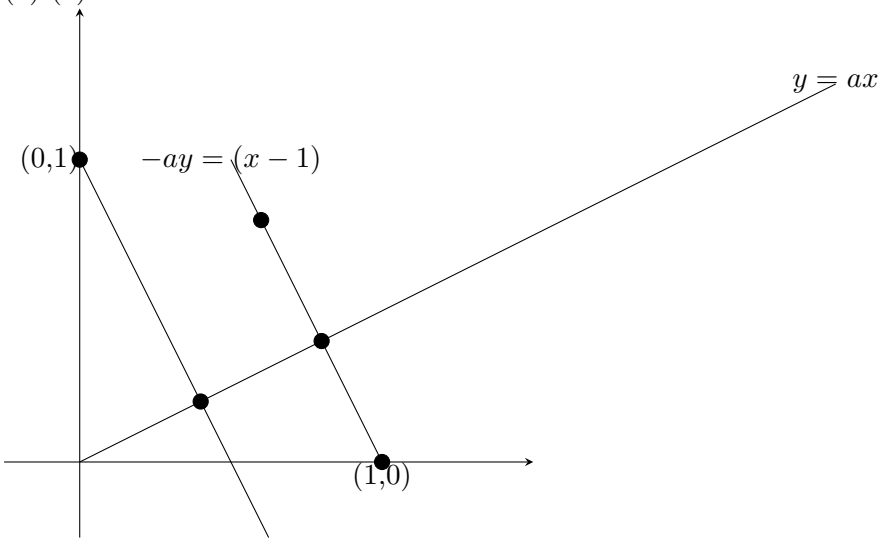
Support class 1

Note that this means the transformation is $R^{-1}TR$. This is the conjugation of $T$ by $R$. **Conjugtion** is a very important operation in mathematics, especially in group theory.
The set of all isometries is a group under composition.
To work out the isometry in the form $A\mathbf{x} + \mathbf{b}$ just compose these maps:

$$
\begin{aligned}
R^{-1}\Big(T(R(\mathbf{x}))\Big) &= R^{-1}\left(T\left(\mathbf{x}-\begin{pmatrix}2\\3\end{pmatrix}\right)\right)\\
&= R^{-1}\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}2\\3\end{pmatrix}\right)\right)\\
&= R^{-1}\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\mathbf{x}-\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}\right)\\
&= R^{-1}\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\mathbf{x}-\begin{pmatrix}3\\-2\end{pmatrix}\right)\\
&= \left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\mathbf{x}-\begin{pmatrix}3\\-2\end{pmatrix}\right)+\begin{pmatrix}2\\3\end{pmatrix}\\
&= \begin{pmatrix}0&1\\-1&0\end{pmatrix}\mathbf{x}+\begin{pmatrix}5\\1\end{pmatrix}
\end{aligned}
$$

Support class 1

Solutions to Support class questions:
(1) (a)




The lines through $(1,0)$ which is perpendicular to the mirror line $y = ax$ is $-ay = (x-1)$. This intersects the mirror line at $-a^2x = (x-1)$, so at the intersection $x = 1/(1+a^2)$. The $y$ coordinate is $ax = a/(1+a^2)$. The vector from $(1,0)$ to $(1/(1+a^2), a/(1+a^2)$ is $(1/(1+a^2)-1, a/(1+a^2)$. We have to add 2 of this to $(1,0)$ to get the reflection of $(1,0)$, which is at

$$(1,0) + 2\Big(\frac{1}{(1+a^2)}(1,a) - (1,0)\Big) = \frac{2(1,a)}{(1+a^2)} - (1,0) = \frac{(1-a^2,2a)}{(1+a^2)}$$

So, the first column of the transformation matrix is the image of $(1,0)$:

$$\begin{pmatrix} \frac{1-a^2}{1+a^2} \\ \\ \frac{2a}{1+a^2} \end{pmatrix}$$

Similarly, the intersection of the line $-a(y-1) = x$ perpendicular the mirror line through $(0,1)$ is at $(a, a^2)/(1+a^2)$. So the image of $(0,1)$ is at

$$(0,1) + 2((a,a^2)/(1+a^2) - (0,1)) = (2a, a^2-1)/(1+a^2)$$

So, the transformation matrix, which has columns the images of $(1,0)$ and $(0,1)$ is

$$A = \begin{pmatrix} \frac{1-a^2}{1+a^2} & \frac{2a}{1+a^2} \\ \\ \frac{2a}{1+a^2} & -\frac{1-a^2}{1+a^2} \end{pmatrix}$$

You can compare this to the form in terms of cos and sin $atan(a)$ etc, and that the determinant is $-1$.
(b) In order to reflect in $y = ax + b$, notice that the line $y = ax + b$ translates to the line $y = ax$ by the map $T : \mathbf{x} \mapsto \mathbf{x} - (0,b)$. So, the required translation is obtained by conjugating the transformation in (a) by $T$, as in the example on the previous page.
We get

$$\mathbf{x} \to A\mathbf{x} - A\begin{pmatrix} 0 \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\mathbf{x} \to \begin{pmatrix} \frac{1-a^2}{1+a^2} & \frac{2a}{1+a^2} \\ \\ \frac{2a}{1+a^2} & -\frac{1-a^2}{1+a^2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} \frac{-2ab}{1+a^2} \\ \\ \frac{b(1-a^2)}{1+a^2} + b \end{pmatrix}$$

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(c) From the previous question, the reflection in $x + y = 1$ is obtained with $a = -1, b = 1$ so we get

$$T : \mathbf{x} \to \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

the reflection in $y = 1$ is obtained with $a = 0, b = 1$

$$R : \mathbf{x} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

The composition is

$$T(R(\mathbf{x})) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(d) The fixed point (or points) is(are) the solution to

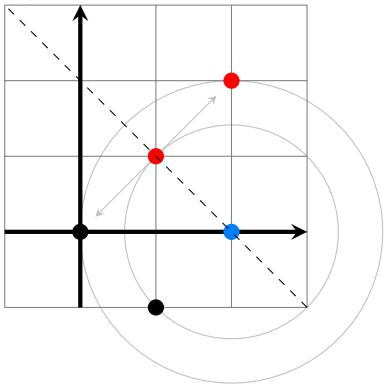
$$\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So $(0, 1)$ is the fixed point.
This can also be seen by noticing that this is the point where the two reflection lines intersect, so is fixed by both these reflections.
This rotation is a rotation of 90 anti-clockwise about $(0, 1)$.
(e) A composition of 2 reflections is a rotation if the reflection lines intersect. Otherwise it is a translation.
(2) Relevant points:



(a) The reflection is in the line $x + y = 2$. We can use the formula from Q1, with $a = -1$ and $b = 2$, to see this is given by:

$$\mathbf{x} \to \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

(b) This is a rotation through 90 degrees clockwise about $(2, 0)$. If we use the formula for a rotation matrix, we know that

Support class 1

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Now to find the expression for the rotation about $(2, 0)$. We can either conjugate $A$ by the map $R : \mathbf{x} \to \mathbf{x} - (2, 0)$; or else solve for $\mathbf{b}$ from knowing that $(0, 2)$ is the fixed point: $(0, 2) = A(0, 2) + \mathbf{b}$; or, we can use the fact that we know that $(0, 0)$ maps to $(2, 2)$, so we must have $\mathbf{b} = (2, 2)$, and the map must be:

$$\mathbf{x} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

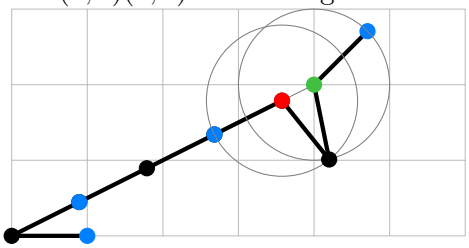
(c) The eigen values of $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ are 1 and $-1$. The corresponding eigen vectors are $(1, -1)$ and $(1, 1)$. You should check you know how to find these; look back at linear algebra notes from year 1 if necessary. The eigen values of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ are $i$ and $-i$. The corresponding eigen vectors are $(1, -i)$ and $(1, i)$.

(d) A reflection matrix has eigen values 1 and $-1$. A matrix corresponding to $\theta$ anti-clockwise rotation has eigen values $\exp(\pm i\theta)$.

(e) A linear map $L$ has an eigen value $\lambda$ with eigen vector $\mathbf{v}$ if $L(\mathbf{v}) = \lambda\mathbf{v}$.

If $T$ is an isometry which is not a linear map, then if $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, if we try to solve to find a vector with $T(\mathbf{v}) = \lambda\mathbf{v}$, we are trying to solve $A\mathbf{v} + \mathbf{b} = \lambda\mathbf{v}$, i.e., $(A - \lambda)\mathbf{v} = -\mathbf{b}$. If $\lambda$ is not an eigen value of $A$, then $(A - \lambda)$ has an inverse, so we can solve to find a corresponding $\mathbf{v}$ with $T(\mathbf{v}) = \lambda\mathbf{v}$. So, we might say that all possible values of $\lambda$ which are not eigen values of $A$ are eigen values of $T$, but this is not very useful. On the other hand, if $\lambda$ is an eigen value of $A$, then we know that the map $A - \lambda I$ has non-zero kernel, spanned by the eigen value. So, by the rank nullity theorem, the image of $A - \lambda I$ has rank at most 1, and if $\mathbf{b}$ is not chosen to be in the image of $A = \lambda I$, then there is no solution to $T(\mathbf{v}) = \lambda\mathbf{v}$, so in general we can't say that $\lambda$ is an eigen value of $T$, if we use the usual definition of eigen value being a vector which is sent to a multiple of itself. So, if we do want to talk about eigen values of isometries, we have to be clear that we must talk about the eigen values of the linear part of the isometry.

(3) This is from [RS, 2.12, page 33]. Note, the "sofa" is being moved in an infinite plane with no obstacles in the way. There are infinitely many ways to do this. Any chain of length one line segments from the line $(0, 0)(1, 0)$ to the target line will give a sequence. For example, consider the method in the figure:



Suppose the goal is to get your "sofa" to a line $PQ$, where $P$ is the closest end to $\mathbf{0}$, marked in green on this figure. Draw a line from $(0, 0)$ to the green point, $P$. Now rotate your sofa about $\mathbf{0}$ to line up along this line. Then repeatedly rotate through 180° until the closest end of your sofa is less than distance one from $P$. This is the red dot. If the red and green dots do not coincide, the draw circles radius 1 about these dots, and mark the point of intersection by $X$, then, rotate about the red dot so the other end is rotated to $X$. Then rotate about $X$ so that end that is at the red dot rotates to the green dot. Finally, rotate about the green point $P$ until the sofa is in the required position.

This argument does not give any explicit maps and coordinates. Do to this, we could, for example, suppose that the end $P$ is at a point $(x, y)$, and the end $Q$ is at $(x + \cos(\theta), y + \sin(\theta))$ for some angle $\theta$. Assume for convenience that $P$ has positive coordinates. Then the first rotation is about the point $(0, 0)$, and the angle of the first rotation is $\phi = \arctan(y/x)$. The next rotations are of 180 degrees, about points at positions $n(\cos(\phi), \sin(\phi))$ for $n = 1, 2, ...$, up to a value of $n$ such that the length from $n(\cos(\phi), \sin(\phi))$ to $P$ is less than 1. It's a bit more work to find the intersection point and values of the last few rotations, so I'm going to skip this.

The main point of this argument is to use the fact that you can rotate about a point $P$ to take any point $Q$ distance 1 from $P$ to another point $Q'$ also distance 1 away from $P$. Given the points $P$, $Q$ and $Q'$, we can find a matrix explicitly by writing $Q - P = (\cos(\phi), \sin(\phi))$ and $Q' - P = (\cos(\phi'), \sin(\phi'))$. Then if $R$ is translation $\mathbf{x} \to \mathbf{x} - P$, and $T$ is the rotation through $\phi' - \phi$, then the required map is the conjugate $R^{-1}TR$; matrices for this can all be written out explicitly, in a similar way to concrete examples given in previous questions.

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We also use here that isometies map lines to lines, so if you know where the end points map to, this detemines where the intermediate points of a line segment map.

(4) A composition of isometries is an isometry, so we just need to check that if $\mathbb{E}^n$ and $\mathbb{E}^m$ are isometric then $n = m$. We will prove in lecture four that an isometry $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is affine, and so $L(\mathbf{z}) = T(\mathbf{z}) - T(\mathbf{0})$ is linear. An isometry is bijective, and so $L$ is a bijection. So the kernel has zero dimension, and the image has dimension $m$. From linear algebra, e.g., from the rank nullity theorem, this means that $n = m$.

(5) We know from a result in lecture 3 that for some $\lambda \in \mathbb{R}$ we have $\mathbf{z} = \mathbf{x} + \lambda(\mathbf{y} - \mathbf{x})$ with $\lambda \in (0, 1)$.
If $\lambda \in (0, 1)$ we are done.
Note!!! I am assuming $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are distinct; if they are not, then we aim to show that $\lambda \in [0, 1]$, but this is trivial, since if $\mathbf{z} = \mathbf{x}$, possibly relabling, then we just take $\lambda = 0$.
So, assuming the points are all distinct, if $\lambda \notin (0, 1)$, then either $\lambda < 0$ or $\lambda > 1$.
Write

$$\mathbf{z} = \lambda_1 \mathbf{x} + \lambda_2 \mathbf{y}$$

where $\lambda_1 + \lambda_2 = 1$, and $\lambda_2 = \lambda$.
If $\lambda_2 = \lambda < 0$, we have $\mathbf{x}$ "in the middle", that is, divide through by $\lambda_1$, and rearrange, and we have

$$\mathbf{x} = \frac{1}{\lambda_1}\mathbf{z} + \frac{-\lambda_2}{\lambda_1}\mathbf{y}$$

Since $\lambda_1 + \lambda_2 = 1$, the sum of these coefficients $\frac{1}{\lambda_1}$ and $\frac{-\lambda_2}{\lambda_1}$ is still 1, but now since $\lambda_1 = 1 - \lambda_2 > 0$ and $-\lambda_2 > 0$, these coefficients both are in the range $(0, 1)$ as required; we can choose which of the points we want to call $\mathbf{x}$ and which $\mathbf{y}$, and we will change the name of the original $\mathbf{x}$ to $\mathbf{z}$, and have the required expression.
If $\lambda > 1$, then we divide through by $\lambda_1 = 1 - \lambda$, and obtain

$$\mathbf{y} = \frac{1}{\lambda_2}\mathbf{z} + \frac{-\lambda_1}{\lambda_2}\mathbf{x}.$$

Again it's easy to check that these coefficients are both in the range $(0, 1)$, so we can relabel the points as required.

(6) An isometry $T$ is distance preserving and a bijection.
The inverse of a bijective map is also bijective (quote this from foundations, you don't need to rewrite a proof).
To prove distance preserving, use that $P$ is distance preserving:

$$d(T^{-1}(P), T^{-1}(Q)) = d(T(T^{-1}(P)), T(T^{-1}(Q))) = d(P, Q)$$

and we're done.

(7) Any map of the form

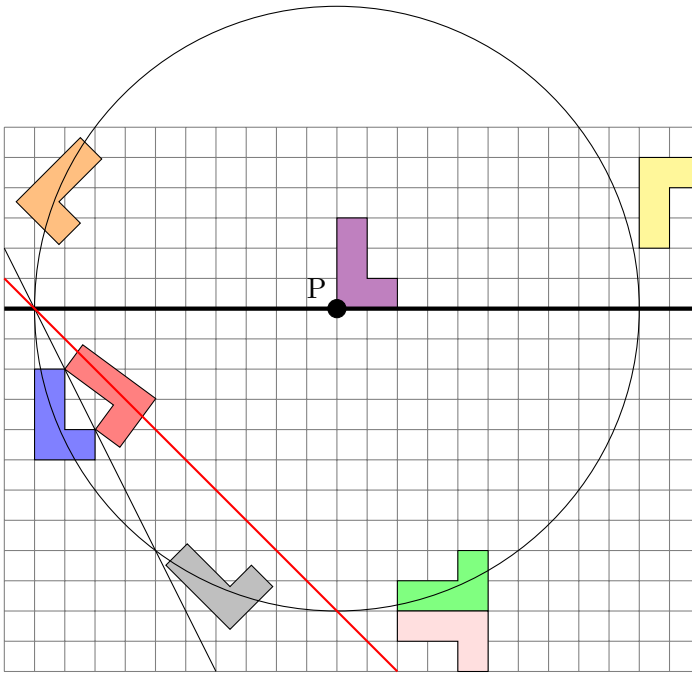
$$\mathbf{x} \to A\mathbf{x} + \mathbf{b}$$

where $A$ is a $2 \times 2$ matrix and $\mathbf{b}$ any vector in $\mathbb{R}^2$, will map lines to lines, but is not an isometry unless $A$ represents a reflection or rotation.

(8)
Support class 1
In the figure, just pick a couple of examples. To compute the transformation explicitly, one method is to translate the first shape so that the lower left corner of the L shape (lower left when rotate to look like an L), is at the origin. Then rotate or reflect to be in the same orientation as the target shape, then translate so that the lower left corner is at the correct position for the target shape.
It may be helpful to use that the center of the rotation mapping the orange, blue, gray, and green shapes to each other is at $(11, 12)$ relative to an origin at the bottom left corner of the grid. The rotations are through multiples of $45°$.

For this question, I'm just looking for a description, e.g.,

| | |
|---|---|
| blue to orange: | rotation 45° clockwise about $P = (11, 12)$ |
| blue to purple: | translation by $(10, 5)$ |
| blue to red: | reflection in the line $y = 14 - 2x$ |
| blue to yellow: | glide reflection: translate by $(20, 0)$, then reflect in the line $y = 12$. |
| blue to grey | rotation 45° anti clockwise about $P = (11, 12)$ |
| blue to green | rotation 90° anti clockwise about $P = (11, 12)$ |
| blue to pink: | glide reflection along the red line $y = 13 - x$; that is, translate by $(10, -10)$ then reflect. |




## 50. MA243 Support Class 2

This was last year's assignment 1:
`https://moodle.warwick.ac.uk/mod/resource/view.php?id=1887360`
Solutions are here:
`https://moodle.warwick.ac.uk/mod/resource/view.php?id=1896687`

Support class 1

## 51. MA243 Support Class 3, week 4, October 2022

(1) (Computation: Decomposition of an isometry into product of reflections)
There is an isometry $T : \mathbb{R}^3 \to \mathbb{R}^3$ which maps points as follows:

$$\begin{aligned} T(\ 0,\ \ 1,\ \ 1) &= (2,1,1) \\ T(\ 0,\ \ 1,-1) &= (2,1,-1) \\ T(\ 1,-1,\ \ 0) &= (0,2,0) \\ T(-1,-1,\ \ 0) &= (0,0,0) \end{aligned}$$

(a) What is $T(\mathbf{0})$?
(b) Find a matrix $A$ and a vector $\mathbf{b}$ such that $T(\mathbf{x}) = A + \mathbf{b}$. Are $A$ and $\mathbf{b}$ unique?
(c) Find the set $\mathrm{Fix}(T)$ of fixed points of $T$.
(d) Write $T$ as a composition of 3 reflections. This means you have to either describe the reflections as maps of the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$, or you can describe the hyperplanes fixed by the reflections.
(e) Is this decomposition unique? Either prove it is, or find a different decomposition.

(2) Prove that for $\mathbf{c} \in \mathbb{R}^n, \mathbf{c} \neq \mathbf{0}$ and $\beta \in \mathbb{R}$, the following set is a hyperplane: $\{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \mathbf{c} \rangle = \beta\}$
(3) Prove that if $R$ is a reflection and $T$ is any isometry, then $T^{-1} \circ R \circ T$ is also a reflection.
(4) If $\rho_\Pi$ is a reflection in the plane $\Pi$, prove that the conjugation of $\rho_\Pi$ by the reflection $T(\mathbf{x}) = \mathbf{x}+\mathbf{c}$, that is, the map $T^{-1} \circ \rho_\Pi \circ T$ is a reflection in $\Pi - \mathbf{c}$.
(5) Suppose that $R$ is a reflection in $\Pi = V + \mathbf{b}$, and $\mathbf{v} \in V$, then show that $R$ commutes with the isometry $T : \mathbf{x} \mapsto \mathbf{x} + \mathbf{v}$.
(6) Prove that a reflection $R$ has order 2, that is, $R \circ R = \mathrm{id}$.
(7) Prove that $O(n)$ is a group.
(8) Prove that $\mathrm{Isom}(X, d)$ is a group for any metric space $(X, d)$
(9) Show that if $L = \mathbf{P} + \mathbb{R}\mathbf{v}$, and $L' = \mathbf{P}' + \mathbb{R}\mathbf{v}'$, then $L = L' \iff \mathbf{v}'$ and $\mathbf{P} - \mathbf{P}'$ are both in $\mathbb{R}\mathbf{v}$.
(10) Show that for $a_1, a_2, \ldots, a_k \in \mathbb{R}^n$ with $\sum_{i=1}^k a_i = 1$, then for an isometry $T$, then for points $\mathbf{P}_1, \ldots \mathbf{P}_k$,

$$T\left(\sum_{i=1}^k a_i \mathbf{P}_i\right) = \sum_{i=1}^k a_i T(\mathbf{P}_i)$$

(a) Since $T$ is an isometry, $T$ preserves lines, and the affine relationship between points on lines (Notes, Proposition 2) so,

$$\begin{aligned} T\left(\frac{1}{2}(0,1,1) + \frac{1}{2}(0,1,-1)\right) &= \frac{1}{2}T(0,1,1) + \frac{1}{2}T(0,1,-1) \\ \Rightarrow T((0,1,0) &= \frac{1}{2}(2,1,1) + \frac{1}{2}(2,1,-1) = (2,1,0) \\ T\left(\frac{1}{2}(1,-1,0) + \frac{1}{2}(-1,-1,0)\right) &= \frac{1}{2}T(1,-1,0) + \frac{1}{2}T(-1,-1,0) \\ \Rightarrow T((0,-1,0) &= \frac{1}{2}(0,2,0) + \frac{1}{2}(0,0,0) = (0,1,0) \end{aligned}$$

From these two we get:

$$\begin{aligned} T\left(\frac{1}{2}(0,1,0) + \frac{1}{2}(0,-1,0)\right) &= \frac{1}{2}T(0,1,0) + \frac{1}{2}T(0,-1,0) \\ \Rightarrow T((0,0,0) &= \frac{1}{2}(2,1,0) + \frac{1}{2}(0,1,0) = (1,1,0) \end{aligned}$$

$$T(\mathbf{0}) = (1,1,0)$$

Support class 3

(b) Now we know $\mathbf{b} = T(\mathbf{0})$, we can write a linear map $L(\mathbf{x}) = T(\mathbf{x}) - T(\mathbf{0})$, and we have

$$\begin{aligned} L(\ 0,\ \ 1,\ \ 1) &= (2,1,1) - (1,1,0) = (1,0,1) \\ L(\ 0,\ \ 1,-1) &= (2,1,-1) - (1,1,0) = (1,0,-1) \\ L(\ 1,-1,\ \ 0) &= (0,2,0) - (1,1,0) = (-1,1,0) \end{aligned}$$

Now from this, writing these as columns of an input and output matrix, we have

$$A\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

So

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$T(\mathbf{x}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

(c) To find $\mathrm{Fix}(T)$, we have to solve $T(\mathbf{x}) = \mathbf{x}$, which means:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} y+1 \\ x+1 \\ z \end{pmatrix}$$

If this had a solution, $x = y + 1 = x + 2$, which is impossible, so

$$\mathrm{Fix}(T) = \varnothing$$

Note that since the matrix $A$ represents a reflection, but there are no fixed points, this means $T$ describes a glide reflection.

(d) First, find a reflection $R$ mapping $T(\mathbf{0})$ to $\mathbf{0}$. Since $T(\mathbf{0}) = (1,1,0)$, the point half way between $\mathbf{0}$ and $T(\mathbf{0})$ is $(1/2, 1/2, 0)$, and the direction of the mirror hyperplane is perpendicular to this. If we take an orthonomal basis consisting of the normalized vector $\mathbf{v} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, and any vectors spanning the space perpendicular to this, such as $(1/\sqrt{2}, -1/\sqrt{2}, 0)$ and $(0,0,1)$, then with respect to this basis, the matrix part of the reflection is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The change of basis matrix from the standard basis is

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So, with respect to the standard basis, the matrix part of $R$ is

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $R$ maps $T(\mathbf{0})$ to $(1,1,0)$, the reflection $R$ is given by

$$R(\mathbf{x}) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Support class 3

This is a reflection in the hyperplane given by

$$\Pi_1 = \{(x, y, z) : x + y = 1\}$$

You can check this by computing $\text{Fix}(R)$

Now we have $R \circ T(\mathbf{0}) = \mathbf{0}$, so $R \circ T$ is a linear map with matrix:

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note, here we can use $T_{A,\mathbf{b}} \circ T_{A',\mathbf{b}'} = T_{AA',A\mathbf{b}'+\mathbf{b}}$, and since we know that $A\mathbf{b}' + \mathbf{b}$ will be zero since the composition is linear, we just compute $AA'$.

The matrix $R \circ T$ can be decomposed as:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These matrices represent a reflection in $y = 0$ followed by a reflection in $x = 0$.

Note, if the matrix was not already diagonal, we could put into normal form, to find the decomposition into a product of reflections and rotations, and then decompose rotations into a product of 2 reflections.

Putting this all together,

$$T = R \circ U \circ V$$

where

$$R \text{ is the reflection in the plane } x + y = 1$$
$$U \text{ is the reflection in the plane } x = 0$$
$$V \text{ is the reflection in the plane } y = 0$$

(e) There are infinitely many ways to decompose $T$ into a product of reflections, for example, since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, we have

$$T = R \circ U' \circ V'$$

where

$$R \text{ is the reflection in the plane } x + y = 1$$
$$U' \text{ is the reflection in the plane } x = y$$
$$V' \text{ is the reflection in the plane } x = -y$$

(2) Prove that for $\mathbf{c} \in \mathbb{R}^n, \mathbf{c} \neq \mathbf{0}$ and $\beta \in \mathbb{R}$, the following set is a hyperplane: $\{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \mathbf{c} \rangle = \beta\}$

A hyperplane is a set of the form $V + \mathbf{b}$ for a vector space of dimension $n - 1$ and a vector $\mathbf{b}$.
Let $V = (\mathbb{R}\mathbf{c})^\perp$, the orthogonal complement to $\mathbb{R}\mathbf{c}$. This has dimension $n - 1$. We assume this result from linear algebra; the proof involves extending a basis for $\mathbb{R}\mathbf{c}$ to an orthogonal basis for $\mathbb{R}^n$. Let $\mathbf{b} = \frac{\beta}{||\mathbf{c}||}\mathbf{c}$. We will show that

$$\{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \mathbf{c} \rangle = \beta\} = V + \mathbf{b}$$

$$\begin{aligned} \mathbf{v} \in V + \mathbf{b} &\iff \mathbf{v} - \mathbf{b} \in V \\ &\iff \mathbf{v} - \mathbf{b} \in (\mathbb{R}\mathbf{c})^\perp \\ &\iff \langle \mathbf{v} - \mathbf{b}, \mathbf{c} \rangle = 0 \\ &\iff \langle \mathbf{v}, \mathbf{c} \rangle = \langle \mathbf{b}, \mathbf{c} \rangle = \beta \end{aligned}$$

Support class 3

(3) Prove that if $R$ is a reflection and $T$ is any isometry, then $T^{-1} \circ R \circ T$ is also a reflection.

Let $\Pi$ be the hyperplane fixed by $R$. Let $\Pi = V + \mathbf{c}$ for some vector subspace $V$ and some vector $\mathbf{c}$. Then

$$\begin{aligned}\mathbf{v} \in \mathrm{Fix}(T^{-1} \circ R \circ T) &\iff T^{-1} \circ R \circ T(\mathbf{v}) = \mathbf{v} \\ &\iff R \circ T(\mathbf{v}) = T(\mathbf{v}) \\ &\iff T(\mathbf{v}) =\in \mathrm{Fix}(R) = \Pi \\ &\iff \mathbf{v} =\in T^{-1}(\Pi)\end{aligned}$$

If we can show that $T^{-1}(\Pi)$ is a hyperplane, then we are done.
We know that the inverse of an isometry is also an isometry, and that isometries are given in terms of an orthogonal matrix and a vector.
Suppose that $T^{-1}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for some orthogonal matrix $A$ and some vector $\mathbf{b}$.
Then $T^{-1}(\Pi) = A(\Pi) + \mathbf{b} = A(V + \mathbf{c}) + \mathbf{b} = A(V) + A\mathbf{b} + \mathbf{c}$.
We claim that $A(V)$ is an $n-1$ dimensional vector space. This is because if $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ is a basis for $V$, then since $A$ is an invertible linear map, $A(\mathbf{v}_1), \ldots, A(\mathbf{v}_{n-1})$ are also linearly independent, and their span is containedequal to the span of $T(V)$. Hence they are a basis of $A(V)$, and so $A(V)$ is an $n-1$ dimensional subspace of $\mathbb{R}^n$.
Hence $A(V) + A\mathbf{b} + \mathbf{c}$ is a hyperplane, and so $T^{-1} \circ R \circ T$ is a reflection.

(4) If $\rho_\Pi$ is a reflection in the plane $\Pi$, prove that the conjugation of $\rho_\Pi$ by the reflection $T(\mathbf{x}) = \mathbf{x} + \mathbf{c}$, that is, the map $T^{-1} \circ \rho_\Pi \circ T$ is a reflection in $\Pi - \mathbf{c}$.

By the previous question, $T^{-1} \circ \rho_\Pi \circ T$ is a reflection.

$$\begin{aligned}\mathbf{v} \in \mathrm{Fix}(T^{-1} \circ \rho_\Pi \circ T) &\iff T^{-1} \circ \rho_\Pi \circ T(\mathbf{v}) = \mathbf{v}) \\ &\iff \rho_\Pi \circ T(\mathbf{v}) = T(\mathbf{v}) \\ &\iff T(\mathbf{v}) \in \mathrm{Fix}(\rho_\Pi) = \Pi \\ &\iff \mathbf{v} + \mathbf{c} \in \Pi \\ &\iff \mathbf{v} \in \Pi - \mathbf{c}\end{aligned}$$

Hence $T^{-1} \circ \rho_\Pi \circ T$ is a reflection in $\Pi - \mathbf{c}$

(5) Suppose that $R$ is a reflection in $\Pi = V + \mathbf{b}$, and $\mathbf{v} \in V$, then show that $R$ commutes with the isometry $T : \mathbf{x} \mapsto \mathbf{x} + \mathbf{v}$.

By the previous question, $T^{-1}RT$ is the reflection in the plane $\Pi - \mathbf{v} = V + \mathbf{b} - vv = V - \mathbf{v} + \mathbf{b} = T(V) + \mathbf{b} = V + \mathbf{b} = \Pi$, where we have used that for any vector $\mathbf{v}$ in $V$, $V - \mathbf{v} = V$, which follows since a vector subspace is closed under addition of vectors in that subspace, and since the inverse of subtracting $\mathbf{v}$ is adding $\mathbf{v}$, the isometry $T$ is a bijection on $V$, so $T(V) = V$.
Now use the fact that a reflection in a hyperplane is unique. Hence since $R$ and $T^{-1}RT$ are both reflections in $\Pi$, they must be equal, so $R = T^{-1}RT$. From this we obtain $TR = RT$, i.e., these isometries commute.
This is why, if we use a translation as in Question (4), to obtain the reflection in $\Pi$, it doesn't make any difference which element of $V + \mathbf{c}$ we translate by - they will all give the same result.

(6) Prove that a reflection $R$ has order 2, that is, $R \circ R = \mathrm{id}$.

There are various ways to prove this result. We will use the formula for the reflection given by

$$R(\mathbf{P}) = \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v} \rangle \mathbf{v}$$

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where $\Pi = \mathbf{b} + (\mathbb{R}\mathbf{v})^{\perp}$ and $||\mathbf{v}|| = 1$. So,

$$\begin{aligned} R(R(\mathbf{P})) &= R(\mathbf{P}) - 2\langle R(\mathbf{P}) - \mathbf{b}, \mathbf{v}\rangle \mathbf{v} \\ &= \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v} - 2\langle \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v} \\ &= \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v} - 2\langle -2\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v}, \mathbf{v}\rangle \mathbf{v} \\ &= \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v} + 4\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \langle \mathbf{v}, \mathbf{v}\rangle \mathbf{v} \\ &= \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v} + 4\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v} \\ &= \mathbf{P} \end{aligned}$$

(7) Prove that $O(n)$ is a group.

We need to show that $O(n)$ contains an identity, inverses, and is closed under multiplication.
(identity): $I$ is orthogonal, since $I^T I = I^2 = I$.
(inverses): If $A \in O(n)$, then $A^T A = I$, so $A^T$ is the inverse of $A$, and since $(A^T)^T = A$, so $(A^T)^T A^T = I$, so we have $A^T$ is also orthogonal
(closure): If $A$ and $B$ are orthogonal, then $(AB)^T(AB) = (B^T A^T)(AB) = B^T(A^T A)B = B^T B = I$, where we have used that matrix multiplcation is associative, and that $(AB)^T = A^T B^T$. (If you have not seen this in linear algebra or have forgotten, and need help to work it out, ask your support class teacher)

(8) Prove that $\mathrm{Isom}(X, d)$ is a group for any metric space $(X, d)$

We need to show that $\mathrm{Isom}(X, d)$ contains an identity, inverses, and is closed under composition.
(identity): The identity map is clealy an isomerty.
(inverses): Since an isometry $T$ is by definition bijective, it has an inverse $T^{-1}$. We have $d(T(\mathbf{x}), T(\mathbf{y})) = d(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$ since $T$ is an isometry. Replacing $\mathbf{x}$ by $T^{-1}(\mathbf{x})$ and $\mathbf{y}$ by $T^{-1}\mathbf{y}$ in this statement shows that $T^{-1}$ is also distance preserving.
(closure): Let $T$ and $R$ be isometries in $\mathrm{Isom}(X, d)$. Since the composition of bijections is a bijection, $T \circ R$ is a bijection.
Since $T$ and $R$ are distance preserving, we have

$$d(T \circ R(\mathbf{x}), T \circ R(\mathbf{y})) = d(T(R(\mathbf{x})), T(R(\mathbf{y}))) = d(R(\mathbf{x}), R(\mathbf{y})) = d(\mathbf{x}, \mathbf{y})$$

So $T \circ R$ is also an isometry.

(9) Show that if $L = \mathbf{P} + \mathbb{R}\mathbf{v}$, and $L' = \mathbf{P}' + \mathbb{R}\mathbf{v}'$, then $L = L' \iff \mathbf{v}'$ and $\mathbf{P} - \mathbf{P}'$ are both in $\mathbb{R}\mathbf{v}$.

$$\mathbf{P} + \mathbb{R}\mathbf{v} = \mathbf{P}' + \mathbb{R}\mathbf{v}' \iff \mathbb{R}\mathbf{v} = \mathbb{R}\mathbf{v}' + (\mathbf{P}' - \mathbf{P})$$

In order for $\mathbb{R}\mathbf{v} = \mathbb{R}\mathbf{v}' + (\mathbf{P}' - \mathbf{P})$, since $\mathbf{0} \in \mathbb{R}\mathbf{v}$, we must have $\mathbf{0} \in \mathbb{R}\mathbf{v}' + (\mathbf{P}' - \mathbf{P})$, so this implies $\mathbf{P}' - \mathbf{P}$ is a multiple of $\mathbf{v}'$. This implies that $\mathbb{R}\mathbf{v}' + (\mathbf{P}' - \mathbf{P}) = \mathbb{R}\mathbf{v}'$, and so $\mathbb{R}\mathbf{v} = \mathbb{R}\mathbf{v}'$ from which we obtain that $\mathbf{v}'$ is a multiple of $\mathbf{v}$, and so if $L = L'$ then $\mathbf{v}'$ and $\mathbf{P} - \mathbf{P}'$ are both in $\mathbb{R}\mathbf{v}$.
Conversely, if $\mathbf{v}' = \alpha^{-1}\mathbf{v}$ and $\mathbf{P} - \mathbf{P}' = \beta\mathbf{v}$, where by defintion of lines, $\mathbf{v}$ and $\mathbf{v}'$ are non-zero, so $\alpha$ and $\beta$ are also non-zero. Then $\mathbf{P} + \mathbb{R}\mathbf{v} = \mathbf{P}' + \beta\mathbf{v}' + \mathbb{R}\alpha\mathbf{v}' = \mathbf{P}' + \mathbb{R}\mathbf{v}'$, so the converse also holds.

(10) Show that for $a_1, a_2, \ldots, a_k \in \mathbb{R}^n$ with $\sum_{i=1}^k a_i = 1$, then for an isometry $T$, then for points $\mathbf{P}_1, \ldots \mathbf{P}_k$,

$$T\left(\sum_{i=1}^k a_i \mathbf{P}_i\right) = \sum_{i=1}^k a_i T(\mathbf{P}_i)$$

We prove this result by induction. The case for $k = 1$ is trivial, since then $a_1 = 1$ and the result just says $T(\mathbf{P}_1) = T(\mathbf{P}_1)$.
We know that isometries preserve lines, and satisfy

(*) $$T(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda T(\mathbf{x}) + \mu T(\mathbf{y})$$

provided $\lambda + \mu = 1$.
Suppose we have proven the result for $k = m$.

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Now suppose $a_1+\cdots+a_m+a_{m+1}=1$. Then we have $a_1+\cdots+a_m=1-a_{m+1}$, so $\frac{a_1}{1-a_{m+1}}+\cdots+\frac{a_m}{1-a_{m+1}}=1$, so

$$\begin{aligned}
T\left(\sum_{i=1}^{m+1} a_i\mathbf{P}_i\right) &= T\left(\sum_{i=1}^{m} a_i\mathbf{P}_i + a_{m+1}P_{m+1}\right)\\
&= T\left((1-a_{m+1})\left(\sum_{i=1}^{m}\frac{a_i}{1-a_{m+1}}\mathbf{P}_i\right) + a_{m+1}P_{m+1}\right)\\
&= (1-a_{m+1})T\left(\sum_{i=1}^{m}\frac{a_i}{1-a_{m+1}}\mathbf{P}_i\right) + a_{m+1}T(P_{m+1}) \text{ by } (*)\\
&= (1-a_{m+1})T\sum_{i=1}^{m}\frac{a_i}{1-a_{m+1}}T(\mathbf{P}_i) + a_{m+1}T(P_{m+1}) \text{ by inductive hypothesis}\\
&= \sum_{i=1}^{m+1} a_iT(\mathbf{P}_i),
\end{aligned}$$

which is the required result, and so the result hold for all $k \geq 1$ by induction.

## 52. MA243 Support Class 4

Assignment 2 from last year:
`https://moodle.warwick.ac.uk/mod/resource/view.php?id=1901885`
Solutions:
`https://moodle.warwick.ac.uk/mod/resource/view.php?id=1901887`
From April exam 2021:
`https://moodle.warwick.ac.uk/mod/url/view.php?id=1903386`
Solutions:
`https://moodle.warwick.ac.uk/mod/url/view.php?id=1903382`

Support class 5

## 53. MA243 Support Class 5, week 6, October 2022

(1) Prove Lemma 24 from class: $O(1,1)$ and $O^+(1,1)$ are groups.
(2) Prove Lemma 25 from class: (1) the standard basis is a Lorentz orthonormal basis; (2) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are Lorentz orthonormal vectors in $\mathbb{R}^n$ then they from a basis, and in this case (3) $\langle \mathbf{v}_i, \mathbf{v}_j \rangle_L = J_{i,j}$
(3) There are 16 matrices of the form

$$\begin{pmatrix} \pm 2 & \pm\sqrt{3} \\ \pm\sqrt{3} & \pm 2 \end{pmatrix}$$

Decide which of these are Lorentz matrices, which are positive – that is whether they are in $O(1,1)$ and if so whether they are in $O^+(1,1)$. If they are in $O^+(1,1)$ decide whether they correspond to a reflection or translation.
(4) Show that the matrix

$$\begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ \sinh\alpha & -\cosh\alpha \end{pmatrix}$$

describes a reflection on $\mathcal{H}^1$. That is, find a fixed point, and sketch the action of this matrix on $\mathcal{H}^1$. Use this matrix to find a reflection of $\mathcal{H}^2$.
(5) Prove that the Lorentzian inner product is bilinear
(6) If $A$ is in $O(1,n)$, show that $A^{-1} = JA^TJ$
(7) If $A$ is in $O(1,n)$, show that $A^T \in O(1,n)$
(8) Show that $\mathcal{H}^1$ is parameterised by the map $\mathbb{R}^1 \to \mathcal{H}^1$:

$$t \mapsto (\cosh(t), \sinh(t))$$

and that this map is an isometry from $\mathbb{R}^1$ to $\mathcal{H}^1$.
(9) Show that there are positive time-like vectors in $\mathbb{R}^n$ which form a basis of $\mathbb{R}^n$.
(10) (this is Q3c in the 2018 exam)
Let $P = (1,0,0) \in \mathcal{H}^2$. Write down an example of an isometry $f : \mathcal{H}^2 \to \mathcal{H}^2$ such that the distance $d_{\mathcal{H}^2}(f^n(P), P)$ is not bounded for $n \to \infty$. In other words, the iterates of $P$ under $f$ move arbitrarily far away from $P$ . Prove that your example has the required property.
[Hint: try this for $\mathcal{H}^1$ first.]
(11) Do question 3 from the April 2021 exam
(12) Show that $\mathcal{H}^2$ is parameterised by the map from $\mathbb{R}^2$ in polar coordinates given by

$$f : (r,\theta) \mapsto (\cosh(r), \sin(\theta)\sinh(r), \cos(\theta)\sinh(r)), \quad r \in [0,\infty), \theta \in [0, 2\pi)$$

Show that this map is not an isometry by showing that $d_{\mathbb{R}^2}((1,0),(1,\pi/2)) \neq d_{\mathcal{H}^2}(f(1,0), f(1,\pi/2))$. Find some points $p, q \in \mathbb{R}^2$ such that we do have the distance from $p$ to $q$ preserved by $f$.
(13) show that there is no possible isometry from $\mathbb{R}^2$ to $\mathcal{H}^2$, by following the method in [RS] exercise 3.13, page 58, computing the circumference of a circle in $\mathcal{H}^2$ with centre $(1,0,0)$, radius. You will need to use integration for this. Also it might help to do exercise 3.1 first for comparison.
(14) Show that the Lorentz inner product is bilinear and symmetric. Show that the Lorentz norm is symmetric. The Lorentz norm is complex on time like vectors, and zero on light like vectors, but is positive on space like vectors. Discuss whether the Lorentz norm is a norm on space like vectors.
(15) make sure you understand the polarisation identity for the Lorentz norm/inner product.
(16) If there is any time remaining: Exercises from 3.1 in course text book [R], at discretion of support class teacher.

## 54. MA243 Support Class 6

Discuss the solutions to ungraded questions Q1 and Q5 listed on homework 3 sheet. (Questions and solutions given in homework section)
Last year assignment 3:
`https://moodle.warwick.ac.uk/mod/resource/view.php?id=1910575`

Support class 5

Solutions:
https://moodle.warwick.ac.uk/mod/resource/view.php?id=1910584

## 55. MA243 Support Class 5, week 6, October 2022

Support class 5 solutions

(1) Prove Lemma 24 from class: $O(1,1)$ and $O^+(1,1)$ are groups.

By definition, $O(1,1)$ is the set of matrices $A$ which satisfy $A^TJA = J$, where

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

To check this is a group we check:
(1) Identity: $I \in O(1,1)$. This is because $I^TJI = J$.
(2) Inverses: if $A^TJA = J$, then $JA^TJA = I$, so the inverse of $A$ is $A^{-1} = JA^TJ$
We need to check this is in $O(1,1)$:

$$(JA^TJ)^{-1}J(JA^TJ) = J(A^T)^{-1}JJA^TJ = J(A^T)^{-1}A^TJ = JJJ = J$$

(We have used that $J^{-1} = J$.)
(3) Closure: If $A^TJA = J$ and $B^TJB = J$ then

$$(AB)^TJ(AB) = B^TA^TJAB = B^TJB = J$$

We have used associativity of matrix multiplication, and that $(AB)^T = B^TA^T$.
Hence $O(1,1)$, with matrix multiplcation, satisfies the group axioms and so is a group.

By definition, elements of $O^+(1,1)$ are elements of $O(1,1)$ which map positive time like vectors bijectively to positive time like vectors.
(1) Identity: this maps positive time like vectors to them selves, so this is fine.
(2) If $A$ is positive time like, then since $A$ maps positive time like vectors bijectively to positive time like vectors, it's inverse must also do the same – inverses exist since we just showed that $O(1,1)$ is a group.
(3) Closure: If $A$ and $B$ both map positive time like vectors bijectively to positive time like vectors, then this must also be true of their composition.

(2) Prove Lemma 25 from class: (1) the standard basis is a Lorentz orthonormal basis; (2) If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are Lorentz orthonormal vectors in $\mathbb{R}^n$ then they from a basis, and in this case (3) $\langle \mathbf{v}_i, \mathbf{v}_j \rangle_L = J_{i,j}$

(1) A Lorentz orthonomal basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ satisfies

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle_L = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } 2 \leq i = j \leq n \\ -1 & \text{if } i = j = 1 \end{cases}$$

Recall that $\langle \mathbf{x}, \mathbf{y} \rangle_L = -x_1y_1 + x_2y_2 + \cdots + x_ny_n$.
We have

$$\langle \mathbf{e}_1, \mathbf{e}_1 \rangle_L = -1 \cdots 1 + 0 + \cdots + 0 = -1$$
$$\langle \mathbf{e}_j, \mathbf{e}_j \rangle_L = 0 + \cdots + 0 + 1 \cdots 1 + 0 + \cdots 0 = 1$$
$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle_L = 0 + 0 + \cdots + 0 = 0$$

So the standard basis if Lorentz orthonormal.
(2) Suppose that $\mathbf{v}_1, \dots, \mathbf{v}_n$ are Lorentz orthonormal. Since there are $n$ of them, if they are linearly independent, then they span $\mathbb{R}^n$, so form a basis.
Suppose that

$$\sum a_i\mathbf{v}_i = \mathbf{0}$$

Then we can take the Lorentz inner product:

$$\left\langle \sum a_i\mathbf{v}_i, \mathbf{v}_1 \right\rangle_L = -a_1 \langle \mathbf{0}, \mathbf{v}_1 \rangle_L = 0$$

and for $j > 1$:

$$\left\langle \sum a_i \mathbf{v}_i, \mathbf{v}_j \right\rangle_L = a_j \langle \mathbf{0}, \mathbf{v}_j \rangle_L = 0$$

Hence all coefficents are zero, so there is no non-trivial relationship between the $\mathbf{v}_i$ and so the are linearly independent, and so form a basis.
(3) This follows from the definition of Lorentz orthonormal and the definition of $J$.

Support class 5 solutions

(3) There are 16 matrices of the form

$$\begin{pmatrix} \pm 2 & \pm\sqrt{3} \\ \pm\sqrt{3} & \pm 2 \end{pmatrix}$$

Decide which of these are Lorentz matrices, which are positive – that is whether they are in $O(1,1)$ and if so whether they are in $O^+(1,1)$. If they are in $O^+(1,1)$ decide whether they correspond to a reflection or translation.

Let the columns of a matrix $C = (c_{ij})$ be $\mathbf{c}_1$ and $\mathbf{c}_2$.

| matrix | $\langle \mathbf{c}_1, \mathbf{c}_1 \rangle_L$ | $\langle \mathbf{c}_1, \mathbf{c}_1 \rangle_L$ | $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle_L$ | $c_{1,1}$ | $O(1,1)$ | $O^+(1,1)$ | transformation |
|---|---|---|---|---|---|---|---|
| $\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}$ | $-1$ | $1$ | $0$ | $+ve$ | ✓ | ✓ | translation |
| $\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & -2 \end{pmatrix}$ | $-1$ | $1$ | $-4\sqrt{3}$ | $+ve$ | ✗ | ✗ | |
| $\begin{pmatrix} 2 & \sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix}$ | $-1$ | $1$ | $-4\sqrt{3}$ | $+ve$ | ✗ | ✗ | |
| $\begin{pmatrix} 2 & \sqrt{3} \\ -\sqrt{3} & -2 \end{pmatrix}$ | $-1$ | $1$ | $0$ | $+ve$ | ✓ | ✓ | reflection |
| $\begin{pmatrix} 2 & -\sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}$ | $-1$ | $1$ | $4\sqrt{3}$ | $+ve$ | ✗ | ✗ | |
| $\begin{pmatrix} 2 & -\sqrt{3} \\ \sqrt{3} & -2 \end{pmatrix}$ | $-1$ | $1$ | $0$ | $+ve$ | ✓ | ✓ | reflection |
| $\begin{pmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix}$ | $-1$ | $1$ | $0$ | $+ve$ | ✓ | ✓ | translation |
| $\begin{pmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & -2 \end{pmatrix}$ | $-1$ | $1$ | $4\sqrt{3}$ | $+ve$ | ✗ | ✗ | |

The remining matrices are these time $-1$, so none of them are in $O^+(1,1)$, and $A \in O(1,1) \iff -A \in O(1,1)$.

(4) Show that the matrix

$$\begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ \sinh\alpha & -\cosh\alpha \end{pmatrix}$$

describes a reflection on $\mathcal{H}^1$. That is, find a fixed point, and sketch the action of this matrix on $\mathcal{H}^1$. Use this matrix to find a reflection of $\mathcal{H}^2$.

We can check (using hyperbolic trig identities) that:

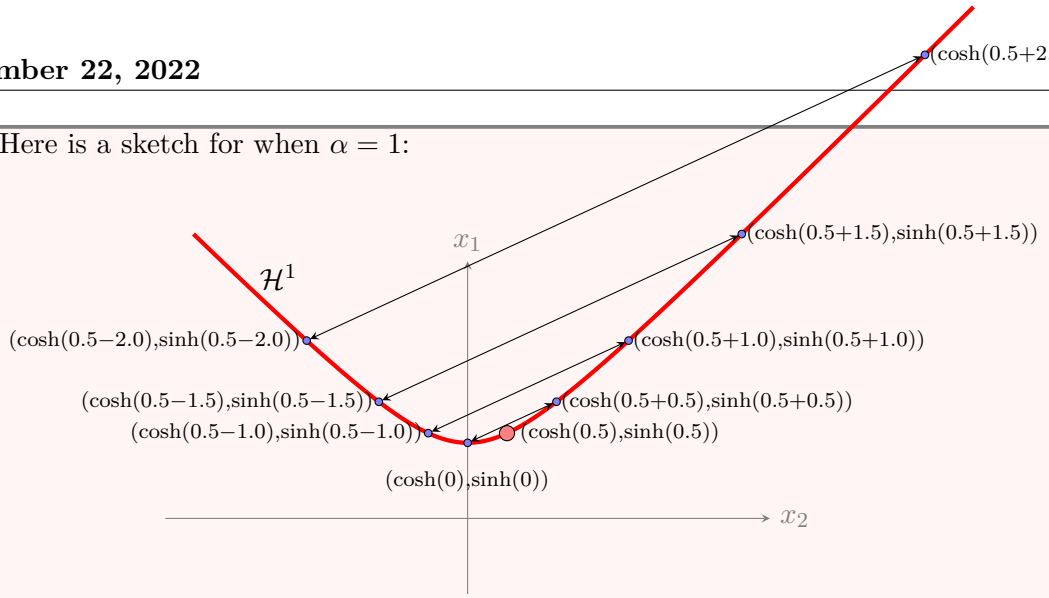
$$\begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ \sinh\alpha & -\cosh\alpha \end{pmatrix} \begin{pmatrix} \cosh(\alpha/2) \\ \sinh(\alpha/2) \end{pmatrix} = \begin{pmatrix} \cosh\alpha\cosh(\alpha/2) - \sinh\alpha\sinh(\alpha/2) \\ \sinh\alpha\cosh(\alpha/2) - \cosh\alpha\sinh(\alpha/2) \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\alpha/2) \\ \sinh(\alpha/2) \end{pmatrix}$$

So $(\cosh(\alpha/2), \sinh(\alpha/2))$ is a fixed point.
Also using the hyperbolic trig identities,
a point at $(\cosh(\beta), \sinh(\beta))$ will be mapped
to a point at $(\cosh(\alpha - \beta), \sinh(\alpha - \beta))$

Support class 5 solutions

Here is a sketch for when $\alpha = 1$:




A reflection of $\mathcal{H}^2$ can be given by

$$\begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 \\ \sinh\alpha & -\cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

there are also other reflections of $\mathcal{H}^2$, such as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{pmatrix}$$

and conjugates of both of these by any other isometries of $\mathcal{H}^2$.

(5) Prove that the Lorentzian inner product is bilinear

This is because the Lorentz inner product is given by $\langle \mathbf{v}, \mathbf{w} \rangle_L = \mathbf{v}^T J \mathbf{w}$, and matrix multiplication is a linear operation, so fixing either $\mathbf{v}$ or $\mathbf{w}$ gives a linear map.

Support class 5 solutions

(6) If $A$ is in $O(1,n)$, show that $A^{-1} = JA^TJ$
This has been covered in the solution to Q1 above.

(7) If $A$ is in $O(1,n)$, show that $A^T \in O(1,n)$
This is Lemma 27 in the notes.

(8) Show that $\mathcal{H}^1$ is parameterised by the map $\mathbb{R}^1 \to \mathcal{H}^1$:

$$t \mapsto (\cosh(t), \sinh(t))$$

and that this map is an isometry from $\mathbb{R}^1$ to $\mathcal{H}^1$.
Sketch: This map is a bijection because of properties of cosh and sinh (see notes for summary). That is, we will assume that cosh maps from $\mathbb{R}$ to $[1,\infty)$, and is a bijection from $[0,\infty)$ to $[1,\infty)$, and sinh is a bijection from $\mathbb{R}$ to $\mathbb{R}$, and that $\cosh^2 - \sinh^2 = 1$. These are exercises in analysis. The bijectivity of these functions gives the bijectivity of this parameterisation, and the identity and definition of $\mathcal{H}^1$ gives the fact that the map does map to $\mathcal{H}^1$.
This map is an isometry, because

$$d_{\mathcal{H}^1}((\cosh(t),\sinh(t)),(\cosh(r),\sinh(r))) = \cosh^{-1}(-(-\cosh(t)\cosh(r)+\sinh(t)\sinh(r)))$$
$$= \cosh^{-1}\cosh(t-r) = |t-r|$$

where we have used that cosh is an even function.

(9) Show that there are positive time-like vectors in $\mathbb{R}^n$ which form a basis of $\mathbb{R}^n$.
This is Lemma 28 in the notes

(10) (this is Q3c in the 2018 exam)
Let $P = (1,0,0) \in \mathcal{H}^2$. Write down an example of an isometry $f : \mathcal{H}^2 \to \mathcal{H}^2$ such that the distance $d_{\mathcal{H}^2}(f^n(P), P)$ is not bounded for $n \to \infty$. In other words, the iterates of $P$ under $f$ move arbitrarily far away from $P$ . Prove that your example has the required property.
[Hint: try this for $\mathcal{H}^1$ first.]
See exam solutions from maths web page

(11) Do question 3 from the April 2021 exam
See exam solutions from maths web page

(12) Show that $\mathcal{H}^2$ is parameterised by the map from $\mathbb{R}^2$ in polar coordinates given by

$$f : (r,\theta) \mapsto (\cosh(r), \sin(\theta)\sinh(r), \cos(\theta)\sinh(r)), \quad r \in [0,\infty), \theta \in [0,2\pi)$$

Show that this map is not an isometry by showing that $d_{\mathbb{R}^2}((1,0),(1,\pi/2)) \neq d_{\mathcal{H}^2}(f(1,0), f(1,\pi/2))$. Find some points $p, q \in \mathbb{R}^2$ such that we do have the distance from $p$ to $q$ preserved by $f$.
The parameterization is Lemma 31.
Computation: (coordintes in $\mathbb{R}^2$ are polar.) $d_{\mathbb{R}^2}((1,0),(1,\pi/2)) = \sqrt{2} \sim 1.414$
$d_{\mathcal{H}^2}(f(1,0), f(1,\pi/2)) = \cosh^{-1}(-\langle(\cosh(1),\sinh(1),0),(\cosh(1),0,\sinh(1))\rangle_L) = \cosh^{-1}(cosh(1)^2) \sim 1.5133$
Since the map in Q8 is an isometry, we can just take any $p, q$ on the intersection of $\mathcal{H}^2$ with $x_3 = 0$.

(13) show that there is no possible isometry from $\mathbb{R}^2$ to $\mathcal{H}^2$, by following the method in [RS] exercise 3.13, page 58, computing the circumference of a circle in $\mathcal{H}^2$ with centre $(1,0,0)$, radius. You will need to use integration for this. Also it might help to do exercise 3.1 first for comparison.

(14) Show that the Lorentz inner product is bilinear and symmetric. Show that the Lorentz norm is symmetric. The Lorentz norm is complex on time like vectors, and zero on light like vectors, but is positive on space like vectors. Discuss whether the Lorentz norm is a norm on space like vectors.

(15) make sure you understand the polarisation identity for the Lorentz norm/inner product.

(16) If there is any time remaining: Exercises from 3.1 in course text book [R], at discretion of support class teacher.

## 56. MA243 Support class 7, week 8, supplementary material

Similar questions to assignment 4, for discussion in support class

Support class 7

(1) Show that for any isometry $T_A$ of $\mathcal{H}^2$, with $A \in O^+(1,2)$, which fixes a space-like vector, there is a Lorentz orthonormal basis with respect to which for some $\beta \in \mathbb{R}$,

$$A = \begin{pmatrix} \cosh(\beta) & \pm\sinh(\beta) & 0 \\ \sinh(\beta) & \pm\cosh(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Describe the transformations the two cases correspond to.

(2) Find the number of planes in $\mathbb{P}^3(\mathbb{F}_5)$, where $\mathbb{F}_5$ is the finite field with 5 elements.

(3) Find the number of planes in $\mathbb{P}^4(\mathbb{F}_5)$, where $\mathbb{F}_5$ is the finite field with 5 elements.

(4) (a) Write down all the projective linear maps $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ taking

$$(1:0) \mapsto (1:0)$$
$$(0:1) \mapsto (0:1)$$

(b) Find the unique projective linear map taking the standard frame of reference $(1:0), (0:1), (1:1)$ to $(1:0), (0:1), (1:2)$ respectively.

(5) (a) Write down all the projective linear maps $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ taking

$$(1:0) \mapsto (1:1)$$
$$(0:1) \mapsto (1:0)$$

(b) Find the unique projective linear map taking the standard frame of reference $(1:0), (0:1), (1:1)$ to $(1:1), (1:0), (0:1)$ respectively.

(6) Show that $(1:1:0), (1:0:1), (0:1:1), (1:1:1)$ is a frame of reference for $\mathbb{P}^2$.

# Solutions

(1) Show that for any isometry $T_A$ of $\mathcal{H}^2$, with $A \in O^+(1,2)$, which fixes a space-like vector, there is a Lorentz orthonormal basis with respect to which for some $\beta \in \mathbb{R}$,

$$A = \begin{pmatrix} \cosh(\beta) & \pm\sinh(\beta) & 0 \\ \sinh(\beta) & \pm\cosh(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Describe the transformations the two cases correspond to.

Solution:

Suppose that $\mathbf{w}$ is a space like vector fixed by $T_A$. Let

$$V := (\mathbb{R}\mathbf{w})^{\perp_L} = \{\mathbf{x} \in \mathbb{R}^3 : \langle \mathbf{x}, \mathbf{w}\rangle_L = 0\}$$

I want to show that $T_A$ is given by the matrix (1) on $\mathbb{R}\mathbf{w}$ and by $\begin{pmatrix} \cosh(\beta) & \pm\sinh(\beta) \\ \sinh(\beta) & \pm\cosh(\beta) \end{pmatrix}$ on $V$, and that these spaces have direct sum all of $\mathbb{R}^3$, and a Lorentz orthnormal basis.

(1) First show that $V$ has dimension 2: Consider the map from $\mathbb{R}^3$ to $\mathbb{R}$ given by

$$\mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{w}\rangle_L$$

this map has image $\mathbb{R}$, since it's not the zero map (e.g., $\langle \mathbf{w}, \mathbf{w}\rangle_L > 0$, and is linear, since $\langle \cdot, \cdot \rangle_L$ is bilinear.

Hence by the rank nullity theorem, the kernel, which is the set $V = (\mathbb{R}\mathbf{w})^{\perp_L}$, is a two dimensional subspace of $\mathbb{R}^3$.

(2) Show that $V$ and $\mathbb{R}\mathbf{w}$ have direct sum $\mathbb{R}^3$: Since $\langle \mathbf{w}, \mathbf{w}\rangle_L \neq 0$, $\mathbf{w} \notin (\mathbb{R}\mathbf{w})^{\perp_L}$, and so $\mathbf{w}$ is linearly independent of the vectors in the subspace $(\mathbb{R}\mathbf{w})^{\perp_L}$, and so the sum of these spaces must be three dimensional, and so all of $\mathbb{R}^3$.

(3) Now I want to show that $V$ contains a time-like vector.

Suppose that $\mathbf{w}$ is given in spherical coordinates by $(r, \theta, \phi)$, so in Euclidean coordinates, (with vertical coordinate $z = x_1$ written first) this is $(r\cos(\phi), r\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta))$ where $-r^2\cos(\phi)^2 + (r\sin(\phi)\cos(\theta))^2 + (r\sin(\phi)\sin(\theta))^2 > 0$, so $r^2(\sin(\phi^2 - \cos(\phi)^2 > 0$, so $|\sin(\phi)| > |\cos(\phi)|$, i.e., $\phi \in (\pi/4, 3\pi/4)$. We may assume $\phi \leq \pi/2$, since if not, this is true for $-\mathbf{w}$, which spans the same vector space, so we can switch $\mathbf{w}$ and $-\mathbf{w}$. Suppose $\phi \leq \pi/2$, then the vector

$$\mathbf{a} = (r\cos(\pi/2 - \phi), r\sin(\pi/2 - \phi)\cos(\theta), r\sin(\pi/2 - \phi)\sin(\theta))$$

is Lorentz orthogonal to $\mathbf{w}$ and time-like, since vectors Lorentz orthogonal to $\mathbf{w}$ are those orthogonal in the usual sense to $J\mathbf{w}$, which is

$$J\mathbf{w} = (r\cos(\pi - \phi), r\sin(\pi - \phi)\cos(\theta), r\sin(\pi - \phi)\sin(\theta))$$

and the vector $\mathbf{a}$ is orthogonal to $J\mathbf{w}$. We can also check that

$$\langle \mathbf{w}, \mathbf{a}\rangle_L = -r\cos(\phi)r\cos(\pi/2-\phi)+r\sin(\phi)\cos(\theta)r\sin(\pi/2-\phi)\cos(\theta)+r\sin(\phi)\sin(\theta)r\sin(\pi/2-\phi)\sin(\theta)$$

$$= r^2(-\cos(\phi)\cos(\pi/2 - \phi) + \sin(\phi)\sin(\pi/2 - \phi)) = r^2(-\cos(\phi + \pi/2 - \phi)) = -r^2\cos(\pi/2) = 0,$$

so $\mathbf{a} \in V$.

Since $\phi \in (\pi/4, \pi/2)$, we have $\pi/2 - \phi \in (0, \pi/4)$, so $\pi - 2\phi \in (0, \pi/2)$ and so

$$\langle \mathbf{a}, \mathbf{a}\rangle_L = -r^2\cos(\pi/2 - \phi)^2 + r^2\sin(\pi/2 - \phi)^2 = -r^2\cos(\pi - 2\phi) < 0$$

Hence $\mathbf{a}$ is a time-like.

By scaling, we may assume without loss of generality (since scaling preserves the spaces these elements span) that $||\mathbf{w}||_L = -1$ and $||\mathbf{a}||_L = 1$.

Now we can use the Lorentz orthnomaliztion process to obtain a basis for $V$ consisting of $\mathbf{a}$ and some other vector $\mathbf{v}$ in $V$, which by definition of $V$ is Lorentz orthogonal to $\mathbf{w}$. So the basis $\mathbf{a}, \mathbf{v}, \mathbf{w}$ is a Lorentz orthonormal basis for $\mathbb{R}^3$.

(4) Now show that $V$ is preserved by $T_A$: This is because $T_A$ preserves the Lorentz metric, and so any vector which is Lorentz orthogonal to $\mathbf{w}$ must be mapped to a vector Lorentz orthogonal to $T_A(\mathbf{w}) = \mathbf{w}$, and so is in $V$ by definition of $V$.

(5) We are writing our matrix with respect to the basis $\mathbf{a}, \mathbf{v}, \mathbf{w}$. Since $V$ and $\mathbf{w}$ are preserved, we know that the matrix has the form

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$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $T_A$ preserves $V$, and $T_A$ is a Lorentz transformation, $T_A$ must map the basis to another Lorentz orthonormal basis, and so $\langle (a, c, 0), (a, c, 0) \rangle_L = -1$ and so since we know that this is a point on $\mathcal{H}^1$, by definition of $\mathcal{H}^1$, and we know how to parameterize $\mathcal{H}^1$ by $a = \cosh(\beta), c = \sinh(\beta)$ for some $\beta \in \mathbb{R}$, this gives the first column of the matrix. Now the second column is Lorentz orthogonal to the first, and so has the form $(b, d, 0)$ with $-b\cosh(\beta) + d\sinh(\beta) = 0$, and so $b = \lambda\sinh(\beta), d = \lambda\cosh(\beta)$, for some $\lambda \in \mathbb{R}$, but since the Lorentz norm of the second column is 1, we must have $\lambda = \pm 1$, hence the result.

The first of these transformations is a Lorentz translation. The second is a (Lorentz) reflection.

Solution: Every plane $\Pi$ in $\mathbb{P}^3(\mathbb{F}_5)$ is the projective image of some three dimensional subspace $V$ of $\mathbb{F}_5^4$, so we need to count these.

A subspace $V$ can be given by three basis elements. Suppose $\mathbf{w}_1$ is the first element. Since $\mathbb{F}_5^4$ has $5^4$ elements, of which $5^4 - 1$ are non-zero, there are $5^4 - 1$ possible choices for $\mathbf{w}_1$.

Now the second basis element must be chosen from the points not in the span of $\mathbf{w}_1$. The span of $\mathbf{w}_1$ contains 5 elements, and so there are $5^4 - 5$ elements for the second choice $\mathbf{w}_2$.

Now $\mathbf{w}_1$ and $\mathbf{w}_2$ span a space given by elements of the form $\alpha w_1 + \beta w_2$, for $\alpha, \beta \in \mathbb{F}_5$, and so there are $5^2$ possible vectors in this span, and so $\mathbf{w}_3$ must be chosen from the remaining $5^4 - 5^2$ elements.

So, this gives

$$(5^4 - 1)(5^4 - 5)(5^4 - 5^2)$$

possible ways to choose a basis.

However, different choices may give rise to the same vector space. But two different choices are related by a change of basis matrix.

let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis of $V$, then any change of basis matrix must have the form of an invertible three by three matrix with coefficients in $\mathbb{F}_5$.

To find such a matrix, pick the first column to be any nonzero vector $\mathbf{v}_1$ in $\mathbb{F}_5^3$. There are $5^3 - 1$ ways to do this. The second column must be any vector not in the span of the first, so there are $5^3 - 5$ ways to choose this, and finally $5^3 - 5^2$ ways to choose the third column. Hence there are $(5^3 - 1)(5^3 - 5)(5^3 - 5^5)$ ways to choose the basis.

This gives

$$\frac{(5^4 - 1)(5^4 - 5)(5^4 - 5^2)}{(5^3 - 1)(5^3 - 5)(5^3 - 5^2)} = \frac{(5^4 - 1)5(5^3 - 1)5^2(5^2 - 1)}{(5^3 - 1)5(5^2 - 1)5^2(5 - 1)} = \frac{5^4 - 1}{5 - 1} = 156$$

different three dimensional subspaces, and so there are 156 planes in $\mathbb{P}^3(\mathbb{F}_5)$.

(2) (a) Write down all the projective linear maps $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ taking

$$(1 : 0) \mapsto (1 : 0)$$
$$(0 : 1) \mapsto (0 : 1)$$

(b) Find the unique projective linear map taking the standard frame of reference $(1 : 0), (0 : 1), (1 : 1)$ to $(1 : 0), (0 : 1), (1 : 2)$ respectively.

## Solution:

(a) These are the maps

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

for any $a, b \in \mathbb{R}$, both non-zero.

(b) Under the map in (a), the point $(1:1)$ maps to $(a:b)$, and therefore, we require $(a:b) = (1:2)$, and so the required unique linear map is the equivalence class of maps

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$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

in $\mathrm{PGL}(\mathbb{R}, 2)$.

(3) (a) Write down all the projective linear maps $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ taking

$$(1:0) \mapsto (1:1)$$
$$(0:1) \mapsto (1:0)$$

(b) Find the unique projective linear map taking the standard frame of reference $(1:0), (0:1), (1:1)$ to $(1:1), (1:0), (0:1)$ respectively.

## Solution:

(a) the required maps have the form

$$\begin{pmatrix} a & b \\ a & 0 \end{pmatrix}$$

for any $a, b \in \mathbb{R}$, both non-zero.

(b) This map maps $(1:1)$ to $(a+b, a)$, so we require $a + b = 0$, and the maps are those of the form

$$\begin{pmatrix} a & -a \\ a & 0 \end{pmatrix}$$

which are all equivalent to

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

in $\mathrm{PGL}(2)$.

(4) Show that $(1:1:0), (1:0:1), (0:1:1), (1:1:1)$ is a frame of reference for $\mathbb{P}^2$.

## Solution:

For a projective frame of reference, any set of three of these must be linearly independent.

Suppose that

$$a(1,1,0) + b(1,0,1) + c(0,1,1) + d(1,1,1) = 0$$

i.e.,

$$a(1,1,0) + b(1,0,1) + c(0,1,1) = -d(1,1,1)$$
$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

which implies that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = d \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{d}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{d}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So we must have $a = b = c = d/2$, as unique solution for any give $d$, and hence if one coeffient is zero, they all are, and so there is no combination of only three of these terms giving a zero result, and hence they are a frame of reference.

NOTE it is acceptable in homework to compute the inverse of a matrix with a compute algebra package, and just state what the result is, provided you make clear what you are doing.

## 57. MA243 Support class 9, week 10

In week 9 and 10 support class, please go over the assignment 4 solutions.
Also, look at the moodle quiz questions for weeks 8, 9 and 10, discuss these in class.

Support class 9

(1) Prove that any two distinct planes in $\mathbb{P}^3(\mathbb{R})$ intersect in a line.
(2) We can include $\mathbb{R}^3$ in $\mathbb{P}^3(\mathbb{R})$ by the map

$$\mathbb{R}^3 \to \mathbb{P}^3$$

$$(x, y, z) \mapsto (x : y : z : 1)$$

Make sure you understand why this is an injection.
Any point which is in $\mathbb{P}^3 \setminus \mathbb{R}^3$ is called a "point at infinity" (with respect to this inclusion).
How many lines in $\mathbb{P}^3$ are "lines at infinity", that is how many lines $L \subset \mathbb{P}^3$ are such that $L \subset \mathbb{P}^3 \setminus \mathbb{R}^3$.

(Possible answers: $0, 1, \infty$)
(3) Do two lines in $\mathbb{P}^3$ always intersect? (Proof or counterexample).
(4) In $\mathbb{P}^2$, with points having coordinates $(x : y : z)$,
   (a) find the value of the point on the lines $x + y + z = 0$ and $x - z = 0$.
   (b) find the equation of the line passing through the points $(1 : 1 : 0)$ and $(0 : 1 : 1)$.
(5) (From RS 5.5 pg 89) Prove that 3 lines in $\mathbb{P}^n$ that intersect in pairs (i.e., any two have an intersection point) either all pass through the same intersection point (are concurrent), or all lie in the same plane (are coplanar) (Hint: use dimension of intersection)
(6) (From RS 5.6 pg 89) Prove that for three mutually non-intersecting lines in $\mathbb{P}^4$ not all contained in any hyperplane, there is a unique lines meeting all three lines. (Hint: Consider the span of two of the lines.)
(7) Show that if $P, Q, R, S$ are four points, with cross ratio $\lambda$ when given in that order, there are in general 6 different possible values for the cross ratio, depending on the order they are taken in, namely

$$\lambda, \frac{1}{\lambda}, 1 - \lambda, 1 - \frac{1}{\lambda}, \frac{1}{1 - \lambda}, \frac{\lambda}{1 - \lambda}.$$

(8) Using the previous question, determine whether the points $(0 : 1), (1 : 1), (2 : 1), (3 : 1)$ can be mapped by a projective linear transformation to $(0 : 1), (1 : 1), (1 : 0), (4 : 3)$, in any order.
You do not need to determine the transformation if one exists.

# Solutions:

(1) The proof comes from the dimension formula. Let the planes be $\Pi_1$, $\Pi_2$. Since they are distinct, the span is all of $\mathbb{P}^3$, and so

$$\dim(\Pi_1 \cap \Pi_2) = \dim(\Pi_1) + \dim(\Pi_2) - \dim(\langle \Pi_1, \Pi_2 \rangle) = 2 + 2 - 3 = 1$$

which is the dimension of a line.

(2) There are infinitely many lines at infinity. Any set of points of the form $(t : at : r : 0)$, for fixed $a$, and $(t : r) \in \mathbb{P}^1$ is a line at infinity; this is the line correspoinding to the plane containing points $(t, at, r, 0)$ in $\mathbb{R}^4$, parameterised by $(t, r) \in \mathbb{R}^2$. For different values of $a$, we get a different plane. These are not the only possible lines, but give an easy example; you could give a different solution.

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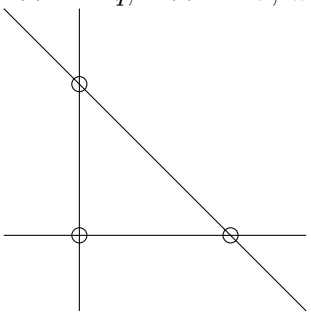
(3) Two lines in $\mathbb{P}^3$ may be disjoint. E.g., $(t : r : 0 : 0)$ and $(0 : 0 : t : r)$ where $(t : r) \in \mathbb{P}^1$.

Note that by the dimension formula, if the lines are disjoint, then their span must be all of $\mathbb{P}^3$. In this example, points $(1 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0)$ are contained in the first line, and $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$ are contained in the second line, so since together this comes from a basis for $\mathbb{R}^4$, these points span $\mathbb{P}^3$.

(4) (a) The solution is $(1 : -2 : 1)$.

(b) The solution is $x + z = y$. Note that in $\mathbb{P}^2$ a line is given by a single equation, which must be the equation for a plane in $\mathbb{R}^3$, since lines are projections of planes through the origin in $\mathbb{R}^3$ to $\mathbb{P}^2$.

(5) Let the lines be $L, M, N$. Suppose they don't all intersect in the same point. Let $L \cap M = p, M \cap N = q, N \cap L = r$, which will be three distinct points.



Each line is spanned by the two intersection points it contains, since any two distinct points on a line span that line, since any two vectors in different directions span the plane they are contained in.

So, the span of all of $p, q, r$ must contain all the lines, and the span of three distinct non-colinear points is a plane.

Note that if the span is a line, then the lines are colinear, and so the same line, and so concurrent.

(6) Let the lines be $L, M, N$.

It is given that no two intersect, so by the dimension formula, the span of any two is a three dimensional space:

$$\dim \langle L, M \rangle = \dim L + \dim M - \dim L \cap M = 1 + 1 - (-1) = 3$$

Let $V_{LM} = \langle L, M \rangle$. Now the span of all three must be the whole space, since otherwise it would be contained in a hyperplane. So

$$4 = \dim \langle V_{LM}, N \rangle = \dim V_{LM} + \dim N + \dim V \cap N = 3 + 1 + \dim V \cap N$$

Hence $\dim V \cap N = 0$, and so $N$ meets the plane $V_{LM}$ in a unique point, $P_N$. Similarly, we have unique points $P_M$ where $M$ meets the plane spanned by $L, N$ and $P_L$ where $L$ meets the plane spanned by $N, M$.

I want to show that these points lie on a line, and that this is the unique line through $L, M, N$.

The uniqueness is pretty easy: Suppose $K$ is a line through $L, M, N$. Since these lines pairwise contain no common point by hypothesis, if the point on $K \cap L$ is $p_1$ and the point on $K \cap M$ is $p_2$, then since $p_1 \neq p_2$, since $L \cap M = \varnothing$, $K = \langle p_1, p_2 \rangle$, and so $K$ must lie in the plane $\langle L, M \rangle$. But we want $K$ to contain a point of $N$, and $N$ only intersects $\langle L, M \rangle$ at $P_N$, so $K$ must pass through $P_N$. Similarly, $K$ must pass through $P_L$ and $P_M$, so, if these points lie on a common line $K$, then it is the unique line meeting $L, M, N$.

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Now we need to check existance: Now set $K = \langle P_L, P_M \rangle$. We want to show this line also passes through $P_N$. Let's compute the dimension of the intersection of $K$ with $N$. Let's work out what the span of $K$ and $N$ is. These are two lines. Any line has dimension 1. The span of two lines can only have dimension 1 if the two lines are equal: by the dimension formula, we would have

$$1 = \dim \langle K, N \rangle = \dim K + \dim N - \dim K \cap N \Rightarrow \dim K \cap N = 1 \Rightarrow N = K$$

But $K$ cannot equal $N$ since $K$ contains points $P_L, P_M$ not in $N$. So $\dim \langle K, N \rangle = 2$, and by the dimension formular, we have

$$\dim K \cap N = \dim K + \dim N - \dim \langle K, N \rangle = 1 + 1 - 2 = 0$$

So the intersection $K \cap N$ is a unique point, in paricular, $K$ does meet $N$, and so is the required line meeting all three lines $L, M, N$.

**Remark:** You could probably give an easier proof by using a projective linear transformation to put the lines in nice coordinates.

**Remark:** We require the hypothesis that the lines do not intersect each other. For example, for the lines of the form $\{(0:0:0:\lambda:\mu) : \lambda, \mu \in \mathbb{R}\}$, $\{(\lambda:\mu:0:0:0) : \lambda, \mu \in \mathbb{R}\}$, $\{(\lambda:0:\mu:0:0) : \lambda, \mu \in \mathbb{R}\}$ the second two lines instersect at $P = (1:0:0:0:0)$, so for any point $Q = (0:0:0:a:1)$ for some fixed $a \in \mathbb{R}$, there is a line through $P, Q$, which has points of the form $(\lambda:0:0:a\mu:\mu)$ for $\lambda, \mu \in \mathbb{R}$, and these are different lines for different values of $a$. On the other hand, these three lines contain the points $(0:0:0:1:0), (0:0:0:0:1)$ (on first line) $(0:1:0:0:0), (1:0:0:0:0)$ (on second line) and $(0:0:1:0:0)$ on third line, so the lift to $\mathbb{R}^5$ contain a basis for $\mathbb{R}^5$, and so these points span $\mathbb{P}^4$.

(7) If $\{P, Q; R, S\} = \lambda$, we need to show that by reordering the points, we can obtain any of the following values:

$$\lambda, \frac{1}{\lambda}, 1-\lambda, 1-\frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{1-\lambda}.$$

We will assume without loss of generality that $P, Q, R, S$ are numbers in $\mathbb{R}$; (since if any is $\infty$ we can apply the projective linear map $t \mapsto 1/t$, and if any is zero, we can apply $t \mapsto t + N$ for sufficiently large $N$ so that all $P, Q, R, S$ are positive. These maps can all be considered as maps on $\mathbb{P}^1$ by interpretting $t$ as $(t:1)$ and $\infty$ as $(1:0)$.)

So, by Proposition 20 in the lecture notes, we have

$$\lambda = \left(\frac{P-R}{P-S}\right)\left(\frac{Q-S}{Q-R}\right)$$

Switching $P, Q$ gives

$$\{Q, P; R, S\} = \left(\frac{Q-R}{Q-S}\right)\left(\frac{P-S}{P-R}\right) = \frac{1}{\lambda}$$

Consider switching $P, S$. We have

$$\{P, Q; R, S\} + \{S, Q; R, P\} = \left(\frac{P-R}{P-S}\right)\left(\frac{Q-S}{Q-R}\right) + \left(\frac{S-R}{S-P}\right)\left(\frac{Q-P}{Q-R}\right)$$

$$= \frac{(P-R)(Q-S) - (S-R)(Q-P)}{(P-S)(Q-R)} = \frac{PQ + RS - RQ - PS - SQ - RP + SP + RQ}{PQ + SR - PR - SQ}$$

$$= \frac{PQ + RS - SQ - RP}{PQ + SR - PR - SQ} = 1$$

So we have

$$\{S, Q; R, P\} = 1 - \lambda$$

Composing the two permutations, we now see we can obtain the values

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$$\frac{1}{1-\lambda},\ 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}$$

And finally inverting the last expression, we obtain

$$\frac{\lambda}{\lambda - 1}$$

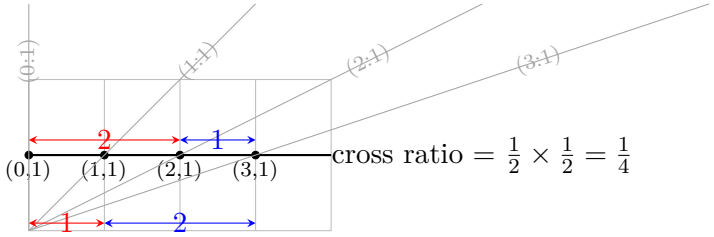
Is it possible to obtain any more values for the cross ratio of a permutation of $P, Q, R, Q$? If we switch $P$ with $Q$ and $R$ with $S$ simulantiously, then the expression will have the same value. Also if we switch $P, R$ and $S, Q$ the value is unchanged. The composition of these is to switch $P, S$ and $Q, R$ also leaves the value unchanged.

The permutations of $P, Q, R, S$ are elements of the symmetry group (permutation group) $S_4$. This group has $4! = 24$ elements. These elements "act" on the value of the cross ratio. But 4 elements leave the value unchanged. These, written in permutation notation, (with () being identity permutation) are in

$$V = \{(),\ (P, Q)(R, S),\ (P, R)(Q, S),\ (P, S)(Q, R)\}.$$

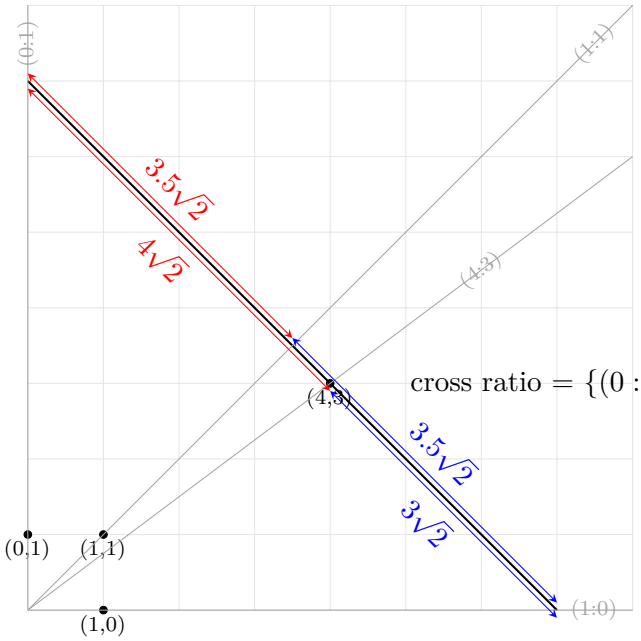
Since $24/4 = 6$, there can only be 6 different values for the cross ratio up to permutations of $P, Q, R, S$, since every value can be obtained 4 times by composing with these permutations. E.g., $(Q, S)$ is the permutation we used above to obtain $\lambda \mapsto \frac{1}{\lambda}$, but we could also apply $(P, Q)(R, S) \cdot (Q, S)$, or compose with any other element of $V$, to also obtain $\frac{1}{\lambda}$.

(8) The cross ratios can be computed by considering the corresponding points in $\mathbb{R}^2$:




$$\{(0:1), (3:1); (1:1); (2:1)\} = 1$$

Support class 9




cross ratio $= \{(0:1),(1:0);(1:1),(4:3)\} = \frac{3.5}{4} \times \frac{3}{3.5} = \frac{3}{4}$

Since $1 - \frac{1}{4} = \frac{3}{4}$, by the previous question, there is a map sending the first set to the second set of numbers. In fact, since we know that $\lambda \mapsto 1 - \lambda$ is acheived by switching $P$ and $S$ from the previous question, we know that we would have

$$\{(4:3),(1:0);(1:1),(0,1)\} = \frac{1}{4}$$

so this would tell us which points to map to which. (I can write up details if there is a demand for this!)

Actually, in class we did not prove that there was a map when the cross ratios are the same, only that there is none if they are not the same, but this result is also true. So stictly speaking, you can only answer this question if you either find the projective transformation explicitly (pretty easy), or prove the converse of Proposition 21 from class (harder). Interested students can look this up themselves, and less interested students probably don't have time to explore every interesting alleyway we have peered into.

## 58. MA243 Homework 1

Due via moodle, at noon, Thursday, week 2, 13th October 2022.
All questions count. The total is 25. You must show your working, proof, or some kind of explanation of justification, for every question. If you get the right answers with no explanation you get half marks.
In these questions $\mathbb{R}^n$ means $\mathbb{E}^n$, i.e., $\mathbb{R}^n$ with the standard Euclidean metric.

(1) (Question about lines in $\mathbb{R}^n$)[1 point] Prove that if $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ are distinct, and

$$\mathbf{z} = \lambda_1 \mathbf{x} + \lambda_2 \mathbf{y},$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$ then

$$d(\mathbf{x}, \mathbf{z}) : d(\mathbf{y}, \mathbf{z}) = \lambda_2 : \lambda_1$$

(2) (Question about isometries of $\mathbb{R}$) [8 points]

(a) Prove from first principles (i.e., only use the definition of an isometry and facts about real numbers, not a big result about all isometries of $\mathbb{R}^n$ for any $n$) that all isometries of $\mathbb{R}^1$ have the form

$$x \mapsto ax + b,$$

where $a = \pm 1$

(b) There is a non-trivial isometry $T$ of $\mathbb{R}$ fixing 3. Find $a$ and $b$ so that $T(x) = ax + b$.

(c) An isometry of $\mathbb{R}$ is called a reflection if it is non-trivial, and fixes one point. Find two reflections $T$ and $S$, giving the points they fix, and writing them in the form $x \mapsto ax + b$, such that $T \circ S(x) = x + 10$

(d) Are the reflections $T$ and $S$ found in (c) unique? If they are, prove this, if they are not give another such decomposition.

(3) (Question about isometries of $\mathbb{R}^2$)[16 points] If $\mathbf{P}$ and $\mathbf{Q}$ are two distinct points in $\mathbb{R}^2$, the line segment $\mathbf{PQ}$ means all the points on the line between these points:

$$\mathbf{PQ} = \{\lambda_1 \mathbf{P} + \lambda_2 \mathbf{Q} : \lambda_1, \lambda_2 \in [0, 1], \lambda_1 + \lambda_2 = 1\}$$

Addition of points is addition of the corresponding vectors relative to $\mathbf{0}$, in the usual way.

You may assume without proof that an isometry has the form $T(x) = Ax + \mathbf{b}$ where $A$ is an orthogonal matrix and $\mathbf{b}$ is a vector (you will be expected to know how to prove this result for exams though)

(a) Show that the image of $\mathbf{PQ}$ under an isometry $T$ is the line segement $T(\mathbf{P})T(\mathbf{Q})$

(b) Prove that if $T$ and $R$ are isometries of any metric space $(X, d)$, then the composition $T \circ R$ is also an isometry.

(c) Show that if $\mathbf{PQ}$ and $\mathbf{P'Q'}$ are line segments of equal length, i.e., $d(\mathbf{P}, \mathbf{Q}) = d(\mathbf{P'}, \mathbf{Q'})$, then there exists an isometry $T$ with $T(\mathbf{PQ}) = \mathbf{P'Q'}$.
Is this isometry unique? If it is, prove it, if not, describe another one with the same property.

(d) Let $A = (0, 0)$, $B = (5, 0)$, $C = (-2, 4)$, $D = (1, 8)$.

(i) Describe a rotation $R$ of $\mathbb{R}^2$ that maps $AB$ to $CD$.

(A) Write your rotation in the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$,

(B) find the centre of the rotation, i.e., a solution to $R(\mathbf{x}) = \mathbf{x}$

(C) find the eigen values of $A$

(ii) Describe a reflection $F$ of of $\mathbb{R}^2$ that maps $AB$ to $CD$.

(A) Write your reflection in the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$,

(B) find a line fixed by this reflection

(C) find the eigen values and vectors of $A$

## 59. MA243 Homework 1 solutions

In these questions $\mathbb{R}^n$ means $\mathbb{E}^n$, i.e., $\mathbb{R}^n$ with the standard Euclidean metric.

(1) (Question about lines in $\mathbb{R}^n$) Prove that if $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ are distinct, and

$$\mathbf{z} = \lambda_1 \mathbf{x} + \lambda_2 \mathbf{y},$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$ then

$$d(\mathbf{x}, \mathbf{z}) : d(\mathbf{y}, \mathbf{z}) = |\lambda_2| : |\lambda_1|$$

$$\begin{aligned} d(\mathbf{x}, \mathbf{z}) &= ||\mathbf{x} - \mathbf{z}|| = ||\mathbf{x} - \lambda_1 \mathbf{x} - \lambda_2 \mathbf{y}|| \\ &= ||\lambda_2(\mathbf{x} - \mathbf{y})|| = |\lambda_2| ||\mathbf{x} - \mathbf{y}|| \end{aligned}$$

Similarly $d(\mathbf{x}, \mathbf{z}) = |\lambda_1| ||\mathbf{x} - \mathbf{y}||$, which gives the result.

(2) (Question about isometries of $\mathbb{R}$)

Homework solutions 1

(a) Prove from first principles (i.e., only use the definition of an isometry and facts about real numbers, not a big result about all isometries of $\mathbb{R}^n$ for any $n$) that all isometries of $\mathbb{R}^1$ have the form

$$x \mapsto ax + b,$$

where $a = \pm 1$

(b) There is a non-trivial isometry $T$ of $\mathbb{R}$ fixing 3. Find $a$ and $b$ so that $T(x) = ax + b$.

(c) An isometry of $\mathbb{R}$ is called a reflection if it is non-trivial, and fixes one point. Find two reflections $T$ and $S$, giving the points they fix, and writing them in the form $x \mapsto ax + b$, such that $T \circ S(x) = x + 10$

(d) Are the reflections $T$ and $S$ found in (c) unique? If they are, prove this, if they are not give another such decomposition.

(a) [3] One way to prove this:

Let $T$ be the isometry.

First suppose $T(0) = 0$, then by definition of isometry, for any $x \in \mathbb{R}$,

$$|x| = d(x, 0) = d(T(x), T(0)) = d(T(x), 0) = |T(x)|$$

so $T(x) = \pm x$.

Suppose $T(1) = 1$, Suppose for some $x \in \mathbb{R}$, we had $T(x) = -x$. Now we also have $d(x, 1) = d(T(x), T(1)) = d(-x, 1) = |1 + x|$. But $d(x, 1) = |x - 1|$, and we can only have $|x + 1| = |x - 1|$ if $x = 0$, since $|x + 1| = |x - 1| \iff (x + 1) = \pm(x - 1) \iff 1 = -1$ or $2x = 0$, but $1 = -1$ is impossible.

So, if $T(1) = 1$, then the map is $T(x) = x$.

Now suppose $T(1) = -1$. Then the map $R(x) = -T(x)$ satisfies $R(x) = 0, R(1) = 1$, and so by the previous argument $R(x) = x$ and so $T(x) = -x$.

So, if $T(0) = 0$, the map is $x \to \pm x$.

Now suppose $T(0) = b$ Now consider the map $R(x) = T(x) - b$. Then $R(0) = 0$, so by the previous argument $R(x) = \pm x$. And so $T(x) = R(x) + b = \pm x + b$

2 point for showing $a = \pm 1$, one point for relating general case to linear case

(b) [1] If $T(3) = 3$ and $T(x) = \pm x + b$, then $3 = \pm 3 + b$ so either $b = 0$, but this is the trivial map, or $b = 6$, and the map is $T(x) = 6 - x$

(c) [3] This could be done by writing the composition of two reflections, e.g., compostion of $T(x) = ax + b$ and $S(x) = cx + d$ is $T(S(x)) = a(cx + d) + b = acx + ad + b = x + 10$ and solving

$$ac = 1$$

$$ad + b = 10$$

For a reflection, we must have $a = c = -1$. So $b - d = 10$.

E.g., we could take $b = 10, d = 0$.

So the reflections in this case are

$$T(x) = 10 - x$$
$$S(x) = -x$$

To find the fixed points, we solve $T(x) = x$ and $R(x) = x$ For $T$ the fixed point is 5, for $S$ the fixed point is 0.

2 point for finding the maps, one point for finding their fixed points

(d) [1] Any other pair $d, b$ with $b - d = 10$ is fine, e.g., the transformations

$$T(x) = -x$$
$$S(x) = -10 - x$$

which fix 0 and $-5$. There are infinitely many possble solutions.

(3) (Question about isometries of $\mathbb{R}^2$) If $\mathbf{P}$ and $\mathbf{Q}$ are two distinct points in $\mathbb{R}^2$, the line segment $\mathbf{PQ}$ means all the points on the line between these points:

$$\mathbf{PQ} = \{\lambda_1\mathbf{P} + \lambda_2\mathbf{Q} : \lambda_1, \lambda_2 \in [0, 1], \lambda_1 + \lambda_2 = 1\}$$



Addition of points is addition of the corresponding vectors relative to $\mathbf{0}$, in the usual way.

You may assume without proof that an isometry has the form $T(x) = Ax + \mathbf{b}$ where $A$ is an orthogonal matrix and $\mathbf{b}$ is a vector. (You will be expected to know how to prove this result for the exam.)

(a) Show that the image of $\mathbf{PQ}$ under an isometry $T$ is the line segment $T(\mathbf{P})T(\mathbf{Q})$

(b) Prove that if $T$ and $R$ are isometries of any metric space $(X, d)$, then the composition $T \circ R$ is also an isometry.

(c) Show that if $\mathbf{PQ}$ and $\mathbf{P'Q'}$ are line segments of equal length, i.e., $d(\mathbf{P}, \mathbf{Q}) = d(\mathbf{P'}, \mathbf{Q'})$, then there exists an isometry $T$ with $T(\mathbf{PQ}) = \mathbf{P'Q'}$.
Is this isometry unique? If it is, prove it, if not, describe another one with the same property.

(d) Let $A = (0, 0)$, $B = (5, 0)$, $C = (-2, 4)$, $D = (1, 8)$.

(i) Describe a rotation $R$ of $\mathbb{R}^2$ that maps $AB$ to $CD$.

(A) Write your rotation in the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$,

(B) find the centre of the rotation, i.e., a solution to $R(\mathbf{x}) = \mathbf{x}$

(C) find the eigen values of $A$

(ii) Describe a reflection $F$ of of $\mathbb{R}^2$ that maps $AB$ to $CD$.

(A) Write your reflection in the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$,

(B) find a line fixed by this reflection

(C) find the eigen values and vectors of $A$

(a) [4] $T$ maps $\mathbf{P}$ to $T(\mathbf{P})$ and $\mathbf{Q}$ to $T(\mathbf{Q})$, so we just have to show all other points on the line $\mathbf{PQ}$ map to $T(\mathbf{P})T(\mathbf{Q})$.

In the class we prove that $\mathbf{z}$ is on the line segment between $\mathbf{x}$ and $\mathbf{y}$ if and only if $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$, so, using that isometries preserve distance, we have

$$\begin{aligned} \mathbf{z} \in \mathbf{PQ} &\iff d(\mathbf{P}, \mathbf{Q}) = d(\mathbf{P}, \mathbf{z}) + d(\mathbf{z}, \mathbf{Q}) \\ &\iff d(T(\mathbf{P}), T(\mathbf{Q})) = d(T(\mathbf{P}), T(\mathbf{z})) + d(T(\mathbf{z}), T(\mathbf{Q})) \\ &\iff T(\mathbf{z}) \in T(\mathbf{P})T(\mathbf{Q}) \end{aligned}$$

To show that the image is the whole of $T(\mathbf{P})T(\mathbf{Q})$, just note that $T$ is a bijection, so has an inverse, and $T^{-1}$ maps the line $T(\mathbf{P})T(\mathbf{Q})$ back to $\mathbf{PQ}$, so since the composition of $T$ and $T^{-1}$ is the identity, $T$ maps the point $T^{-1}(\mathbf{z})$ to $\mathbf{z}$ on $T(\mathbf{P})T(\mathbf{Q})$, and the map surjectively maps $\mathbf{PQ}$ to $T(\mathbf{P})T(\mathbf{Q})$.

2 points for using the fact that another (1) point for using it correctly. 1 point for justifying that the image is equal to the line segment $T(\mathbf{P})T(\mathbf{Q})$, not just a subset of this segment.

(b) [2] An isometry is a distance preserving bijection. A bijective map can be defined to be a map with an inverse, so if $T$ and $R$ both have inverses, $T^{-1}$ and $R^{-1}$ then the inverse of the composition $T \circ R$ also exists, and is $R^{-1} \circ T^{-1}$. Thus the composition is a bijection.

Now show the composition is distance preserving:

$$\begin{aligned} d(T \circ R(\mathbf{x}), T \circ R(\mathbf{y})) &= d(T(R(\mathbf{x})), T(R(\mathbf{y}))) \\ &= d(R(\mathbf{x}), R(\mathbf{y})) \text{ since } T \text{ is distance preserving} \\ &= d(\mathbf{x}, \mathbf{y}) \text{ since } T \text{ is distance preserving} \end{aligned}$$

1 point for justification of being a bijection, 1 point for proof of distance preserving

(c) [4] There are several ways to do this question.

Suppose $\mathbf{PQ}$ has length $h$, so this segment is the same length as the segment from $\mathbf{0}$ to $(h, 0)$. The isometry $T(\mathbf{x}) = \mathbf{x} - \mathbf{P}$ maps $\mathbf{P}$ to $\mathbf{0}$.

Let $\mathbf{Q}' = \mathbf{Q} - \mathbf{P}$, which is the image of $\mathbf{Q}$ under $T$. So, $T$ applied to $\mathbf{PQ}$ is a line segment from $\mathbf{0}$ to $\mathbf{Q}'$.

Homework solutions 1

Since $T$ is an isometry, and so preserves distance, $d(\mathbf{0}, \mathbf{Q}') = h$, which means that $\mathbf{Q}' = h(\cos\theta, \sin\theta)$ for some $\theta \in [0, 2\pi)$, and a rotation through $-\theta$, given by the isometry

$$R : \mathbf{x} \to \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \mathbf{x}$$

maps $\mathbf{0}$ to $\mathbf{0}$ and $\mathbf{Q}'$ to $(h, 0)$.

So the composition $U = R \circ T$ maps $\mathbf{PQ}$ to $\mathbf{0}(h, 0)$. (We have proved that the composition of isometries is an isometry)

Similarly there is an isometry $V$ mapping $\mathbf{P}'\mathbf{Q}'$ to $\mathbf{0}(h, 0)$. The composition of the isometries $V^{-1}$ and $U$, namely, $V^{-1} \circ U$ is the required isometry.

This isometry is not unique, because we could map either $\mathbf{P}$ to $\mathbf{P}'$ and $\mathbf{Q}$ to $\mathbf{Q}'$ or $\mathbf{P}$ to $\mathbf{Q}'$ and $\mathbf{Q}$ to $\mathbf{P}'$

If the result is proved in this way, (1) point for involving a translation (1) point for involving a rotation (1) point for involving composition of functions (1) point for justifying that the map is not unique. Points may need to be assigned in some other way if a different method is used. Correct result always gets 4 points

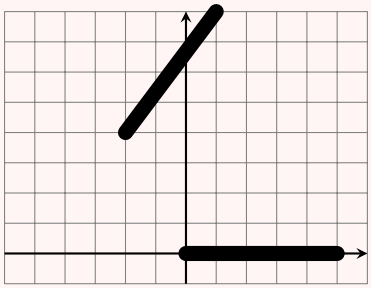
(d) (i) [3]

(ii) [3]

We can either solve this algebraically, or geometrically. One algebraic method would be to write out $T(\mathbf{x}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x} + \begin{pmatrix} e \\ f \end{pmatrix}$ and substitute the values for $T(\mathbf{x})$ and $\mathbf{x}$ in order to solve for $a, b, c, d, e, f$, also using that $A$ is orthogonal.

We could also find the solution using a composition of maps, as in part (c).

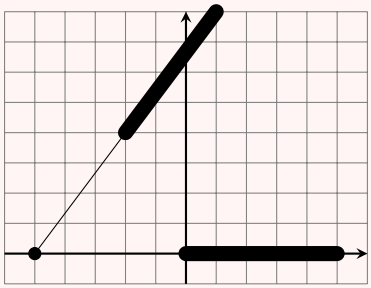
Alternatively we can use a geometric approach. First draw a picture of the line segments:



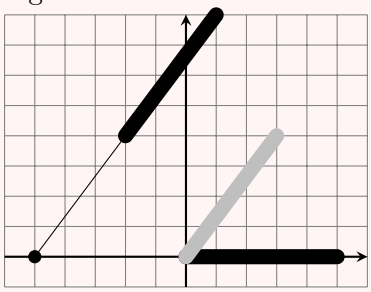
This is chosen to be an easy example, since when we extend the lines they intersect at equal distance from the intersection point, so this gives the centre of the rotation,

at

$$(-5, 0)$$



In order to find $A$ and $\mathbf{b}$, we can think of this as a rotation followed by a translation; this figure shows the result after the rotation:





The rotation taking the line $\mathbf{0}(5, 0)$ to $\mathbf{0}(3, 4)$ takes $(5, 0)$ to $(3, 4)$. Since it is a linear map, it takes $(1, 0)$ to $(\frac{3}{5}, \frac{4}{5})$. This means the matrix $A$ has first column given by this vector:

$$A = \begin{pmatrix} \frac{3}{5} & ? \\ \frac{4}{5} & ? \end{pmatrix}$$

Since we know that a rotation matrix has the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, this tells us the rest of $A$ (though we could also just draw the image of the other axis at right angles to the gray line, and work out the image of $(0, 1)$ to write as the second column).

The translation from the gray line to the line from $(-2, 4)$ to $(1, 8)$ takes $\mathbf{0}$ to $(-2, 4)$, so $\mathbf{b} = (-2, 4)$. So the map is given by

$$\mathbf{x} \to \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

You can check this works algebraically by substituing $\mathbf{x} = \mathbf{0}$ and $(5, 0)$ to check the images are in the right place.

The centre of the rotation, as shown in this figure, is $(-5, 0)$. You can also check this algebraically, by checking $T(-5, 0) = (-5, 0)$.

The eigen values are the roots of the characteristic polynomial,

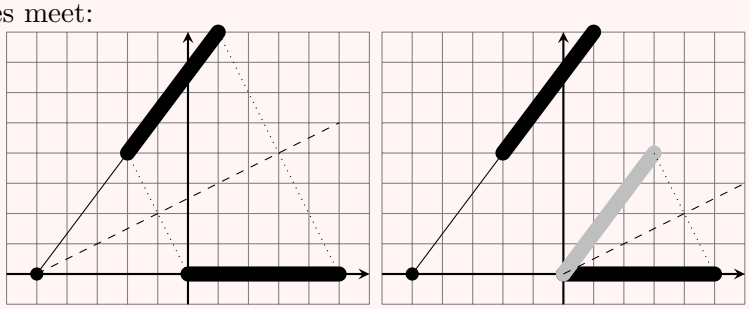
$$\left| \begin{pmatrix} \frac{3}{5} - t & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} - t \end{pmatrix} \right| = (3/5 - t)^2 + (4/5)^2 = t^2 - 6/5t + 1$$

which has roots

$$\frac{3}{5} \pm i\frac{4}{5}$$

Note that this is a special case of the eigen values for a rotation matrix being $\cos\theta \pm i\sin\theta$.

For the reflection, we can draw the reflection line through the line where these lines meet:



From the first of these two pictures, we can see that the line fixed by the reflection is

$$y = \frac{1}{2}x + \frac{5}{2}$$

From the second, we see that as before, the first column of the matrix $A$ is the same as for the rotation matirx case, but since it's a reflection, we have a different second column. The value of $\mathbf{b}$ is also the same. So the transformation is

$$\mathbf{x} \to \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

A reflection matrix in $\mathbb{R}^2$ always has eigen values

$$1 \text{ and } -1.$$

You can multiply out the characteristic polynomial to check; it is $t^2 - 1$.

1 point for each of (A), (B), (C) in each case; points given whichever method is used provided the answers are correct, and there is some explanation

Homework solutions 1

## 60. MA243 Homework 2

Due via moodle, at noon, Thursday, week 4, 27th October 2022.
Only questions (1)-(4) will be marked. The total is 25. You must show your working, proof, or some kind of explanation of justification, for every question. You can quote results from the class, but make sure it's clear which result you are using. If you get the right answers with no explanation you get half marks. In these questions $\mathbb{R}^n$ means $\mathbb{E}^n$, i.e., $\mathbb{R}^n$ with the standard Euclidean metric.

(1) (Computation: Decomposition of an isometry into product of reflections)
There is an isometry $T : \mathbb{R}^3 \to \mathbb{R}^3$ which maps points as follows:

$$\begin{aligned} T(\ 1,0,0) &= (2,0,\ \ 1)\\ T(\ 0,1,0) &= (2,1,\ \ 0)\\ T(\ 0,0,1) &= (1,0,\ \ 0)\\ T(-1,0,0) &= (2,0,-1) \end{aligned}$$

(a) What is $T(\mathbf{0})$?
(b) Find a matrix $A$ and a vector $\mathbf{b}$ such that $T(\mathbf{x}) = A + \mathbf{b}$. Are $A$ and $\mathbf{b}$ unique?
(c) Find the set $\mathrm{Fix}(T)$ of fixed points of $T$.
(d) Write $T$ as a composition of 2 reflections. This means you have to either describe the reflections as maps of the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$, or you can describe the hyperplanes fixed by the reflections.
(e) Is this decomposition unique? Either prove it is, or find a different decomposition.

Homework 2

(2) (Basis free description of the reflection in a hyperplane) Let $\Pi$ be a hyperplane in $\mathbb{R}^n$ defined by $\Pi = \mathbf{b} + (\mathbb{R}\mathbf{v})^\perp$, where $||\mathbf{v}|| = 1$. Prove that the reflection in $\Pi$ is given by

$$\rho_\Pi(\mathbf{P}) = \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v}\rangle \mathbf{v}$$

You should show that
(a) This map is an isometry
(b) This map is non-trivial
(c) This map fixes $\Pi$

(3) Prove that if $\Pi$ is a hyperplane, and $T$ an isometry, then $T(\Pi)$ is also a hyperplane
(4) In $\mathbb{R}^2$, the composition of a translation and a non-trivial rotation is a rotation. If $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ and $R(\mathbf{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mathbf{x}$, what is the centre of rotation of $T \circ R$? What is the centre of rotation of $R \circ T$? Answer as a formula in terms of $\theta$ and $\mathbf{b}$.

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Optional unmarked extras for practice and for the support class

(5) Prove that for $\mathbf{c} \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, the following set is a hyperplane: $\{\mathbf{v} \in \mathbb{R}^n : \langle \mathbf{v}, \mathbf{c}\rangle = \beta\}$
(6) Prove that if $R$ is a reflection and $T$ is any isometry, then $T^{-1} \circ R \circ T$ is also a reflection.
(7) If $\rho_\Pi$ is a reflection in the plane $\Pi$, prove that the conjugation of $\rho_\Pi$ by the reflection $T(\mathbf{x}) = \mathbf{x}+\mathbf{c}$, that is, the map $T^{-1} \circ \rho_\Pi \circ T$ is a reflection in $\Pi - \mathbf{c}$.
(8) Suppose that $R$ is a reflection in $\Pi = V + \mathbf{b}$, and $\mathbf{v} \in V$, then show that $R$ commutes with the isometry $T : \mathbf{x} \mapsto \mathbf{x} + \mathbf{v}$.
(9) Prove that a reflection $R$ has order 2, that is, $R \circ R = \mathrm{id}$.
(10) Prove that $O(n)$ is a group.
(11) Prove that $\mathrm{Isom}(X, d)$ is a group for any metric space $(X, d)$
(12) Show that if $L = \mathbf{P} + \mathbb{R}\mathbf{v}$, and $L' = \mathbf{P'} + \mathbb{R}\mathbf{v'}$, then $L = L' \iff \mathbf{v'}$ and $\mathbf{P} - \mathbf{P'}$ are both in $\mathbb{R}\mathbf{v}$.
(13) Show that for $a_1, a_2, \ldots, a_k \in \mathbb{R}^n$ with $\sum_{i=1}^k a_i = 1$, then for an isometry $T$, then for points $\mathbf{P}_1, \ldots \mathbf{P}_k$,

$$T\left(\sum a_k \mathbf{P}_k\right) = \sum a_k T(\mathbf{P}_k)$$

## 61. MA243 Homework 2: Solutions

Due via moodle, at noon, Thursday, week 4, 27th October 2022.
Only questions (1)-(4) will be marked. The total is 25. You must show your working, proof, or some kind of explanation of justification, for every question. You can quote results from the class, but make sure it's clear which result you are using. If you get the right answers with no explanation you get half marks. In these questions $\mathbb{R}^n$ means $\mathbb{E}^n$, i.e., $\mathbb{R}^n$ with the standard Euclidean metric.

(1) (Computation: Decomposition of an isometry into product of reflections)
There is an isometry $T : \mathbb{R}^3 \to \mathbb{R}^3$ which maps points as follows:

$$\begin{aligned} T(\ 1,0,0) &= (2,0,\ \ 1)\\ T(\ 0,1,0) &= (2,1,\ \ 0)\\ T(\ 0,0,1) &= (1,0,\ \ 0)\\ T(-1,0,0) &= (2,0,-1) \end{aligned}$$

(a) What is $T(\mathbf{0})$?
(b) Find a matrix $A$ and a vector $\mathbf{b}$ such that $T(\mathbf{x}) = A + \mathbf{b}$. Are $A$ and $\mathbf{b}$ unique?
(c) Find the set $\mathrm{Fix}(T)$ of fixed points of $T$.
(d) Write $T$ as a composition of 2 reflections. This means you have to either describe the reflections as maps of the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$, or you can describe the hyperplanes fixed by the reflections.
(e) Is this decomposition unique? Either prove it is, or find a different decomposition.

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(a) [2] Since $T$ is an isometry, $T$ preserves lines, and the following relationship holds:

$$T((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) = (1-\lambda)T(\mathbf{x}) + \lambda T(\mathbf{y})$$

So,

$$\begin{aligned} T(0,0,0) &= T(\frac{1}{2}(1,0,0) + \frac{1}{2}(-1,0,0))\\ &= \frac{1}{2}T(1,0,0) + \frac{1}{2}T(-1,0,0)\\ &= \frac{1}{2}\Big((2,0,1) + (2,0,-1)\Big) = (2,0,0) \end{aligned}$$

(b) [2] (1) point for giving $\mathbf{b}$; (1) point for $A$ We know that $T(\mathbf{x}) = A\mathbf{x} + T(\mathbf{0})$, that is,

$$\mathbf{b} = T(\mathbf{0}) = (2,0,0)$$

and we must have

$$\begin{aligned} A(1,0,0) &= (\ \ 0,0,1)\\ A(0,1,0) &= (\ \ 0,1,0)\\ A(0,0,1) &= (-1,0,0) \end{aligned}$$

The images of the basis vectors are the columns of $A$, so

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T(\mathbf{x}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$A$ and $\mathbf{b}$ are unique, since $\mathbf{b}$ must be equal to $T(\mathbf{0})$, and the matrix of a linear transformation is unique.

(c) [2] (1) for writing out the correct equation; (1) for solving correctly

$$\mathrm{Fix}(T) = \{\mathbf{x} \in \mathbb{R}^3 : T(\mathbf{x}) = \mathbf{x}\}$$

So we need to solve

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{aligned} -z + 2 &= x \\ y &= y \\ x &= z \end{aligned}$$

This implies $-z + 2 = x = z$, which implies $2 = 2z$, so $x = z = 1$ and the fixed points are on the line

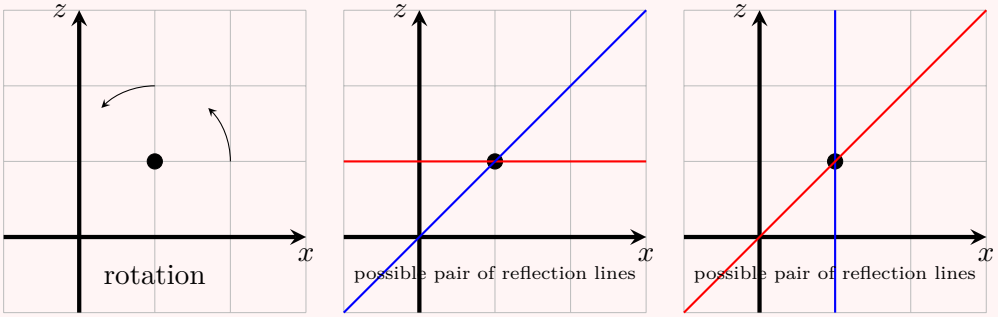
$$\mathrm{Fix}(T) = \{(1, y, 1) : y \in \mathbb{R}\}$$

(d) [4] (4 points for any sensible method, and any valid solution; not necessary to reduce to working in $\mathbb{R}^2$)

There are various method of finding the decomposition into a product of reflections. Since the line $x = z = 1$ is fixed, we know that $T$ describes a rotation about this line, and from the fact that the matrix restricted to the $xz$-plane is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we know that it's a rotation of 90 degrees anti-clockwise when restricted to the $xz$ plane. We might notice that the $y$ coordinate is unchanged by this transformation, so we can work entirely in the $xz$ plane, and add back the $y$ coordinate at the end.




In the figures, we reflect in the red line then the blue line. The two reflection lines must intersect in the centre of rotation, and must be at angle 45 to each other.

The second pair of lines might be more likely to be used, because this corresponds to the method discussed in class / the notes, which maps $T(\mathbf{0})$ to $\mathbf{0}$; this can either be done at the end of the beginning of the sequence of transformation, but we did it in order to reverse $T$, so that means it's the first step of decomposing, or the last step when going in the way of building $T$ from reflections.

the reflection in the red line corresponds to

$$\mathbf{x} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$$

the reflection in the blue line is given b

$$\mathbf{x} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

the reflection in the red line corresponds to

$$R_1\mathbf{x} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{x}$$

The reflection in the blue line corresponds to

$$R_2 : \mathbf{x} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Then $T = R_2 \circ R_1$

(e) [2] (2 points for any other decomposition) For example, if we choose a decomposition corresponding to the middle picture above, the reflection in the red line corresponds to

$$\mathbf{x} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$$

and the reflection in the blue line is

$$\mathbf{x} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Writing this out in terms of a solution to the original problem, by putting the $y$ coordinate back in, we can say that $T$ is equal to the reflection

$$R_1\mathbf{x} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{x}$$

followed by application of the reflection

$$R_2\mathbf{x} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Ie., $T = R_2 \circ R_1$,

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(2) (Basis free description of the reflection in a hyperplane) Let $\Pi$ be a hyperplane in $\mathbb{R}^n$ defined by $\Pi = \mathbf{b} + (\mathbb{R}\mathbf{v})^{\perp}$, where $||\mathbf{v}|| = 1$. Prove that the reflection in $\Pi$ is given by

$$\rho_\Pi(\mathbf{P}) = \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v} \rangle \mathbf{v}$$

You should show that

(a) This map is an isometry
(b) This map is non-trivial
(c) This map fixes $\Pi$

(a) (6 points)

(1) point for knowing definition of isometry and what to check

We want to show that $\rho_\Pi$ preserves distance, and is a bijection.

(2) points for correctly checking bijectivity

(1) $T$ is a bijection: We show that $T$ composed with itself is the identity, so $T$ has a double sided inverse so is bijective:

$$T(T(\mathbf{P})) = T(\mathbf{P}) - 2\langle T(\mathbf{P}) - \mathbf{b}, \mathbf{v} \rangle \mathbf{v}$$

$$= \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v} \rangle \mathbf{v} - 2\langle \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v} \rangle \mathbf{v} - \mathbf{b}, \mathbf{v} \rangle \mathbf{v}$$

$$= \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b} + \mathbf{P} - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v} \rangle \mathbf{v} - \mathbf{b}, \mathbf{v} \rangle \mathbf{v}$$

$$= \mathbf{P} - 2\langle 2(\mathbf{P} - \mathbf{b}) - 2\langle \mathbf{P} - \mathbf{b}, \mathbf{v} \rangle \mathbf{v}, \mathbf{v} \rangle \mathbf{v}$$

$$= \mathbf{P} - 4\Big(\langle(\mathbf{P}-\mathbf{b}),\mathbf{v}\rangle - \langle \mathbf{P}-\mathbf{b},\mathbf{v}\rangle \langle \mathbf{v},\mathbf{v}\rangle\Big)\mathbf{v}$$

$$= \mathbf{P} - 4\Big(\langle \mathbf{P}-\mathbf{b},\mathbf{v}\rangle - \langle \mathbf{P}-\mathbf{b},\mathbf{v}\rangle\Big)\mathbf{v}$$

$$= \mathbf{P}$$

(3) points for correctly checking distance preserving. (1) point for correctly computing $\rho_\Pi(\mathbf{P}) - \rho_\Pi(\mathbf{Q})$. (2) points for the computation of $||\rho_\Pi(\mathbf{P}) - \rho_\Pi(\mathbf{Q})||$. The computation might look more complicated if the difference $\mathbf{P}-\mathbf{Q}$ is not written as $\mathbf{R}$, but it will still be marked correct

(2) Let $\mathbf{P}$ and $\mathbf{Q}$ be two points. we want to show that $||\rho_\Pi(\mathbf{P})-\rho_\Pi(\mathbf{Q})|| = ||\mathbf{P}-\mathbf{Q}||$
We have

$$\begin{aligned}\rho_\Pi(\mathbf{P}) - \rho_\Pi(\mathbf{Q}) &= \mathbf{P} - 2\langle \mathbf{P}-\mathbf{b},\mathbf{v}\rangle \mathbf{v} - \mathbf{Q} + 2\langle \mathbf{Q}-\mathbf{b},\mathbf{v}\rangle \mathbf{v} \\ &= \mathbf{P}-\mathbf{Q} - 2\langle \mathbf{P}-\mathbf{Q},\mathbf{v}\rangle \mathbf{v}\end{aligned}$$

Let's write $\mathbf{R} = \mathbf{P}-\mathbf{Q}$ for short, so we have

$$\rho_\Pi(\mathbf{P}) - \rho_\Pi(\mathbf{Q}) = \mathbf{R} - 2\langle \mathbf{R},\mathbf{v}\rangle \mathbf{v}$$

We want to show that $||\mathbf{R}|| = ||\mathbf{R} - 2\langle \mathbf{R},\mathbf{v}\rangle \mathbf{v}||$

$$\begin{aligned}||\mathbf{R} - 2\langle \mathbf{R},\mathbf{v}\rangle \mathbf{v}||^2 &= \langle \mathbf{R} - 2\langle \mathbf{R},\mathbf{v}\rangle \mathbf{v}, \mathbf{R} - 2\langle \mathbf{R},\mathbf{v}\rangle \mathbf{v}\rangle \\ &= \langle \mathbf{R},\mathbf{R}\rangle - 4\langle \mathbf{R},\langle \mathbf{R},\mathbf{v}\rangle \mathbf{v}\rangle + 4\langle \mathbf{R},\mathbf{v}\rangle^2 \langle \mathbf{v},\mathbf{v}\rangle \\ &= \langle \mathbf{R},\mathbf{R}\rangle - 4\langle \mathbf{R},\mathbf{v}\rangle \langle \mathbf{R},\mathbf{v}\rangle + 4\langle \mathbf{R},\mathbf{v}\rangle^2 \\ &= \langle \mathbf{R},\mathbf{R}\rangle - 4\langle \mathbf{R},\mathbf{v}\rangle \langle \mathbf{R}-\mathbf{R},\mathbf{v}\rangle \\ &= \langle \mathbf{R},\mathbf{R}\rangle - 4\langle \mathbf{R},\mathbf{v}\rangle \langle \mathbf{0},\mathbf{v}\rangle \\ &= \langle \mathbf{R},\mathbf{R}\rangle \\ &= ||\mathbf{R}||^2\end{aligned}$$

So we're done. Various different arguments are possible, e.g., you could show that this function is linear, then choose an orthogonal basis and find the description in terms of a matrix and vector. You could modify the above argument by writing $\mathbf{R}$ as a sum of an element of $\mathbb{R}\mathbf{v}$ and an element of $(\mathbb{R}\mathbf{v})^\perp$. Any correct method carefully explained is OK.

(b) (1 point) We just need to find some point that is not fixed, e.g.:

$$\rho_\Pi(\mathbf{b}+\mathbf{v}) = \mathbf{b}+\mathbf{v} - 2\langle \mathbf{b}+\mathbf{v}-\mathbf{b},\mathbf{v}\rangle \mathbf{v} = \mathbf{b}+\mathbf{v}-2\mathbf{v} = \mathbf{b}-\mathbf{v} \neq \mathbf{b}+\mathbf{v}$$

(c) (2 points: one for knowing form of point in $\Pi$, one for correct computation)
A point in $\Pi$ has the form $\mathbf{b}+\mathbf{w}$ where $w \in (\mathbb{R}\mathbf{v})^\perp$, so $\langle \mathbf{v},\mathbf{w}\rangle = 0$.
We find

$$\begin{aligned}\rho_\Pi(\mathbf{b}+\mathbf{w}) &= \mathbf{b}+\mathbf{w} - 2\langle (\mathbf{b}+\mathbf{w})-\mathbf{b},\mathbf{v}\rangle \mathbf{v} \\ &= \mathbf{b}+\mathbf{w} - 2\langle \mathbf{w},\mathbf{v}\rangle \mathbf{v} = \mathbf{b}+\mathbf{w}-\mathbf{0} = \mathbf{b}+\mathbf{w}\end{aligned}$$

which shows that any point in $\Pi$ is fixed.

(3) Prove that if $\Pi$ is a hyperplane, and $T$ an isometry, then $T(\Pi)$ is also a hyperplane

(1) point to compute $T(\Pi)$ correctly, (1) point for knowing the definition of a hyperplane
$T\mathbf{x} = A\mathbf{x}+\mathbf{b}$ for some orthonomal $A$ and vector $\mathbf{b}$.
Let $\Pi = V + \mathbf{c}$. We know that $A$ has an inverse, $A^T$, and since $A$ is a linear map, it maps $V$ to another subvector space of the same dimension, so $W = A(V)$ is a hyperplane.
Now $T(\Pi) = T(V+\mathbf{c}) = A(V+\mathbf{c}) + \mathbf{b} = A(V) + A\mathbf{c} + \mathbf{b} = W + \mathbf{d}$ where $\mathbf{d} = A\mathbf{c}+\mathbf{b}$. Hence $T(\Pi)$ has the correct form so we can conclude it is a hyperplane.

(4) In $\mathbb{R}^2$, the composition of a translation and a non-trivial rotation is a rotation. If $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ and $R(\mathbf{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mathbf{x}$, what is the centre of rotation of $T \circ R$? What is the centre of rotation of $R \circ T$? Answer as a formula in terms of $\theta$ and $\mathbf{b}$.

To find the fixed point of $T \circ R$ we need to solve $T \circ R\mathbf{x} = \mathbf{x}$, that is:
$\mathbf{x} = T(R(\mathbf{x})) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mathbf{x} + \mathbf{b}$. We can rewrite this as

$$\begin{pmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{pmatrix} \mathbf{x} = \mathbf{b}$$

$$\mathbf{x} = \begin{pmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{pmatrix}^{-1} \mathbf{b}$$

This matrix has determinant $(1-\cos\theta)^2 + \sin\theta^2 = 1 - 2\cos\theta + 1$ this can only be zero if $\cos\theta = 1$, which only happens if $\theta = 0$, but we said the rotation was nontrivial.
This answer is acceptable, since it's given in terms of $\theta$ and $\mathbf{b}$. Students may multiply it out, which is also OK but not required.

To find the fixed point of $T \circ R$ we need to solve $R \circ T\mathbf{x} = \mathbf{x}$, that is:
$\mathbf{x} = T(R(\mathbf{x})) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} (\mathbf{x} + \mathbf{b})$. We can rewrite this as

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$$\begin{pmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{pmatrix} \mathbf{x} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mathbf{b}$$

Now the solution is

$$\mathbf{x} = \begin{pmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mathbf{b}$$

This is acceptable to leave the solution in this form. As before the matrix is invertible, so this is well defined.

## 62. MA243 Homework 3

Due via moodle, at noon, Thursday 17th November, week 7, 2022.
Only questions with a red star ★ will be marked.
The other questions can be covered in the support classes.

(1) ★ [6 points] Prove that there is an isometry $V \cap S^n \to S^1$ for any two dimensional sub vector space $V$. Hint: Find an orthogonal matrix taking $V$ to $\mathbb{R}^2$. You may assume that $V$ has an orthonormal basis, and that any orthonormal basis of $V$ may be extended to an orthonormal basis of $\mathbb{R}^{n+1}$.

(2) A circle is a set of points a fixed distance $r$ from a fixed point $\mathbf{P}$
   (a) Prove that any circle on the sphere $S^2$ is the intersection of the sphere with a plane.
   (b) Find a formula for the length of the circumference of a circle of radius $r$ (measured by the spherical metric) on $S^2$.
   You may assume that the circumference of a circle of radius $r$ in Euclidean space is $2\pi r$.
   Assuming that distance of any path is preserved by an isometry, show that the sphere $S^2$ can not be isometric to the plane.
   (c) Find a formula for the area of a circle of radius $r$ measured on the sphere. Note that because the spherical metric is given by the Euclidean length of a path on the sphere, the area can be computed using usual methods of integration in spherical coordinates, with the area element you have seen in Geometry and motion.
   (d) Is a circle on $\mathcal{H}^2$ given by $\mathcal{H}^2$ intersected by a plane? Either prove or find a counter example.

(3) ★ [6 points] The following matrix is orthogonal, and represents an isometry of $S^2$.

$$A = \frac{1}{3}\begin{pmatrix} 1 & -2 & ? \\ 2 & ? & -2 \\ 2 & 2 & 1 \end{pmatrix}$$

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   (a) Determine the missing entries.
   (b) Find the set of fixed points of $T_A$, the map $T_A(\mathbf{x}) = A\mathbf{x}$.
   (c) determine whether $T_A$ is a reflection, rotation, or rotation followed by reflection in a plane perpendicular to the axis of rotation.

(4) ★ [7 points] Prove that from the definitions that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transformation, then $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ is a Lorentz orthonormal basis. (You may not assume that $T$ is a linear map.)

(5) The following matrix is Lorentz orthogonal, and represents an isometry of $\mathcal{H}^2$:

$$\begin{pmatrix} 5/3 & 4/5 & 16/15 \\ 4/3 & ? & 4/3 \\ ? & 4/5 & -3/5 \end{pmatrix}$$

   (a) Determine the missing entries.
   (b) Find the set of fixed points of $T_A$.
   (c) determine whether $T_A$ is a
      (i) reflection (fixes a hyperbolic line in $\mathcal{H}^2$),
      (ii) rotation, (fixes a point in $\mathcal{H}^2$),
      (iii) Lorentz translation along an axis $L$ (no fixed points, orientation preserving) or
      (iv) Lorentz glide (a translation along axis $L$ followed by by reflection in a plane perpendicular to the axis of rotation. (no fixed points, orientation reversing)

(6) ★ [6 points] Prove that if $R$ is an isometry on $\mathcal{H}^n$, and we define

$$\widehat{R}(\mathbf{x}) = -i||\mathbf{x}||_L R\left(\frac{\mathbf{x}}{-i||\mathbf{x}||_L}\right)$$

for positive time-like vectors $\mathbf{x}$, then $\widehat{R}$ preserves the Lorentz inner product on positive time-like vectors, i.e., if $\mathbf{x}$ and $\mathbf{y}$ are positive time-like, then

$$\left\langle \widehat{R}(\mathbf{x}), \widehat{R}(\mathbf{y}) \right\rangle_L = \langle \mathbf{x}, \mathbf{y} \rangle_L$$

### 63. MA243 Homework 3: Solutions to unmarked questions

(2)

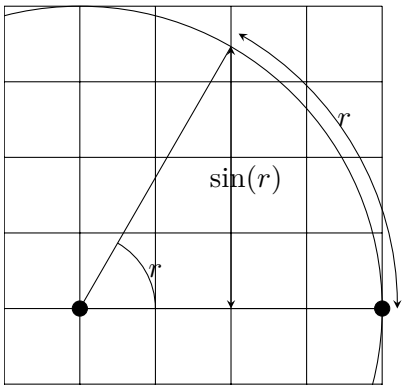
(a) Suppose $\mathbf{P}$ is the centre of a circle, $C$ on $S^1$, of radius $r \leq \pi$.

Since isometries preserve distance by definition, we may assume that $\mathbf{P} = (1, 0, 0)$.

(Details: This is because since $\mathbf{P} \in S^1$, we know $||\mathbf{P}|| = 1$, so we can extend to an othonormal basis of $\mathbb{R}^3$ (orthonormalisation, from Linear Algebra), and the three orthonomal basis elements form columns of an orthonormal matrix, which maps $(1, 0, 0)$ to $\mathbf{P}$. By Corollary 5 from MA243, we know that there is an isomorphism between $\mathrm{Isom}(S^1)$ and $O(n+1)$, and so this matrix does indeed define an isometry, and we can assume that $\mathbf{P} = (1, 0, 0)$.)

Suppose a point $\mathbf{x}$ on $C$ has coordinates $(a, b, c)$ and is distance $r$ from $(1, 0, 0)$. Since $\mathbf{x} \in S^1$, we have $a^2 + b^2 + c^2 = 1$. Since $\mathbf{x} \in C$, by definition of distance on $S^1$, we have $\cos(r) = \langle (1, 0, 0), (a, b, c) \rangle = a$. So, there is no extra condision on $b, c$, and $C$ is the set of points on $S^1$ where $x = a$, which is the intersection of the plane $x = a$ with $S^1$.

(b) If $C$ has radius $r$, as measured on the surface of the sphere, then the radius of this circle in the ambient Euclidean space is $\sin(r)$, and so the length of the circumference of $C$ is $2\pi \sin(r)$.




Note in particular, the ratio between the circumferance and the radius is

$$\frac{2\pi \sin(r)}{r}$$

For very small $r$, this means the ratio is quite close to $2\pi$. Notice that the ratio does not depend on the sphere, but on the radius $r$, so the ratio of a circle's radius to its circumference is not constant on the sphere.

(c) Apply a spherical isometry as before, and we may assume the circle has center at $(0, 0, 1)$. Since (by results from Geometry and Motion) the area element for the integral is $r^2 \sin\phi d\phi d\theta$ in terms of spherical coordinates, and setting $r = 1$,

Then we have that the surface area of the circle radius $r$ is:

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{r} \sin\phi d\phi d\theta = 2\pi \left[-\cos(\phi)\right]_0^r = 2\pi(1 - \cos(r))$$

You are recommended to make a sketch.

Note that this means the ratio between the area and the square of the radius is

$$\frac{2\pi(1 - \cos(r))}{r^2}$$

For very small $r$, $\cos(r) \sim 1 - r^2/2$, so the ratio is approximately $\pi$ when $r$ is small enough.

(d) In the same way as for part (a), we may now assume that $P = (1, 0, 0)$, using Lemma 30 in the notes. Now the hyperbolic distance from a point $(a, b, c)$ to $(1, 0, 0)$ is given by definition

$$\cosh^{-1}(-\langle (a, b, c), (1, 0, 0)\rangle) = \cosh^{-1}(a)$$

and so the circle is defined by intersection of $\mathcal{H}^2$ with the plane given by $x_1 = a$.

Since we have shown (Corollary 10) that isometries of the hyperbolic plane are given by linear maps, any isomety must map a plane to a plane, and so all circles on $\mathcal{H}^2$ are given by the intersection of $\mathcal{H}^2$ with a plane.

(5)

(a) We need to choose entries so that $A^T J A = J$. This is simple algebra.

It turns out the missing entries are 0 and 1, so the matrix is

$$\begin{pmatrix} 5/3 & 4/5 & 16/15 \\ 4/3 & 1 & 4/3 \\ 0 & 4/5 & -3/5 \end{pmatrix}$$

(b) To find the fixed points, we have to solve $A\mathbf{x} = \mathbf{x}$. This is the same as finding an eigen vector with eigen value 1. Is there such an eigen values? Compute the characteristic polynomial: (fine to do this using a computer; the point is to know how to interpret the results of a computation (calculators not allowed in exams though)...

$$\left| - \begin{pmatrix} 5/3-\lambda & 4/5 & 16/15 \\ 4/3 & 1-\lambda & 4/3 \\ 0 & 4/5 & -3/5-\lambda \end{pmatrix} \right| = \lambda^3 - 31/15\lambda^2 - 31/15\lambda + 1$$

We can see that $\lambda = 1$ is not a root of this polynomial.

So

$$\mathrm{Fix}(T_A) = \varnothing$$

(c) There are no fixed points, so $T_A$ can not be a reflection or a rotation.

Since the determinant is $-1$, the transformation must reverse orientation, and is a Lorentz glide reflection.

In order to find the direction of this glide translation, we must find the vector with eigen value $-1$. Solving $A\mathbf{v} = -\mathbf{v}$ (usual usual methods from linear algbera) gives $\mathbf{v} = (1, 2, -4)$. The Lorentz orthognal plane,

$$\Pi = \{(x, y, z) : -x + 2y - 4z = 0\}$$

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defines a line $L = \Pi \cap \mathcal{H}^2$.

Points $\mathbf{x}$ on $\Pi$ satisfy $\langle \mathbf{v}, \mathbf{x} \rangle_L = 0$. Since $T_A$ is a hyperbolic isometry (since $A \in O^+(1, 2)$), we have

$$0 = \langle T_A(\mathbf{x}), T_A(\mathbf{v}) \rangle_L = \langle T_A(\mathbf{x}), -\mathbf{v} \rangle_L = - \langle T_A(\mathbf{x}), \mathbf{v} \rangle_L$$

so $T_A(\mathbf{x})$ is also on $L$, so although $L$ is not fixed point wise, the whole line $L$ is fixed, and so $T_A$ is a translation along direction $L$, followed by reflection in $L$.

**important note:** I have given the computation of direction of the Lorentz glide, but the question does not ask this; this is just for completeness. If a question asks "is something a reflection, rotation, translation?", you only need to say which one it is, you don't have to say the centre of rotation, etc, unless that is explicitly asked. However, you do need to give justification for your answer, which might include explicitly stating the centre of rotation, or might be determined otherwise. Always make sure to read questions carefully to decide how much detail to give, but at least some justification will be needed for full marks. If you give a non-standard proof, then you will need to include more details to explain what you are doing, and make sure it's totally clear to the marker that you are not just following the standard proof but leaving out details. You won't be marked down for giving extra details, so if unsure, include more – or for homework, just ask.

## 64. MA243 Homework 3: Solutions

Due via moodle, at noon, Thursday, week 7, 17th November 2022.

(1) Prove that there is an isometry $V \cap S^n \to S^1$ for any two dimensional sub vector space $V$. Hint: Find an orthogonal matrix taking $V$ to $\mathbb{R}^2$. You may assume that $V$ has an orthonormal basis, and that any orthonormal basis of $V$ may be extended to an orthonormal basis of $\mathbb{R}^{n+1}$.

Let $V$ have an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2$, and extend this to an orthonormal basis $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ for $\mathbb{R}^{n+1}$.

The condition for a point $\mathbf{x}$ to be in $S^n$ is that

$$\langle \mathbf{x}, \mathbf{x}, \rangle = \mathbf{x}^T \cdot \mathbf{x} = 1$$

[1 point for condition to be in $S^n$ in terms of inner product; this can also be given as a condition $||\mathbf{x}|| = 1$, any usable condition, showing knowledge of the definition, is OK]

Any point in $\mathbb{R}^{n+1}$ has the form $\mathbf{x} = \sum_{i=1}^{n+1} a_i \mathbf{v}_i$.

Any point in $V$ has the form $\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2$.

Let $\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2 \in V \cap S^n$, then since $\mathbf{x} \in S^n$,

$$\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^T \cdot \mathbf{x} = 1 = a^2 + b^2 = 1$$

To show that $V \cap S^n$ and $S^1$ are isometric, we need to find a distance preserving bijection between them. [1 point for knowing what an isometry is, or what isometric means]

Define a map [1 point for definition of map]

$$\phi : S^n \cap V \to S^1$$

$$\phi : a\mathbf{v}_1 + b\mathbf{v}_2 \to (a, b)$$

Since by definition $S^1$ is the set of points $\mathbf{y}$ in $\mathbb{R}^2$ with $||\mathbf{y}|| = 1$, the image is in $S^1$, and the map is surjective, since for any $(a, b) \in S^1$, the point we have $a^2 + b^2 = 1$, so the point $\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2$ is in $V \cap S^n$, and maps to $(a, b)$. [1 point for surjectivity]

The map is injective since [1 point for injectivity]

$$\phi(a\mathbf{v}_1 + b\mathbf{v}_2) = \phi(c\mathbf{v}_1 + d\mathbf{v}_2) \iff (a, b) = (c, d) \iff a\mathbf{v}_1 + b\mathbf{v}_2 = c\mathbf{v}_1 + d\mathbf{v}_2$$

where in the second equality we use the fact that $\mathbf{v}_1, \mathbf{v}_2$ is a basis of $V$, so a vector in $V$ is expressed uniquely in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Hence $\phi$ is a bijection.

Now we need to check that $\phi$ preserves distance. [1 point for showing $\phi$ preserves distance; should include showing knowledge of the definition of distance on $S^n$] The distance between two points on $S^n$ is given by

$$d(\mathbf{x}, \mathbf{y}) = \cos^{-1} \langle \mathbf{x}, \mathbf{y} \rangle$$

Let $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$ and $\mathbf{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2$. All other coefficients $x_i$ and $y_i$ of $\mathbf{v}_i$ for $i \geq 2$ are zero. Then since $\mathbf{v}_i$ are orthonormal,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n+1} x_i y_i = x_1 y_1 + x_2 y_2 = \langle (x_1, x_2), (y_1, y_2) \rangle$$

So, we have

$$d_{S^n}(\mathbf{x}, \mathbf{y}) = \cos^{-1} \langle \mathbf{x}, \mathbf{y} \rangle = \cos^{-1}(\langle (x_1, x_2), (y_1, y_2) \rangle) = d_{S^1}(\phi(\mathbf{x}), \phi(\mathbf{y})$$

Hence the metric is also preserved, so $\phi$ is an isometry.

(3) ★ [6 points] The following matrix is orthogonal, and represents an isometry of $S^2$.

$$A = \frac{1}{3} \begin{pmatrix} 1 & -2 & ? \\ 2 & ? & -2 \\ 2 & 2 & 1 \end{pmatrix}$$

(a) Determine the missing entries.

(b) Find the set of fixed points of $T_A$, the map $T_A(\mathbf{x}) = A\mathbf{x}$.

(c) determine whether $T_A$ is a reflection, rotation, or rotation followed by reflection in a plane perpendicular to the axis of rotation.

(a) [2 points; can be divided as one for knowing definition of orthogonal, and one for getting correct values] Suppose

$$A = \frac{1}{3}\begin{pmatrix} 1 & -2 & x \\ 2 & y & -2 \\ 2 & 2 & 1 \end{pmatrix}$$

The condition on begin orthogonal is $A^TA = I$, so we compute:

$$A^TA = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ -2 & y & 2 \\ x & -2 & 1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & -2 & x \\ 2 & y & -2 \\ 2 & 2 & 1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & \dots & \dots \\ 2y+2 & \dots & \dots \\ x-2 & \dots & \dots \end{pmatrix} = I$$

Since we are told that the matrix is orthogonal, we don't need to compute more, we can already see that

$$y = -1, x = 2$$

the matrix is

$$A = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$$

(b) [3 points: one for knowing definition of fixed points, one for getting correct eigen vector, and one for scaling correctly to get two points on the sphere]
The fixed points of $T_A$ are points $\mathbf{x}$ where $A\mathbf{x} = \mathbf{x}$, ie., $(A - I)\mathbf{x} = 0$, so solutions of

$$\begin{pmatrix} -2/3 & -2/3 & 2/3 \\ 2/3 & -4/3 & -2/3 \\ 2/3 & 2/3 & -2/3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Subtracting row 2 from row 3 gives:

$$\begin{pmatrix} -2/3 & -2/3 & 2/3 \\ 2/3 & -4/3 & -2/3 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So we have $y = 0$, and then from the original row 3, $(2x + 2y - 2z)/3 = 0$, we get $x = z$. We expect $(1, 0, 1)$ is the eigen value, and checking the first row shows this is the case. $(1, 0, 1)$ is not on the sphere, so we need to scale by its length, giving two fixed points:

$$\pm(1, 0, 1)/\sqrt{2}$$

(c) [1 point, statement of correct transformation, don't need to give the amount of rotation etc] The transformation is a rotation, about the axis in direction $(1, 0, 1)$. This is because it only has twp fixed points, whereas reflections have a fixed great circle, and rotation followed by reflection in plane perpendicular to rotation axis has no fixed points.

(2) Prove that from the definitions that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transformation, then $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ is a Lorentz orthonormal basis. (You may not assume that $T$ is a linear map.)

Since $IJI = J$, we have

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle_L = J_{i,j}$$

which is the condition for a set to be Lorentz orthonormal, so $\mathbf{e}_i$ are Lorentz orthnormal.

[1 point to mention $\mathbf{e}_i$ are Lorentz orthnormal]

By definition, a Lorentz transformation preserves the Lorentz inner product, and so

[1 point to know/use definition of a Lorentz transformation]

$$\langle T(\mathbf{e}_i), T(\mathbf{e}_j) \rangle_L = \langle \mathbf{e}_i, \mathbf{e}_j \rangle_L = J_{i,j}$$

and so $T(\mathbf{e}_i)$ also form a Lorentz orthonormal set.

[1 point to show $T(\mathbf{e}_i)$ are Lorentz orthnormal]

[1 point to explain that we only have to check linear independence] Since there are $n$ vectors, we know from linear algebra, that if they are linearly independent, then they also span $\mathbb{R}^n$, and so must form a basis.

Suppose there is a linear combination

[1 point for writing out linear combination in terms of the $T(\mathbf{e}_i)$]

$$\sum_{i=1}^{n} a_i T(\mathbf{e}_i) = \mathbf{0}$$

Taking the Lorentz inner product with $T(\mathbf{e}_1)$ gives

[1 point for computation of Lorentz inner product with $T(\mathbf{e}_1)$, in particular getting the sign right]

$$\left\langle \sum_{i=1}^{n} a_i T(\mathbf{e}_i), T(\mathbf{e}_1) \right\rangle_L = \langle \mathbf{0}, T(\mathbf{e}_1) \rangle_L = 0$$

So by bilinearity of the Lorentz inner product,

$$\sum_{i=1}^{n} a_i \langle T(\mathbf{e}_i), T(\mathbf{e}_1) \rangle_L = a_1 \langle T(\mathbf{e}_1), T(\mathbf{e}_1) \rangle_L + 0 = -a_1 = 0$$

[1 point for computation of Lorentz inner product with $T(\mathbf{e}_j)$, for $j > 1$ in particular getting the sign right]

$$\left\langle \sum_{i=1}^{n} a_i T(\mathbf{e}_i), T(\mathbf{e}_j) \right\rangle_L = \langle \mathbf{0}, T(\mathbf{e}_j) \rangle_L = 0$$

So by bilinearity of the Lorentz inner product,

$$\sum_{i=1}^{n} a_i \langle T(\mathbf{e}_i), T(\mathbf{e}_j) \rangle_L = a_j \langle T(\mathbf{e}_j), T(\mathbf{e}_j) \rangle_L + 0 = a_j = 0$$

Hence all $a_i = 0$ for $i = 1, \ldots, n$, and so the only possible linear combination of $T(\mathbf{e}_i)$ which is zero, is the zero combination, and so $T(\mathbf{e}_i)$ are linearly independent, and hence a basis because there are $n$ of them.

[Other proofs may be acceptable, but they must be from the definitions, not quoting theorems that use this result!]

(3) Prove that if $R$ is an isometry on $\mathcal{H}^n$, and we define

$$\widehat{R}(\mathbf{x}) = -i||\mathbf{x}||_L R\left(\frac{\mathbf{x}}{-i||\mathbf{x}||_L}\right)$$

for positive time-like vectors $\mathbf{x}$, then $\widehat{R}$ preserves the Lorentz inner product on positive time-like vectors, i.e., if $\mathbf{x}$ and $\mathbf{y}$ are positive time-like, then

$$\left\langle \widehat{R}(\mathbf{x}), \widehat{R}(\mathbf{y}) \right\rangle_L = \langle \mathbf{x}, \mathbf{y} \rangle_L$$

Let $\mathbf{x}$ and $\mathbf{y}$ be positive time like, i.e.,
[1 point for knowing definition of positive time-like, or some kind of knowledge of what the set up is]
if $\mathbf{x} = (x_1, \ldots, x_{n+1})$ and $\mathbf{y} = (y_1, \ldots, y_{n+1})$, then $x_1, y_1 > 0$, and $\langle \mathbf{x}, \mathbf{x} \rangle_L < 0, \langle \mathbf{y}, \mathbf{y} \rangle_L < 0$
Note that $||\mathbf{x}||_L = i\alpha$ for some positive real number $\alpha$, by definition of $||\mathbf{x}||_L = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_L}$ as the positive imaginary root when the Lorentz inner product is negative. Therefore, $\frac{\mathbf{x}}{-i||\mathbf{x}||_L}$ and $\frac{\mathbf{y}}{-i||\mathbf{y}||_L}$ is also positive, and by bilinearity of the inner product, $\frac{\mathbf{x}}{-i||\mathbf{x}||_L}$ and $\frac{\mathbf{y}}{-i||\mathbf{y}||_L}$ are in $\mathcal{H}^n$, and so the expression is well defined. [Not expected to justify the expession is well defined; this can be assumed]
Now compute:

$$\begin{aligned}
\left\langle \widehat{R}(\mathbf{x}), \widehat{R}(\mathbf{y}) \right\rangle_L &= \left\langle -i||\mathbf{x}||_L R\left(\frac{\mathbf{x}}{-i||\mathbf{x}||_L}\right), -i||\mathbf{y}||_L R\left(\frac{\mathbf{y}}{-i||\mathbf{y}||_L}\right) \right\rangle_L \\
&= i||\mathbf{x}||_L i||\mathbf{y}||_L \left\langle R\left(\frac{\mathbf{x}}{-i||\mathbf{x}||_L}\right), R\left(\frac{\mathbf{y}}{-i||\mathbf{y}||_L}\right) \right\rangle_L \\
&= i||\mathbf{x}||_L i||\mathbf{y}||_L \left\langle \frac{\mathbf{x}}{-i||\mathbf{x}||_L}, \frac{\mathbf{y}}{-i||\mathbf{y}||_L} \right\rangle_L \\
&= i||\mathbf{x}||_L i||\mathbf{y}||_L \frac{1}{i||\mathbf{x}||_L i||\mathbf{y}||_L} \langle \mathbf{x}, \mathbf{y} \rangle_L = \langle \mathbf{x}, \mathbf{y} \rangle_L
\end{aligned}$$

Here we have used for each of these equalities:

(a) substitution of definition of $\widehat{R}$
(b) bilinearity of Lorentz inner product
(c) Fact that $R$ is by definition an isometry, which by definition means that $\cosh^{-1}(\langle R(\mathbf{v}), R(\mathbf{w}) \rangle_L = \cosh^{-1}(\langle \mathbf{v}, \mathbf{w} \rangle_L$ for points on $\mathcal{H}^n$, and so by applying cosh, this implies that $R$ preserves the Lorentz inner product.
(d) another use of bilinearity

[1 point for each line of this computation, and 1 point for explanations; if it's done slightly differently, the same kinds of justification should be used and mentioned clearly for full points; so, 1 point for substitution of definition, 2 points for the uses of bilinearity, 1 point for the use of $R$ being an isometry, and 1 point for mentioning these facts clearly as well as applying them ]

## 65. MA243 Homework 4

Due via moodle, at noon, Thursday 1st December, week 9, 2022.
You are allowed to assume any result which we have proved in the course, as long as you make clear which results you are assuming, but you should not assume results we have not proved, e.g., you can't assume that all transformations of $\mathcal{H}^2$ can be written in the form as given in the table in the notes for Lecture 21. You can assume results about Euclidean isometries of $\mathbb{R}^2$.

(1) Show that for any isometry $T_A$ of $\mathcal{H}^2$, with $A \in O^+(1,2)$, which fixes a time-like vector, there is a Lorentz orthonormal basis with respect to which for some $\theta \in [0, 2\pi)$,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \pm\sin(\theta) \\ 0 & \sin(\theta) & \mp\cos(\theta) \end{pmatrix}$$

Describe the transformations the two cases correspond to.

(2) Find the number of points, lines, and planes in $\mathbb{P}^3(\mathbb{F}_3)$, where $\mathbb{F}_3$ is the finite field with 3 elements.

(3) (a) Write down all the projective linear maps $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ taking

$$(1:0:0) \mapsto (1:0:2)$$
$$(0:1:0) \mapsto (1:0:0)$$
$$(0:0:1) \mapsto (1:1:1)$$

(b) Show that $(1:0:2), (1:0:0), (1:1:1), (3:1:3)$ is a frame of reference for $\mathbb{P}^2$.

(c) Find the unique projective linear map taking the standard frame of reference $(1:0:0), (0:1:0), (0:0:1), (1:1:1)$ to $(1:0:2), (1:0:0), (1:1:1), (3:1:3)$ respectively.

Homework 4

## 66. MA243 Homework 4 Solutions

Due via moodle, at noon, Thursday 1st December, week 9, 2022.
You are allowed to assume any result which we have proved in the course, as long as you make clear which results you are assuming, but you should not assume results we have not proved, e.g., you can't assume that all transformations of $\mathcal{H}^2$ can be written in the form as given in the table in the notes for Lecture 21. You can assume results about Euclidean isometries of $\mathbb{R}^2$.

(1) Show that for any isometry $T_A$ of $\mathcal{H}^2$, with $A \in O^+(1,2)$, which fixes a time-like vector, there is a Lorentz orthonormal basis with respect to which for some $\theta \in [0, 2\pi)$,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \pm\sin(\theta) \\ 0 & \sin(\theta) & \mp\cos(\theta) \end{pmatrix}$$

Describe the transformations the two cases correspond to.

Step 1: show that the fixed point can be taken to be $(1,0,0)$ Suppose $\mathbf{x} \in \mathbb{R}^3$ is a time-like vector which is fixed by $T_A$. then by definition of time-like vectors, $\langle \mathbf{x}, \mathbf{x}, \rangle_L < 0$, so $\sqrt{-\langle \mathbf{x}, \mathbf{x}, \rangle_L}$ is a positive real number, and so $\mathbf{x}/\sqrt{-\langle \mathbf{x}, \mathbf{x}, \rangle_L}$ is also a vector in $\mathbb{R}^3$. Since the Lorentz inner product is bilinear,

$$\left\langle \frac{\mathbf{x}}{\sqrt{-\langle \mathbf{x}, \mathbf{x}, \rangle_L}}, \frac{\mathbf{x}}{\sqrt{-\langle \mathbf{x}, \mathbf{x}, \rangle_L}} \right\rangle_L = \frac{1}{-\langle \mathbf{x}, \mathbf{x} \rangle_L} \langle \mathbf{x}, \mathbf{x} \rangle_L = -1$$

A fixed point corresponds to an eigen vector with eigen value 1. Any non-zero multiple of an eigen vector is also an eigen vector, with the same eigen value. So we can replace $\mathbf{x}$ by $\mathbf{x}/\sqrt{-\langle \mathbf{x}, \mathbf{x}, \rangle_L}$, which is also a fixed point of $T_A$. If necessary, we can replace $\mathbf{x}$ by $-1$, which does not change $\langle \mathbf{x}, \mathbf{x} \rangle_L$, since by bilinearity, $\langle -\mathbf{x}, -\mathbf{x} \rangle_L = (-1)^2 \langle \mathbf{x}, \mathbf{x} \rangle_L$, so we can assume that the first coordinate of $\mathbf{x}$ is positive, and so $\mathbf{x} \in \mathcal{H}^2$, so we have a fixed point of the isometry $T_A$ which lies on $\mathcal{H}^2$.

Homework 4

Now we know (by Lemma 30 from lecure notes) that there is a change of basis matrix $B \in O^+(1,n)$ such that $B\mathbf{x} = \mathbf{e}_1 = (1,0,0)$. Since $O^+(1,2)$ is a group, and $A, B \in O^+(1,2)$, so is $BAB^{-1}$

Writing $A$ with respect to the new Lorentz orthonormal basis is equivalent to replacing $A$ by $BAB^{-1}$.

So, now we may assume that $A$ fixes $\mathbf{e}_1 = (1,0,0)$. (it's fine to use $A$ and $T_A$ interchangably.)

Step 2: the orthogonal complement of $(1,0,0)$ is fixed by $T_A$

Since $A \in O^+(1,2)$, by defnition of Lorentz transformations, $A$ preserves the Lorentz inner product. Now, since we have made a change of basis, we assume that the fixed vector is $\mathbf{x} = \mathbf{e}_1$. This means that

$$\langle \mathbf{e}_1, \mathbf{y} \rangle_L = 0 \iff \langle T_A(\mathbf{e}_1), T_A(\mathbf{y}) \rangle_L = 0$$

so since $\mathbf{e}_1$ is fixed by $A$,

$$\langle \mathbf{e}_1, \mathbf{y} \rangle_L = 0 \iff \langle \mathbf{e}_1, T_A(\mathbf{y}) \rangle_L = 0,$$

so the Lorentz orthogonal complement of $\mathbf{e}_1$ must be fixed.

Step 3: State/show the orthogonal complement of $(1,0,0)$ is spanned by $\mathbf{e}_2, \mathbf{e}_3$

The Lorentz orthogonal complement of $\mathbf{e}_1$ is the space spanned by $\mathbf{e}_2, \mathbf{e}_3$. This follows from the fact that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a Lorentz orthonormal basis, by Lemma 25. For $\mathbf{y} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$, we have

$$\langle \mathbf{y}, \mathbf{e}_1 \rangle = a = 0 \iff \mathbf{y} = b\mathbf{e}_2 + c\mathbf{e}_3$$

So the space $V$ spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$ is fixed by $A$.

Step 4: State/show that the Lorentz inner product on the orthogonal complement of $(1,0,0)$ coincides with the Euclidean inner product on this space

Now consider the map $A|_V : V \to V$.

$\mathbf{e}_2, \mathbf{e}_3$ is a basis for this space, and on this space, we have $\langle \mathbf{e}_i, \mathbf{e}_j \rangle_L = 0$ for $i \neq j$ and 1 for $i = j$, i.e.,

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle_L = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$$

where $\langle \mathbf{e}_i, \mathbf{e}_j \rangle$ is the usual Euclidean inner product.

Step 5: Explain how this means that $A$ is a Euclidean isometry when restricted to the space spanned by $\mathbf{e}_2, \mathbf{e}_3$

So $A|_V$ preserves the Euclidean inner product on $V$, and so by Proposition 4, $A|_V$ is an isometry of the Euclidean space spanned by $\mathbf{e}_2, \mathbf{e}_3$.

Step 6: Quote Theorem 3 to give the form of the matrix on $\mathbf{e}_2, \mathbf{e}_3$. (if students prove this from scratch, or make some other justification or refer to something else also OK)

By Theorem 3, this means that with respect to the basis $\mathbf{e}_2, \mathbf{e}_3$, $A|_V$ is repesented by a matrix of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Step 7: Put together the action on $\mathbf{e}_1$ and on $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$

With respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, we must have

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \pm\sin\theta \\ 0 & \sin\theta & \mp\cos\theta \end{pmatrix}$$

Step 8: Brief description

On $\mathbb{R}^3$ the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

corresponds to a rotation around the axis $(1,0,0)$. Restricted to $\mathcal{H}^2$, this is a rotation about the point $(1,0,0)$. The angle rotated through is $\theta$.

On $\mathbb{R}^3$ the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -\cos\theta \end{pmatrix}$$

corresponds to a reflection in the plane $x_3 = \tan(\theta/2)x_2$. Restricted to $\mathcal{H}^2$, this is a reflection in the line which projects $x_3 = \tan(\theta/2)x_2$, and can be described by the set of points of the form $(\cosh(\alpha), \sinh(\alpha)\cos(\theta/2), \sinh(\alpha)\sin(\theta/2))$.

(full marks just for saying which is reflection and which is rotation, don't need to specify axes and planes etc).

(2) Find the number of points, lines, and planes in $\mathbb{P}^3(\mathbb{F}_3)$, where $\mathbb{F}_3$ is the finite field with 3 elements.

[3 points for number of points] $\mathbb{F}_3$ has 3 elements, so $(\mathbb{F}_3)^4$ has $3^4 = 81$ elements, so taking away zero, $((\mathbb{F}_3)^4)^*$ has 80 elements.
The points of $\mathbb{P}^3(\mathbb{F}_3)$ are the lines in $(\mathbb{F}_3)^4$. Each line has the form $\mathbf{0}, \mathbf{v}, 2\mathbf{v}$ for some non-zero vector in $\mathbb{F}_3$, and lines only intersect at $\mathbf{0}$. So there are 2 non-zero points on each line, and so there are

$$40$$

points in $\mathbb{P}^3(\mathbb{F}_3)$.
[3 points for number of planes] By defintion, every line in $\mathbb{P}^3(\mathbb{F}_3)$ corresponds to a plane in $(\mathbb{F}_3)^4$.
Each plane is spanned by 2 linearly independent vectors, and each plane contains $3^2 = 9$ vectors, so 8 non-zero vectors Each line in a plane contains two non-zero vectors, so there are $8/2 = 4$ lines. Any choice of 2 of these lines spans the plane, and there are 4 choose 2 $= 6$ possible choices.
In $(\mathbb{F}_3)^4$ there are a total of 40 lines. There are $40 \times 39/2$ ways to choose any two, and since there are 6 ways to take a pair of lines generating each plane, this means there are

$$\frac{40 \times 39}{2 \times 6} = 130$$

planes in $(\mathbb{F}_3)^4$, and so 130 lines in $\mathbb{P}^3(\mathbb{F}_3)$.
[2 points for number of planes]
Finally, for the number of planes, every hyperplane in $(\mathbb{F}_3)^4$, is orthogonal to a vector in $(\mathbb{F}_3)^4$, so the number of plane is equal to the number of points, which is

$$40$$

Proof that a hyperplane is given by all vectors orthogonal to a given vector (not required for this question, since I didn't ask for a proof, but you should know a proof).

A hyperplane in $k^4$, for any field $k$, is a three dimensional subspace, which means it's spanned by three linearly independent vectors, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. You can use the usual orthonormalistion process to find an orthonormal basis with first three elements spanning this space, and last vector, $\mathbf{v}_0$ orthogonal to these. Then we must have that the plane spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is equal to the set of vectors orthogonal to $\mathbf{v}_0$, and we are done.

Homework 4

**Note:** The numbers of $k$ dimensional subspaces in a $n$-dimensional space over a field of $q$ elements are given by the the Gaussian binomial coefficients, $[n, k]_q$.

(3) (a) Write down all the projective linear maps $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ taking

$$(1:0:0) \mapsto (1:0:2)$$
$$(0:1:0) \mapsto (1:0:0)$$
$$(0:0:1) \mapsto (1:1:1)$$

(b) Show that $(1:0:2), (1:0:0), (1:1:1), (3:1:3)$ is a frame of reference for $\mathbb{P}^2$.

(c) Find the unique projective linear map taking the standard frame of reference $(1:0:0), (0:1:0), (0:0:1), (1:1:1)$ to $(1:0:2), (1:0:0), (1:1:1), (3:1:3)$ respectively.

(a) [3 points] The required linear maps all have the form

$$A_{a,b,c} = \begin{pmatrix} a & b & c \\ 0 & 0 & c \\ 2a & 0 & c \end{pmatrix}$$

where $a, b, c$ are any non-zero real numbers. (Also, as projective linear maps, $A_{a,b,c} = A_{\lambda a, \lambda b, \lambda c}$ for any non-zero $\lambda$, but it's not requred to say this.)

(b) [3 points]

To show that $[\mathbf{v}_0], [\mathbf{v}_1], [\mathbf{v}_2], [\mathbf{v}_3]$ is a projective frame of reference, we need to show that any 3 of $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$. For this, it is sufficient to show that the only non-trivial relationship of the form $a\mathbf{v}_0 + b\mathbf{v}_1 + c\mathbf{v}_2 + d\mathbf{v}_3 = \mathbf{0}$ must have all of $a, b, c, d$ non-zero.

Set $\mathbf{v}_0 = (1, 0, 2), \mathbf{v}_1 = (1, 0, 0), \mathbf{v}_2 = (1, 1, 1)$ and $\mathbf{v}_3 = (3, 1, 3)$.

Then $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$ span $\mathbb{R}^3$, since we can write the standard basis as $\mathbf{e}_1 = \mathbf{v}_2$, $\mathbf{e}_3 = (\mathbf{v}_1 - \mathbf{v}_2)/2$, and $\mathbf{e}_2 = \mathbf{v}_3 - \mathbf{v}_2 - (\mathbf{v}_1 - \mathbf{v}_2)/2$.

Since there are three of them, they are a basis for $\mathbb{R}^3$, and there is no non-trivial relation between $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$.

Also, since they are a basis, there is a unique way to write $\mathbf{v}_3$ in terms of them, which is:

$$\mathbf{v}_3 = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2$$

Since all the coefficients are non-zero, this means that $[\mathbf{v}_0], [\mathbf{v}_1], [\mathbf{v}_2], [\mathbf{v}_3]$ is a projective frame of reference.

(c) [2 points]

Since we have $\mathbf{v}_3 = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2$, the required matrix is:

$$A_{a,b,c} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

which is unique as an element of $PGL(3)$.

## 67. Weekly Moodle Quizzes

Practice these quizes for a quick summary of key ideas from each week.

Moodle Quizzes

Question 1
Not yet answered
Marked out of 1.00

$(1,2,3),(3,4,5),(5,6,7)$ is a basis for $\mathbf{R}^3$

Select one:
- True
- False

Question 2
Not yet answered
Marked out of 1.00

All vector spaces have a finite basis

Select one:
- True
- False

Question 3
Not yet answered
Marked out of 1.00

It is possible for a vector space to have more than one basis

Select one:
- True
- False

Question 4
Not yet answered
Marked out of 1.00

$\mathbf{C}$ is a two dimensional real vector space

Select one:
- True
- False

Question 5
Not yet answered
Marked out of 1.00

If a vector space has a basis with $n$ elements and another basis with $m$ elements, then $n = m$. Here $n, m \in \mathbf{N}$.

Select one:
- True
- False

Moodle Quizzes

Question **6**

Not yet answered

Marked out of 1.00

If $V$ is a finite dimensional vector space, and $S_1$ is a set of linearly independent vectors and $S_2$ is a set of vectors that span $V$, then $S_1$ cannot contain more vectors than $S_2$

Select one:

- True
- False

Question **7**

Not yet answered

Marked out of 1.00

The zero vector space does not have a basis

Select one:

- True
- False

Question **8**

Not yet answered

Marked out of 1.00

If $S$ is a set of vectors spanning a vector space $V$, then any element of $V$ can be written uniquely in terms of the elements of $S$

Select one:

- True
- False

Question **9**

Not yet answered

Marked out of 1.00

There is a linear map $T$ from $\mathbf{R}^2$ to $\mathbf{R}^2$ with $T(1,1) = (0,1)$ and $T(0,1) = (0,0)$. If there is such a map, what is its matrix?

Select one:

- True
- False

Question **10**

Not yet answered

Marked out of 1.00

There is a linear map $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that $T(1,1) = (7,6)$ and $T(0,0) = (3,2)$

Select one:

- True
- False

Question **11**

Not yet answered

Marked out of 1.00

The length of the orthogonal projection of the vector $(1,7)$ on to the vector $(8,6)$ is

- -10
- 5
- 10
- $\sqrt{50}$
- 50

Moodle Quizzes

Question **12**

Not yet answered

Marked out of 1.00

The area of the parallelogram determined by the vectors $(4, 3)$ and $(1, 7)$. is

- 32
- 10
- 25
- 100
- 50

Previous activity

◄ Week 1 quiz: linear algebra review

Jump to...

Next activity

Lecture 1 slides ▶

https://warwick.ac.uk/

Moodle Quizzes

Question 1

Not yet answered

Marked out of 1.00

Let $S = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ be the circle of radius $1$ in $\mathbf{R}^2$. Now define the distance $d(P, Q)$ between two points $P$ and $Q$ to be given by their usual distance given by the Euclidean metric in $\mathbf{R}^2$.

Then $(S, d)$ is a metric space

Select one:

- True
- False

Question 2

Not yet answered

Marked out of 1.00

Let $S = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ be the circle of radius $1$ in $\mathbf{R}^2$. Now define the distance $d_2(P, Q)$ between two points $P$ and $Q$ to be given by their distance measured along the circle. I.e., if the angle between the line from $0$ to $P$ and the line from $0$ to $Q$ is (\theta\le\pi\), then the distance is $\theta$.

Then $(S, d_2)$ is a metric space.

Select one:

- True
- False

Question 3

Not yet answered

Marked out of 1.00

Let $S = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ be the circle of radius $1$ in $\mathbf{R}^2$.

Define $d_1(P, Q)$ to be the usual Euclidean distance in $\mathbf{R}^2$ between two points.

Define $d_2(P, Q)$ to be given by the distance between $P$ and $Q$ measured along the circle.

Then $(S, d_1)$ and $(S, d_2)$ are isometric metric spaces.

Select one:

- True
- False

Moodle Quizzes

Question 4
Not yet answered
Marked out of 1.00

Every Euclidean space is also a vector space

Select one:
- True
- False

Question 5
Not yet answered
Marked out of 1.00

There is a unique metric $d(x, y)$ on $\mathbf{R}$ which makes $(\mathbf{R}, d)$ into a Euclidean space.

Select one:
- True
- False

Question 6
Not yet answered
Marked out of 1.00

There is an isometry of $\mathbf{R}$ with the usual Euclidean metric (absolute value), given by $f(x) = 2x$.

Select one:
- True
- False

Question 7
Not yet answered
Marked out of 1.00

Every finite dimensional real vector space can be given the structure of a metric space

Select one:
- True
- False

Question 8
Not yet answered
Marked out of 1.00

Any subset of $\mathbf{R}^m$ with the Euclidean metric is a metric space

Select one:
- True
- False

Question 9
Not yet answered
Marked out of 1.00

For any subset $Y \subset \mathbf{R}^m$, $(Y, d)$ is a Euclidean space, where $d$ is the Euclidean metric on $\mathbf{R}^m$ restricted to $Y$.

Select one:
- True
- False

Moodle Quizzes

Question **10**

Not yet answered

Marked out of 1.00

Let $L$ be the line in $\mathbb{R}^2$ where $x + y = 1$.

This line is isometric to $\mathbf{R}$ in a unique way.

Select one:

- True
- False

Question **11**

Not yet answered

Marked out of 1.00

Let $L$ be the line in $\mathbb{R}^2$ where $x + y = 1$. Let $L$ have the metric given by the restriction of the Euclidean metric on $\mathbb{R}^2$ to $L$.

Suppose $T$ is an isometry from $L$ to $\mathbb{R}$, (with the Euclidean metric from \(\mathbb R^2]) on $L$ and the Euclidean metric on $\mathbb{R}$.

Suppose that $T(0, 1) = 0$, then $T$ is uniquely determined

Select one:

- True
- False

Question **12**

Not yet answered

Marked out of 1.00

There is a unique isometry from $\mathbb{E}^1$ to $\mathbb{E}^1$.

Select one:

- True
- False

Question **13**

Not yet answered

Marked out of 1.00

$\mathbb{R}^1$ can be given the structure of a metric space in a unique way, namely, for $x, y \in \mathbb{R}^1$, the metric is given by $d(x, y) = |x - y|$

Select one:

- True
- False

Moodle Quizzes

Question **14**

Not yet answered

Marked out of 1.00

If $d(x, y)$ is a metric on $\mathbf{R}^1$, then there is some real number $a$ such that $d(x, y) = a|x - y|$ for all $x, y \in \mathbb{R}^1$.

Select one:

- True
- False

Question **15**

Not yet answered

Marked out of 1.00

If $T$ is a map from $\mathbb{R}^m$ to $\mathbb{R}^m$ which maps straight lines to straight lines, then $T$ must be an isometry

Select one:

- True
- False

Question **16**

Not yet answered

Marked out of 1.00

Given a Euclidean space $\mathbb{E}^1$, which is identified with $\mathbb{R}^2$ via an isometry $f_1$, and also via another isometry $f_2$, i.e. so that the maps $f_1$ and $f_1$ from $E^1$ are both distance preserving bijections.

If $A$ and $B$ are points in $\mathbb{E}^1$, then because $f_i$ is a bijection, we can define an addition as $A + B = f_i^{-1}(f_i(A) + f_i(B))$ for either $i = 1$ or $2$.

Claim: the result $A + B$ is independent of whether we use $f_1$ or $f_2$

Select one:

- True
- False

Previous activity

◀ Week 2 quiz: Euclidean space, isometries

Jump to...

Next activity

Linear algebra review (mostly covered in support classes) ▶

Moodle Quizzes

https://warwick.ac.uk/

Question **1**

Not yet answered

Marked out of 1.00

Take the line $L$ to be given by $L = \{(x, y) \subset \mathbb{R}^1 : y = 2\}$ with the distance induced by the Euclidean distance in $\mathbb{R}^2$.

$L$ is a Euclidean space; we can also call it a ``model'' of $\mathbb{E}^1$, since 1 dimensional Euclidean space means any line; they are all "the same", but if you want to work with one, you should pick a specific line. Usually you would probably just work with the line $\mathbb{R}^1$, but you might for some reason want to work with a different line.

Any Euclidean space can be given the structure of a vector space by choosing an isometry with $\mathbb{R}^1$. Let's choose the following isometry:

$F : (x, y) \mapsto x + 3$

Now we can use this isometry to add together points, by defining

$P + Q = F^{-1}(F(P) + F(Q)$

Let $P = (3, 2), Q = (4, 2)$, which are points on $L$.

With this definition, what is $P + Q$?

- (7,4)
- (4,2)
- $(10, 2)$
- (7,2)

Question **2**

Not yet answered

Marked out of 1.00

With the addition of points defined as in the previous question on the line $L = \{(x, 2) : x \in \mathbb{R}\}$, what is the value of $\frac{1}{2}(1, 2) + \frac{1}{2}(3, 2)$?

- (2,4)
- (4,2)
- (3,2)
- (4,4)
- (2,2)

Moodle Quizzes

Question **3**

Not yet answered

Marked out of 1.00

Which of the following are orthogonal matrices?

- $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$
- $\begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$
- $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$

Question **4**

Not yet answered

Marked out of 1.00

We know that an isometry in $R^2$ is a composition of at most $3$ reflections.

Suppose an isometry $T$ is not equal to a reflection, but can be written as $T = R \circ S$ for some reflections $R$ and $S$

Then the reflections $R$ and $S$ are unique

Select one:

- True
- False

Moodle Quizzes

Question **5**

Not yet answered

Marked out of 1.00

Select all of the following which are reflections (and not glide reflections).

- [ ] $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$

  (in case this doesn't display correctly, the matrix is [0,0,1; 0,-1,0; 1,0,0] )

- [ ] $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$
- [ ] $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$
- [ ] $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \frac{1}{9} \begin{pmatrix} 1 & 8 & -4 \\ 8 & 1 & 4 \\ -4 & 4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- [ ] $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Question **6**

Not yet answered

Marked out of 1.00

An isometry of $\mathbb{R}^3$ is a reflection if and only if it fixes a hyperplane

Select one:

- True
- False

Question **7**

Not yet answered

Marked out of 1.00

Every isometry of $\mathbb{R}^3$ fixes some hyperplane

Select one:

- True
- False

Moodle Quizzes

Question **8**

Not yet answered

Marked out of 1.00

There is an isometry with eigen vectors $1, 1, 1$ and $1, 2, 3$ with different eigen values

Select one:

- True
- False

Question **9**

Not yet answered

Marked out of 1.00

There is an isometry of $\mathbb{R}^3$ with eigen vectors $(1, 1, 1)$ and $(1, 2, -3)$

Select one:

- True
- False

Question **10**

Not yet answered

Marked out of 1.00

If $A$ is an orthogonal matrix, and $B$ is any matrix, then the matrix $BAB^{-1}$ is also orthogonal

Select one:

- True
- False

Question **11**

Not yet answered

Marked out of 1.00

Euclid's fifth postulate says that given a line and a point not on the line, there is a parallel line through the point.

Given $L : y = 3x + 7$ and $P = (1, 1)$, what is the equation of the line through $P$ parallel to $L$?

- $y = 3x - 2$
- $y = -3x + 4$
- $y = 3x + 7$
- $y = 3x + 1$

Previous activity

◀ Week 3 quiz: Euclidean isometries ("BUG": Q8 should say eigen vectors with different eigen values)

Jump to...

Next activity

Week 3 lecture notes (updated 23.10.22) ▶

1.00

points intersect?

- (0,1)
- (1,0)
- (1,1)
- (1,-1)
- (-1,1)

Question 1
Not yet answered
Marked out of 1.00

A straight line on the sphere is a subset of section of a great circle which is continuous and connected. Given any two points on the sphere, there is a unique straight line between them.

Select one:
- True
- False

Question 2
Not yet answered
Marked out of 1.00

There is a triangle in $S^2$ with all angles greater than $\pi$

Select one:
- True
- False

Question 3
Not yet answered
Marked out of 1.00

A circle on the $2$ sphere $S^2$ is the set of all points a constant distance from a given point, with respect to the spherical metric. This means that all circles on the $2$ sphere, $S^2$ are given by the intersection of the sphere with a plane.

Select one:
- True
- False

Question 4
Not yet answered
Marked out of 1.00

All spheres $S^n_r$ of the same dimension are isometric to each other

Select one:
- True
- False

Question 5
Not yet answered
Marked out of 1.00

There is an isometry of $S^2$ with no fixed points

Select one:
- True
- False

Moodle Quizzes

Question 6
Not yet answered
Marked out of 1.00

A zero dimensional sphere is a point

Select one:
- True
- False

Question 7
Not yet answered
Marked out of 1.00

An isometry which fixes a great circle on $S^2$ i.e., with $T(C) = C$ where $C$ is a great circle, is given by a reflection

Select one:
- True
- False

Question 8
Not yet answered
Marked out of 1.00

The spherical metric on the sphere $S^n_r$ is equal to the restriction of the Euclidean metric on $\mathbf{R}^{n+1}$ to the sphere

Select one:
- True
- False

Question 9
Not yet answered
Marked out of 1.00

A great circle on the sphere is the intersection of any hyperplane with the sphere

Select one:
- True
- False

Question 10
Not yet answered
Marked out of 1.00

There is a triangle in $S^2$ with all three angles greater than $pi/2$

Select one:
- True
- False

Moodle Quizzes

Previous activity
◀ week 4 quiz: spherical geometry

Jump to...

Next activity
Geogebra picture for main theorem of spherical trigonometry ▶



Moodle Quizzes

Question **1**
Not yet answered
Marked out of 1.00

If $T$ is an isometry of the sphere $S^2$, and $P$ and $Q$ are two distinct points which are not antipodal, then specifying $T(P)$ and $T(Q)$ uniquely determines $T$

Select one:
- True
- False

Question **2**
Not yet answered
Marked out of 1.00

The group of isometries of $S^2$ acts transitively on pairs of points distance $\pi/2$ apart - that is, if $P$ and $Q$ are distance $\pi/2$ apart, and $P'$ and $Q'$ are distance $\pi/2$ apart, then there is an isometry with $T(P) = P'$ and \(T(Q)=Q')

Select one:
- True
- False

Question **3**
Not yet answered
Marked out of 1.00

If $T$ is an isometry of the sphere $S^2$, which fixes two points, then there are only two possible isometries $T$ could be. The same is true for $S^3$

Select one:
- True
- False

Moodle Quizzes

Question **4**

Not yet answered

Marked out of 1.00

Which of the following describes the relationship between $\cosh$ and $\cos$?

(you can use the definitions of both given in terms of $exp$)

$\cosh(x) = \ldots$

- [ ] \(\cos(ix))
- [ ] \(-i\cos(-x))
- [ ] \(\cos(x))
- [ ] \(\cos(-x))
- [ ] \(i\cos(x))

Question **5**

Not yet answered

Marked out of 1.00

Which of the following describes the relationship between $\sinh(x)$ and $\sin(x)$?

$\sinh(x) = \ldots$

- [ ] $-i\sin(x)$
- [ ] $i\sin(ix)$
- [ ] $\sin(ix)$
- [ ] $-i\sin(ix)$

Question **6**

Not yet answered

Marked out of 1.00

Which of the following vectors is in the Lorentz orthogonal complement of $(1,1,1)$?

I.e., for which of these $\mathbf{v}$ is $\langle (1,1,1), \mathbf{v} \rangle_L = 0$?

- [ ] (2,-1,-1)
- [ ] (-2,-1,-1)
- [ ] (0,-1,1)
- [ ] (0,1,1)

Question **7**

Not yet answered

Marked out of 1.00

Which of the following vectors are positive time like vectors in $\mathbf{R}^3$?

- [ ] $(3,4,5)$
- [ ] $(5,4,4)$
- [ ] $(5,3,4)$
- [ ] $(4,3,2)$
- [ ] $(5,3,3)$
- [ ] $(-4,3,2)$

Moodle Quizzes

Question **8**

Not yet answered

Marked out of 1.00

Which of the following matrices are Lorentz orthogonal?

- $\begin{pmatrix} \cos(4) & \sin(4) \\ \sin(4) & \cos(4) \end{pmatrix}$
- $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} \cosh(4) & \sinh(4) \\ \sinh(4) & \cosh(4) \end{pmatrix}$
- $\begin{pmatrix} \sqrt{2} & 1 \\ -1 & -\sqrt{2} \end{pmatrix}$
- $\begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Question **9**

Not yet answered

Marked out of 1.00

Which of the following are in $PO(1,1)$?

- $\begin{pmatrix} \cosh(4) & -\sinh(4) \\ \sinh(4) & \cosh(4) \end{pmatrix}$
- $\begin{pmatrix} -\cosh(4) & \sinh(4) \\ -\sinh(4) & \cosh(4) \end{pmatrix}$
- $\begin{pmatrix} \cosh(4) & -\sinh(4) \\ \sinh(4) & -\cosh(4) \end{pmatrix}$
- $\begin{pmatrix} \cosh(4) & \sinh(4) \\ \sinh(4) & \cosh(4) \end{pmatrix}$

Question **10**

Not yet answered

Marked out of 1.00

Which of the following is a parametrisation of the hyperbolic line, given by

$\mathcal{H}^1 = \{(x,y) : y^2 = x^2 + 1\}$?

$(x,y) = \ldots$

- $(\sinh(t), \cosh(t))$
- $(cosh(t), -\sinh(t))$
- $(sin(t), \cos(t))$
- $(-cosh(t), \sinh(t))$
- $(\cosh(t), \sinh(t))$

Moodle Quizzes

Previous activity
◀ Week 5 quiz: more spherical geometry some hyperbolic geometry

Jump to...

Next activity
Hyperbolic line in demos ▶

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Moodle Quizzes

Question **1**
Not yet answered
Marked out of 1.00

Suppose $T$ is the isometry of $\mathbb{R}^2$ given by $\mathbf{x} \mapsto -\mathbf{x}$. Since all vectors in $\mathbb{R}^2$ are eigen vectors with eigen value $1$ for this transformation, the corresponding matrix is $-I$ with respect to any basis, and so any two reflections $\rho_1$ and $\rho_2$ have composition equal to $T$

- [ ] True
- [ ] false

Question **2**
Not yet answered
Marked out of 1.00

The composition of two reflections in $\mathbf{R}^2$ is always a reflection, and the composition of two rotations is always a rotation

Select one:
- True
- False

Question **3**
Not yet answered
Marked out of 1.00

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ always describes a reflection, no matter whether this is with respect to an orthonormal basis or not

Select one:
- True
- False

Question **4**
Not yet answered
Marked out of 1.00

This question is concerning linear isometries, ie., this question assume the isometries mentioned fix $(0, 0)$.

Whereas a reflection in the plane $\mathbb{R}^2$ has a matrix which depends on the orthonormal basis chosen, a rotation in the plane has the same matrix with respect to any choice of orthonormal basis

Select one:
- True
- False

Moodle Quizzes

Question 5
Not yet answered
Marked out of 1.00

If a 2 by 2 rotation matrix has negative determinant, it must represent a reflection

Select one:
- True
- False

Question 6
Not yet answered
Marked out of 1.00

Which of the following vectors are Lorentz orthogonal to (2,1)?

- [ ] (1,-2)
- [ ] (2,1)
- [ ] (1,2)
- [ ] (-2,-4)
- [ ] (-1,2)
- [ ] (-1,-2)

Question 7
Not yet answered
Marked out of 1.00

If $\mathbf{x}$ is a space like vector and $\mathbf{w}$ is a vector Lorentz orthogonal to $\mathbf{x}$ then $\mathbf{w}$ must be time like

Select one:
- True
- False

Question 8
Not yet answered
Marked out of 1.00

The map from hyperboloic 2-space $(\mathcal{H}^2, d_{\mathcal{H}^2}$ to Euclidean two space $(\mathbb{R}^2, d_{\mathbb{R}^2})$ given by $(x_1, x_2, x_3) \mapsto (x_2, x_3)$ is an isometry

Select one:
- True
- False

Question 9
Not yet answered
Marked out of 1.00

The parametrisation map from $\mathbb{R}^2$ to $\mathcal{H}^2$ which maps a point in polar coordinates by:

$(r, \theta) \mapsto (\cosh(r), \cos(\theta)\sinh(r), \sin(\theta)\sinh(r))$

is an isometry between Euclidean space and Hyperbolic space

Select one:
- True
- False

Moodle Quizzes

Question **10**

Not yet answered

Marked out of 1.00

The vectors $(5/3, 4/3, 0)$ and $(4/5, 1, 4/5)$ are two elements of a Lorentz orthonormal basis for $\mathbb{R}^3$.

Which of the following could be the third element of this basis?

- $(0, -4/5, 1)$
- $16/15, 4/3, 4/5)$
- $(-4/5, -1, 4/5$
- $(4/3, 5/3, 0)$
- $(16/15, 4/3, -3/5)$

Previous activity

◄ Week 6 quiz; hyperboloic geometry and a bit of Review of Euclidean geometry

Jump to...

Next activity

Geogebra: Hyperbolic parallel lines ▶

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Moodle Quizzes

Question 1
Not yet answered
Marked out of 1.00

If $P, Q, R$ are distinct points in $\mathcal{H}^2$, and $S$ is a point in the interior of the triangle with vertices at $P, Q, R$, the the sum of the angles $\angle PSQ$, $\angle QSR$, and $\angle RSP$ is less than $2\pi$

Select one:
- True
- False

Question 2
Not yet answered
Marked out of 1.00

If $L$ is a hyperbolic line on $\mathcal{H}^2$, which passes through $(1, 0, 0)$, then the projection of $L$ to the $x_2x_3$ plane is always a straight line in the Euclidean sense.

Select one:
- True
- False

Question 3
Not yet answered
Marked out of 1.00

If the matrix $A$ of a hyperbolic isometry has eigen values $1, (1 - i)/\sqrt{2}, (1 + i)/\sqrt{2}$, then $T_A$ is a rotation

Select one:
- True
- False

Moodle Quizzes

Question 4
Not yet answered
Marked out of 1.00

The intersection of $\mathcal{H}^2$ with the plane $x_1 = 7$ is a hyperbolic line on $\mathcal{H}^2$

Select one:
- True
- False

Question 5
Not yet answered
Marked out of 1.00

Suppose $A$ is the matrix of a hyperbolic isometry of $\mathcal{H}^2$, and $A$ has exactly one eigen value equal to $1$. Then $T_A$ must be a rotation, since it fixes one point in $\mathcal{H}^2$

Select one:
- True
- False

Question 6
Not yet answered
Marked out of 1.00

If $\mathbf{v}$ and $\mathbf{z}$ are any vectors in $\mathbb{R}^n$, then $\mathbf{z} - \langle \mathbf{v}, \mathbf{z} \rangle_L \mathbf{v}$ is Lorentz orthogonal to $\mathbf{v}$

Select one:
- True
- False

Question 7
Not yet answered
Marked out of 1.00

If $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $x_1 > 0$ and $\langle \mathbf{x}, \mathbf{y} \rangle_L = 0$ then $\mathbf{y}$ must be space like

Select one:
- True
- False

Question 8
Not yet answered
Marked out of 1.00

There exist four distinct lines in $\mathbf{H}^2$ all of which are all ultra parallel to each other (if this was an exam question, you would either prove this is impossible, or else find an example)

Select one:
- True
- False

Question 9
Not yet answered
Marked out of 1.00

In $\mathcal{H}^2$, the sum of the angles in a triangle is strictly less than $\pi$, and for any $0 < A < \pi$ there is a triangle in $\mathcal{H}^2$ with angle sum equal to $A$

Select one:
- True
- False

Question 10
Not yet answered
Marked out of 1.00

If $P, Q, R$ is a hyperbolic triangle on $\mathcal{H}^2$, then the hyperbolic angle at $P$ is equal to the dihedral angle between the plane spanned by $P$ and $Q$ and the plane spanned by $P$ and $R$

Select one:
- True
- False

Moodle Quizzes

Previous activity
◄ week 7 quiz (hyperbolic geometry)

Jump to...

Next activity
Geogebra lines example ▶

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Moodle Quizzes

Question **1**
Not yet answered
Marked out of 1.00

Let $\mathbb{F}_3$ be the field of 3 elements (same as numbers modulo $3$. How many points are there in the projective plane $\mathbb{P}^2(\mathbb{F}_3)$?

- [ ] 8
- [ ] 13
- [ ] 26
- [ ] 27
- [ ] 3

Question **2**
Not yet answered
Marked out of 1.00

Any intersection of a plane not through the origin with the hyperbolic plane $\mathcal{H}^2$ in a hyperbolic circle, provided it is non empty, i.e., a set of points some fixed hyperbolic distance from a point P in $\mathcal{H}^2$

Select one:
- True
- False

Question **3**
Not yet answered
Marked out of 1.00

Suppose $A$ is a matrix in $O^+(1,2)$, and $\det(A) = -1$, then the isometry $T_A$ can not be a rotation

Select one:
- True
- False

Question **4**
Not yet answered
Marked out of 1.00

Suppose $A$ is a matrix in $O^+(1,2)$, and $T_A$ has no fixed points in $\mathcal{H}^2$, then $T_A$ must be a translation

Select one:
- True
- False

Moodle Quizzes

Question **5**

Not yet answered

Marked out of 1.00

If $\mathbf{v} \neq \mathbf{0}$ is a vector in $\mathbb{R}^3$, then the set of vectors Lorentz orthonormal to $\mathbf{v}$ is a subvector space of dimension $2$

Select one:

- True
- False

Question **6**

Not yet answered

Marked out of 1.00

Which of the following is a correct statement?

- $(2,4,6) \in (1:2:3)$
- $(1,2,3) = (1:2:3)$
- $(1:2:4) \sim (2:4:8)$
- $(1:2:3) = (4:8:12)$

Question **7**

Not yet answered

Marked out of 1.00

which of the following are projectively equivalent to $(1,2,3)$

- $(1,2,3)$
- $(1:2:3)$
- $(-1,-2,-3)$
- $(2,4,6$

Question **8**

Not yet answered

Marked out of 1.00

There are always infinitely many points at infinity in any projective plane $\mathbb{P}^2(k)$ for any field $k$. Points at infinity are those not in the projection to any given plane in $k^3$, i.e., take a plane not through the origin, then if a point is not on a line which passes through this plane, then with respect to the projection to this plane, the point is "at infinity"

Select one:

- True
- False

Moodle Quizzes

Question **9**

Not yet answered

Marked out of 1.00

The map from $\mathbb{P}^2$ to $\mathbb{P}^2$ given by $(a:b) \mapsto (a+b:a-b)$ is represented by which of the following matrices?

- $\begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$
- $\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Question **10**

Not yet answered

Marked out of 1.00

A line in $\mathbb{R}^2$, where $\mathbb{R}^2$ is considered as a plane in $\mathbb{R}^3$ could project to a circle on a different plane in $\mathbb{R}^3$

Select one:

- True
- False

Previous activity
◄ week 8 quiz

Jump to...

Next activity
lecture 22 slides (mostly will be blackboard) ▶

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Moodle Quizzes

Question **1**

Not yet answered

Marked out of 1.00

Which of the following is correct?

In this question, the inclusion $\mathbb{R}^k \subset \mathbb{P}^n$ is to be read as including an element $(x_1, \dots, x_k)$ of $\mathbb{R}^k$ as the element $(x_1 : \cdots : x_k : 1 : 0 : \cdots : 0)$ in $\mathbb{P}^n$, that is, $k$ possibly non-zero elements, then $1$ then $n-k$ zeros.

The inclusion $\mathbb{P}^k \subset \mathbb{P}^n$ is to be read as including an element $(x_1 : \cdots : x_k)$ as $(x_1 : \cdots : x_k : 0 : \cdots : 0)$

$\infty$ is to be understood as the point $(1 : 0 : 0 \cdots : 0)$, i.e., all zeros except the first element

- $\mathbb{P}^3 = \mathbb{R}^3 \cup \mathbb{P}^2$
- $\mathbb{P}^3 = \mathbb{R}^3 \cup \mathbb{R}^2$
- $\mathbb{P}^3 = \mathbb{R}^3 \cup \mathbb{P}^2 \cup \{\infty\}$
- $\mathbb{P}^3 = \mathbb{R}^3 \cup \mathbb{R}^2 \cup \mathbb{R}^1 \cup \{\infty\}$
- $\mathbb{P}^3 = \mathbb{R}^3 \cup \{\infty\}$

Question **2**

Not yet answered

Marked out of 1.00

The zero of $\mathbb{P}^4$ is the point $(0 : 0 : 0 : 0 : 0)$

Select one:

- True
- False

Moodle Quizzes

Question **3**

Not yet answered

Marked out of 1.00

Which of the following matrices is the inverse of $\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$ in PGL(2)?

- $\begin{pmatrix} 5 & -1 \\ -1 & 1 \end{pmatrix}$
- $\begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}$
- $\frac{1}{5}\begin{pmatrix} 5 & -1 \\ -1 & 1 \end{pmatrix}$
- $\begin{pmatrix} -1 & 1 \\ 1 & -5 \end{pmatrix}$

Question **4**

Not yet answered

Marked out of 1.00

A projective linear map is a linear map from $\mathbb{P}^n$ to $\mathbb{P}^n$ for some positive integer $n$

Select one:

- True
- False

Question **5**

Not yet answered

Marked out of 1.00

The empty projective space is given by $\mathbb{P}(\{\backslash 0\})$, and is considered to have dimension $0$

Select one:

- True
- False

Question **6**

Not yet answered

Marked out of 1.00

Any three distinct points in $\mathbb{P}^1$ form a projective frame of reference for $\mathbb{P}^1$

Select one:

- True
- False

Question **7**

Not yet answered

Marked out of 1.00

Any four distinct points in $\mathbb{P}^2$ form a projective frame of reference for $\mathbb{P}^2$

Select one:

- True
- False

Question **8**

Not yet answered

Marked out of 1.00

1) Given any three distinct points $P_0, P_1, P_2$ and any other three distinct points $Q_0, Q_1, Q_2$, all points in the line $\mathbb{P}^1$, there is a perspectivity, $T$ mapping $P_i$ to $Q_i$ for $i = 0, 1, 2$

2) Given any four distinct points $P_0, P_1, P_2, P_3$ and any other four distinct points $Q_0, Q_1, Q_2, Q_3$, all points in the plane $\mathbb{P}^2$, there is a perspectivity, $T$ mapping $P_i$ to $Q_i$ for $I = 0, 1, 2, 3$, provided that neither set of points all lie on a straight line

3) Given any five distinct points $P_0, P_1, P_2, P_3, P_4$ and any other four distinct points $Q_0, Q_1, Q_2, Q_3, Q_4$, all points in $\mathbb{P}^3$, there is a perspectivity, $T$ mapping $P_i$ to $Q_i$ for $I = 0, 1, 2, 3, 4$, provided that neither set of points all lie on a straight line

Select one:

- True
- False

Moodle Quizzes

Question **9**

Not yet answered

Marked out of 1.00

Which of the following maps sends the standard projective frame of reference of $\mathbb{P}^1$ to $(1:1), (2:3), (1:2)$

- $\begin{pmatrix} -1 & 2 \\ -1 & 3 \end{pmatrix}$
- $\begin{pmatrix} -2 & 2 \\ -2 & 3 \end{pmatrix}$
- $\begin{pmatrix} -1 & 2 \\ -1 & 3 \end{pmatrix}$
- $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

Question **10**

Not yet answered

Marked out of 1.00

Which of the following maps the standard projective frame of reference of $\mathbb{P}^1$ to $(1:2), (1:3), (0:1)$? (in the given order)

- $\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$
- $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$
- $\begin{pmatrix} 2 & -3 \end{pmatrix}$
- $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

Moodle Quizzes

Previous activity
◄ week 9 quiz: projective geometry

Jump to...

Next activity
Slides for week 9 lecture 2 for quick example ▶

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Moodle Quizzes

Question **1**
Not yet answered
Marked out of 1.00

If $P, Q, R, S$ and $P', Q', R', S'$ are points in $P^2$, then there is a projective map taking $P, Q, R, S$ and $P', Q', R', S'$ if and only if the cross ratios are equal: \{P,Q;R,S\} = \{P',Q';R,S'\})

Select one:
- True
- False

Question **2**
Not yet answered
Marked out of 1.00

Suppose $\{P, Q; R, S\} = 3$. What is the value of $\{Q, P; R, S\}$

- 1/3
- 7
- -2
- -3
- 3

Question **3**
Not yet answered
Marked out of 1.00

If the cross ratio $\{P, Q; R, S\} = 3$ then there is some reordering of the points which has cross ratio $-1/2$

Select one:
- True
- False

Question **4**
Not yet answered
Marked out of 1.00

There is a projective linear map taking $(1:0), (0:1), (1:1), (1:2)$ to $(1:2), (2:3), (3:5), (5:8)$, in the given order

Select one:
- True
- False

Question 5
Not yet answered
Marked out of 1.00

Which of the following are the minimal and maximal dimension of the intersection of two three dimensional subspaces of $\mathbb{P}^5$ ?

- [ ] 1,6
- [ ] 0,6
- [ ] 3,6
- [ ] 1,3
- [ ] 0,3

Question 6
Not yet answered
Marked out of 1.00

Two planes in $\backslash\mathrm{Pro}^6$ must always intersect

Select one:
- True
- False

Question 7
Not yet answered
Marked out of 1.00

Three distinct planes in $\mathbb{R}^4$ must always have a point which is contained in all three planes

Select one:
- True
- False

Question 8
Not yet answered
Marked out of 1.00

Three distinct planes in $\mathbb{P}^3$ must always have a point in common to all three planes

Select one:
- True
- False

Question 9
Not yet answered
Marked out of 1.00

There exists a finite axiomatic projective plane which contains two lines, $L, M$, with exactly 3 points lying on $L$ and exactly 4 points lying on $M$

Select one:
- True
- False

Question 10
Not yet answered
Marked out of 1.00

Let $(P, L, I)$ be a finite axiomatic projective plane, such that each line contains exactly 4 points.

Then $P$ must contain $13$ points

Select one:
- True
- False

Moodle Quizzes

Previous activity
◄ Week 10 quiz: Projective maps, cross ratio, dimension formula

Jump to...

Next activity
Desmos Desargues picture ▶



Moodle Quizzes