SAMPLE PATH LARGE DEVIATIONS FOR LAPLACIAN MODELS IN $(1+1)$-DIMENSIONS

STEFAN ADAMS, ALEXANDER KISTER, AND HENDRIK WEBER

Abstract. For Laplacian models in dimension $(1+1)$ we derive sample path large deviations for the profile height function, that is, we study scaling limits of Gaussian integrated random walks and Gaussian integrated random walk bridges perturbed by an attractive force towards the zero-level, called pinning. We study in particular the regime when the rate functions of the corresponding large deviation principles admit more than one minimiser, in our models either two, three, or five minimiser depending on the pinning strength and the boundary conditions. This study complements corresponding large deviation results for gradient systems with pinning for Gaussian random walk bridges in $(1+1)$-dimension ([FS04]) and in $(1+d)$-dimension ([BFO09]), and recently in higher dimensions in [BCF14]. In particular it turns out that the Laplacian cases, i.e., integrated random walks, show richer and more complex structures of the minimiser of the rate functions which are linked to different phases.

1. Introduction and large deviation results

1.1. **The models.** We are going to study models for a $(1+1)$-dimensional random field. The models depend on the so-called potential, a measurable function $V\colon \mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ such that $x\mapsto \exp(-V(x))$ is bounded and continuous and that

$$\int_{\mathbb{R}} \mathrm{e}^{-V(x)}\,\mathrm{d}x<\infty \;\text{ and }\; \int_{\mathbb{R}} x^2\mathrm{e}^{-V(x)}\,\mathrm{d}x=:\sigma^2<\infty \;\text{ and }\; \int_{\mathbb{R}} x\mathrm{e}^{-V(x)}\,\mathrm{d}x=0.$$

The model will be a probability measure given by a Hamiltonian depending on the Laplacian of the random field. The Hamiltonian $\mathcal{H}_{[\ell,r]}(\phi)$, defined for $\ell,r\in\mathbb{Z}$, with $r-\ell\ge 2$, and for $\phi\colon\{\ell,\ell+1,\dots,r-1,r\}\to\mathbb{R}$ by

$$\mathcal{H}_{[\ell,r]}(\phi):=\sum_{k=\ell+1}^{r-1}V(\Delta\phi_k), \tag{1.1}$$

where Δ denotes the discrete Laplacian, $\Delta\phi_k=\phi_{k+1}+\phi_{k-1}-2\phi_k$. Our pinning models are defined by the following probability measures $\gamma_{N,\varepsilon}^{\psi}$ and $\gamma_{N,\varepsilon}^{\psi_f}$ respectively:

$$\begin{aligned}\gamma_{N,\varepsilon}^{\psi}(\mathrm{d}\phi)&=\frac{1}{Z_{N,\varepsilon}(\psi)}\mathrm{e}^{-\mathcal{H}_{[-1,N+1]}(\phi)}\prod_{k=1}^{N-1}(\varepsilon\delta_0(\mathrm{d}\phi_k)+\mathrm{d}\phi_k)\prod_{k\in\{-1,0,N,N+1\}}\delta_{\psi_k}(\mathrm{d}\phi_k),\\ \gamma_{N,\varepsilon}^{\psi_f}(\mathrm{d}\phi)&=\frac{1}{Z_{N,\varepsilon}(\psi_f)}\mathrm{e}^{-\mathcal{H}_{[-1,N+1]}(\phi)}\prod_{k=1}^{N-1}(\varepsilon\delta_0(\mathrm{d}\phi_k)+\mathrm{d}\phi_k)\prod_{k\in\{-1,0\}}\delta_{\psi_k}(\mathrm{d}\phi_k),\end{aligned}\tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary: 60K35; Secondary: 60F10; 82B41.
Key words and phrases. Large deviation, Laplacian models, pinning, integrated random walk, scaling limits, bi-harmonic.

where $N \geq 2$ is an integer, $\varepsilon \geq 0$ is the pinning strength, $\mathrm{d}\phi_k$ (resp. $\mathrm{d}\phi_k^+$) is the Lebesgue measure on $\mathbb{R}$, δ_0 is the Dirac mass at zero and where $\psi \in \mathbb{R}^{\mathbb{Z}}$ is a given boundary condition and $Z_{N,\varepsilon}(\psi)$ (resp. $Z_{N,\varepsilon}(\psi_f)$) is the normalisation, usually called the partition function. The model with Dirichlet boundary condition on the left hand side and free boundary conditions on the right hand side (terminal times for the corresponding integrate random walk) has only the left hand side (initial times) fixed.

Our measures in (1.2) are $(1+1)$-dimensional models for a linear chain of length N which is attracted to the defect line, the x-axis, and the parameter $\varepsilon \geq 0$ tunes the strength of the attraction and one wishes to understand its effect on the field, in the large N limit. The models with $\varepsilon = 0$ have no pinning reward at all and are thus free Laplacian models. By "$(1+1)$-dimensional" we mean that the configurations of the chain are described by trajectories $\{(k, \phi_k)\}_{0 \leq k \leq N}$ of the field, so that we are dealing with directed models. Models with Laplacian interaction appear naturally in the physical literature in the context of semiflexible polymers, c.f. [BLL00, HV09], or in the context of deforming rods in space, cf. [Ant05].

The basic properties of the models were investigated in the two papers [CD08, CD09], to which we refer for a detailed discussion and for a survey of the literature. In particular, it was shown in [CD08] that there is a critical value $\varepsilon_c \in (0, \infty)$ that determines a phase transitions between a *delocalised regime* ($\varepsilon < \varepsilon_c$), in which the reward is essentially ineffective, and a *localised regime* ($\varepsilon > \varepsilon_c$), in which on the other hand the reward has a microscopic effect on the field. For more details see Section 1.2.3 below. All our models are linked to integrals of corresponding random walks ([Sin92, CD08, CD09]). The present paper derives large deviation principles for the macroscopic empirical profile distributed under the measures in (1.2) (Section 1.2). The corresponding large deviation results for the gradient models, that is, the Hamiltonian is a function of the discrete gradient of the field, have been derived in [FS04] for Gaussian random walk bridges in $\mathbb{R}$ and for Gaussian random walks and bridges in higher dimensions in [BFO09]. In [FO10] large deviations for general non-Gaussian random walks in $\mathbb{R}^d, d \geq 1$, have been analysed, and in [BCF14] gradient model in higher (lattice) dimensions have been introduced. Common feature of all these gradient models is the linear scale (in N) for the large deviations (the scaling limit scale is $\sqrt{N}$, for more details see [F]), whereas in the Laplacian case the scale for the scaling limits is $N^{3/2}$ as proved by Caravenna and Deuschel in [CD09]. This scale goes back to the work [Sin92] on integrated random walks. Henceforth the scale for our large deviation principles is N^2 and suggested by the scaling of the Hamiltonian (discrete bi-Laplacian), see Appendix A for further details. Despite the different scale of the large deviations also the proofs are different. As all our Laplacian models are versions of integrated random walk measures our proofs cannot rely on the Markov property as used in the corresponding gradient models. We use standard expansion techniques combined with a correction of so-called single zeroes to double zeroes. This allows us to write the distribution over disjoint intervals separated by a double zero as the product of independent distributions over the disjoint intervals.

Our second major result is a complete analysis of the rate functions, in particular in the critical situation that more than one minimiser for the rate function exist (Section 2). The macroscopic time, observed after scaling, of our integrated random walks runs over the interval $[0, 1]$. In the rate function arising in the large deviation result for mixed Dirchlet and free boundary conditions only the starting point and its gradient (speed) at $t = 0$ is specified, while the rate function for Dirchlet boundary conditions requires us to specify the terminal point (location and speed) at $t=1$ as well. The variational problem for our rate functions shows a much richer structure for the minimiser as for corresponding gradient models in [BFO09]. Without pinning there is a unique bi-harmonic function minimising the macroscopic bi-Laplacian energy, see Appendix A. Once the pinning reward is switched on the integrated random walk (scaled random field) has essentially two different strategies to pick up reward, see Section 2. One strategy is to start picking up the reward earlier despite the energy involved to bend to the zero line with speed zero whereas the other one is to cross the zero level and producing a longer bend before turning to the zero level and picking up reward.

In Section 1.2 we present all our large deviation results which are proved in Section 3, whereas our variational analysis results are given in Section 2 and their proofs are given in Section 2.3. In Appendix A we give some basic details about the bi-harmonic equation and bi-harmonic function along with convergence statements for the discrete bi-Laplacian. In Appendix B some well-known facts about partition functions of Gaussian integrated random walks are collected.

1.2. **Sample path large deviations.**

1.2.1. *Empirical profile.* Let $h_N = \{h_N(t)\colon t \in [0,1]\}$ be the macroscopic empirical profile determined from the microscopic height function ϕ under a proper scaling, that is, defined through some polygonal approximation of $(h_N(k/N) = \phi_k/N^2)_{k\in\Lambda_N}$, where $\Lambda_N = \{-1,0,\dots,N,N+1\}$, so that

$$h_N(t) = \frac{\lfloor Nt\rfloor - Nt + 1}{N^2}\phi_{\lfloor Nt\rfloor} + \frac{Nt - \lfloor Nt\rfloor}{N^2}\phi_{\lfloor Nt\rfloor+1}, \quad t\in[0,1]. \tag{1.3}$$

We consider h_N distributed under the given measures in (1.2) with the following particular choice of scaled boundary conditions $\psi^{(N)}$ in the case of Dirichlet boundary conidtions, defined for any $a,\alpha,b,\beta\in\mathbb{R}$ as

$$\psi^{(N)}(x) = \begin{cases} aN^2 - \alpha N & \text{if } x=-1,\\ aN^2 & \text{if } x=0,\\ bN^2 & \text{if } x=N,\\ bN^2-\beta N & \text{if } x=N+1,\\ 0 & \text{otherwise.}\end{cases} \tag{1.4}$$

For the Dirichlet boundary conditions on the left hand side and free boundary conditions on the right hand side we only specify the boundary in $x=-1$ and $x=0$, and write $\psi_f^{(N)}(-1) = aN^2 - \alpha N$ and $\psi_f^{(N)}(0) = aN^2$, see (1.2). Our different macroscopic models are given in terms of the empirical profile h_N sampled via microscopic height functions under the measures in (1.2) have the boundary value a and gradient α on the left hand side and for the Dirichlet case boundary value b and gradient β on the right hand side. We write $\boldsymbol{r} = (a,\alpha,b,\beta)$ to specify our choice of boundary conditions $\psi^{(N)}$ in the Dirichlet case and $\boldsymbol{a}=(a,\alpha)$ for the mixed Dirichlet and free boundary case.

We denote the Gibbs distributions with $\varepsilon=0$ (no pinning) by $\gamma_N^{\boldsymbol{r}}$ for Dirichlet boundary conditions and by $\gamma_N^{\boldsymbol{a}}$ for Dirichlet boundary conditions on the left hand side and free boundary conditions on the right hand side and their partition functions by $Z_N(\boldsymbol{r})$ and $Z_N(\boldsymbol{a})$, respectively. In Section 1.2.2 we study the large deviation principles without pinning ($\varepsilon=0$) for general integrated random walks with free boundary conditions on the right hand side and show that these results apply to Gaussian integrated walk bridges as well. For the Dirichlet case we consider the space

$$H^2_{\boldsymbol{r}} = \{h \in H^2([0,1])\colon h(0) = a, h(1) = b, \dot{h}(0) = \alpha, \dot{h}(1) = \beta\},$$

and for the left hand side free boundary case the space

$$H^2_{\boldsymbol{a}} = \{h \in H^2([0,1])\colon h(0) = a, \dot{h}(0) = \alpha\}.$$

Here, $H^2([0,1])$ is the usual Sobolev space. We write $\mathcal{C}([0,1];\mathbb{R})$ for the space of continuous functions on $[0,1]$ equipped with the supremum norm. In Section 1.2.3 we present our main large deviation result for the measures with pinning.

1.2.2. *Large deviations for integrated random walks and Gaussian integrated random walk bridges.* We recall the integrated random walk representation in Proposition 2.2 of [CD08]. Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables, with marginal laws $X_1 \sim \exp(-V(x))\mathrm{d}x$, and $(Y_n)_{n\in\mathbb{N}_0}$ the corresponding random walk with initial condition $Y_0 = \alpha N$ and $Y_n = \alpha N + X_1 + \cdots + X_n$. The integrated random walk is denoted by $(Z_n)_{n\in\mathbb{N}_0}$ with $Z_0 = aN^2$ and $Z_n = aN^2 + Y_1 + \cdots + Y_n$. We denote $\mathbb{P}^{\boldsymbol{a}}$ the probability distribution of the above defined processes. Then the following holds.

Proposition 1.1 ([CD08])**.** *The pinning free model $\gamma_N^{\boldsymbol{r}}$ ($\varepsilon = 0$) is the law of the vector $(Z_1, \dots, Z_{N-1})$ under the measure $\mathbb{P}^{\boldsymbol{r}}(\cdot) := \mathbb{P}^{\boldsymbol{a}}(\cdot | Z_N = bN^2, Z_{N+1} = bN^2 - \beta N)$. The partition function $Z_{N,\varepsilon}(\boldsymbol{r})$ is the value at $(\beta N, bN^2 - \beta N)$ of the density of the vector (Y_{N+1}, Z_{N+1}) under the law $\mathbb{P}^{\boldsymbol{r}}$. The model $\gamma_N^{\boldsymbol{a}}$ coincides with the integrated random walk $\mathbb{P}^{\boldsymbol{a}}$.*

The first part of the following result is the generalisation of Mogulskii's theorem [Mog76] from random walks to integrated random walks whereas its second part is the generalisation to Gaussian integrated random walk bridges.

Theorem 1.2. *(a) Let V be any potential of the form above such that $\Lambda(\lambda) := \log \mathbb{E}[\mathrm{e}^{\langle \lambda, X_1\rangle}] < \infty$ for all $\lambda \in \mathbb{R}$, then the following holds. The large deviation principle (LDP) holds for h_N under $\gamma_N^{\boldsymbol{a}}$ on the space $\mathcal{C}([0,1];\mathbb{R})$ as $N \to \infty$ with speed N and the unnormalised good rate function $\mathcal{E}_f$ of the form:*

$$\mathcal{E}_f(h) = \begin{cases} \int_0^1 \Lambda^*(\ddot{h}(t))\,\mathrm{d}t, & \text{if } h \in H^2_{\boldsymbol{a}}, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.5}$$

Here Λ^* denotes the Fenchel-Legendre transform of Λ.

(b) For $V(\eta) = \frac{1}{2}\eta^2$ the following holds. The large deviation principle (LDP) holds for h_N under $\gamma_N^{\boldsymbol{r}}$ on the spaces $\mathcal{C}([0,1];\mathbb{R})$ as $N \to \infty$ with speed N and the unormalised good rate function $\mathcal{E}$ of the form:

$$\mathcal{E}(h) = \begin{cases} \frac{1}{2}\int_0^1 \ddot{h}^2(t)\,\mathrm{d}t, & \text{if } h \in H^2_{\boldsymbol{r}}, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.6}$$

Remark 1.3. *(a) The rate functions in both cases are obtained from the unnormalised rate functions by $I_f^0(h) = \mathcal{E}_f(h) - \inf_{g\in H^2_{\boldsymbol{a}}} \mathcal{E}_f(g)$ for general integrated random walks with potential V respectively by $I^0(h) = \mathcal{E}(h) - \inf_{g\in H^2_{\boldsymbol{r}}} \mathcal{E}(g)$ for Gaussian integrated random walk bridges.*

(b) We believe that the large deviation in Theorem 1.2(b) holds for general potentials V as in (a) as well. For the Gaussian integrated random walk bridges there exist explicit formulae for the distribution, see [GSV05]*. Our main result concerns the large deviations for the pinning model for Gaussian integrated random walk bridges. General integrated random walk bridges will require different techniques.*

1.2.3. *Large deviations for pinning models.* The large deviation principle for the pinning models gets an additional term for the rate functions. Recall the logarithm of the partition function is the free energy, and the difference in the free energies with pinning and without pinning for zero boundary conditions ($\boldsymbol{r} = \mathbf{0}$) will be an important ingredient in our rate functions. We determine $\tau(\varepsilon)$ by its thermodynamic limit

$$\tau(\varepsilon) = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,\varepsilon}(\mathbf{0})}{Z_N(\mathbf{0})}. \tag{1.7}$$

The existence of the limit in (1.7) and its properties have been derived by Caravenna and Deuschel in [CD08], we summarise their result in the following proposition.

Proposition 1.4 ([CD08])**.** *The limit in* (1.7) *exist for every $\varepsilon \geq 0$. Furthermore, there exists $\varepsilon_{\mathrm{c}} \in (0,\infty)$ such that $\tau(\varepsilon) = 0$ for $\varepsilon \in [0,\varepsilon_{\mathrm{c}}]$, while $0 < \tau(\varepsilon) < \infty$ for $\varepsilon \in (\varepsilon_{\mathrm{c}},\infty)$, and as $\varepsilon \to \infty$,*

$$\tau(\varepsilon) = \log \varepsilon (1 - o(1)).$$

Moreover the function τ is real analytic on $(\varepsilon_{\mathrm{c}},\infty)$.

We have the following sample path large deviation principles for h_N under $\gamma^{\boldsymbol{r}}_{N,\varepsilon}$ and $\gamma^{\boldsymbol{a}}_{N,\varepsilon}$, respectively. The unnormalised rate functions are given by Σ^ε and Σ^ε_f, respectively, all of which are of the form

$$\Sigma^\varepsilon(h) = \frac{1}{2}\int_0^1 \ddot{h}^2(t)\,\mathrm{d}t - \tau(\varepsilon)|\{t \in [0,1] \colon h(t) = 0\}|, \tag{1.8}$$

for $h \in H = H^2_{\boldsymbol{r}}$ and $H = H^2_{\boldsymbol{a}}$, respectively, and where $|\cdot|$ stands for the Lebesgue measure.

Theorem 1.5. *The LDP holds for h_N under $\gamma_N = \gamma^{\boldsymbol{r}}_{N,\varepsilon}, \gamma^{\boldsymbol{a}}_{N,\varepsilon}$ respectively on the space $\mathcal{C}([0,1];\mathbb{R})$ as $N \to \infty$ with the speed N and the good rate functions $I = I^\varepsilon$ and $I = I^\varepsilon_f$ of the form:*

$$I(h) = \begin{cases} \Sigma(h) - \inf_{h\in H}\{\Sigma(h)\}, & \text{if } h \in H, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.9}$$

with $\Sigma = \Sigma^\varepsilon$ and $\Sigma = \Sigma^\varepsilon_f$ respectively, and $H = H^2_{\boldsymbol{r}}$ respectively $H = H^2_{\boldsymbol{a}}$. Namely, for every open set $\mathcal{O}$ and closed set $\mathcal{K}$ of $\mathcal{C}([0,1];\mathbb{R})$ equipped with the uniform topology, we have that

$$\begin{aligned} \liminf_{N\to\infty} \frac{1}{N} \log \gamma_N(h_N \in \mathcal{O}) &\geq -\inf_{h\in\mathcal{O}} I(h), \\ \limsup_{N\to\infty} \frac{1}{N} \log \gamma_N(h_N \in \mathcal{K}) &\leq -\inf_{h\in\mathcal{K}} I(h), \end{aligned} \tag{1.10}$$

in each of two situations.

As the limit $\tau(\varepsilon)$ of the difference of the free energies appears in our rate functions it is worth stressing that this has a direct translation in terms of some path properties of the field, see [CD08]. This is the microscopic counterpart of the effect of the reward term in our pinning rate functions. Defining the contact number ℓ_N by

$$\ell_N := \#\{k \in \{1,\dots,N\} \colon \phi_k = 0\},$$

we can easily obtain that for $\varepsilon > 0$ (see [CD08]),

$$D_N(\varepsilon) := \mathbb{E}_{\gamma^{\mathbf{0}}_{N,\varepsilon}}\big[\ell_N/N\big] = \varepsilon\tau'(\varepsilon).$$

This gives the following paths properties. When $\varepsilon > \varepsilon_{\rm c}$, then $D_N(\varepsilon) \to D(\varepsilon) > 0$ as $N \to \infty$, and the mean contact density is non-vanishing leading to localisation of the field (integrated random walk respectively integrated random walk bridge). For the other case, $\varepsilon < \varepsilon_{\rm c}$, we get $D_N(\varepsilon) \to 0$ as $N \to \infty$ and thus the contact density is vanishing in the thermodynamic limits leading to de-localisation.

2. Minimisers of the rate functions

We are concerned with the set $\mathcal{M}^\varepsilon$ of the minimiser of the unnormalised rate functions in (1.8) for our pinning LDPs. Any minimiser of (1.8) is a zero of the corresponding rate function in Theorem 1.5. We let $h^*_{\boldsymbol{r}} \in H^2_{\boldsymbol{r}}$ be the unique minimiser of the energy $\mathcal{E}$ defined in (A.1) (see Proposition A.1), that is, $\mathcal{E}(h) = 1/2\int_0^1 \ddot{h}^2(t)\,\mathrm{d}t$ is the energy of the bi-Laplacian in dimension one. For any interval $I \subset [0,1]$ we let $h^{*,I}_{\boldsymbol{r}} \in H^2_{\boldsymbol{r}}(I)$, where the boundary conditions apply to the boundaries of I, be the unique minimiser of $\mathcal{E}^I(h) = \frac{1}{2}\int_I \ddot{h}^2(t)\,\mathrm{d}t$, and we sometimes write $\boldsymbol{a}, \boldsymbol{b}$ for the boundary condition $\boldsymbol{r}$ with $\boldsymbol{a} = (a,\alpha)$ and $\boldsymbol{b} = (b,\beta)$. Of major interest are the zero sets

$$\mathcal{N}_h = \{t \in [0,1] \colon h(t) = 0\} \quad \text{of any minimiser } h.$$

In Section 2.1 we study the minimiser for the case of Dirichlet boundary conditions on the left hand side and free boundary conditions on the right hand side, whereas in Section 2.2 we summarise our findings for the Dirichlet boundary case on both the right hand and left hand side. In Section 2.3 we give the proofs for our statements.

2.1. **Free boundary conditions on the right hand side.** We consider Dirichlet boundary conditions on the left hand side and the free boundary condition on the right hand side (no terminal condition) only.

Let $\mathcal{M}^\varepsilon_f$ denote the set of minimiser of Σ^ε_f.

Proposition 2.1. *For any boundary condition $\boldsymbol{a} = (a,\alpha)$ on the left hand side the set $\mathcal{M}^\varepsilon_f$ of minimiser of Σ^ε_f is a subset of*

$$\{\overline{h}\} \cup \{h_\ell \colon \ell \in (0,1)\}, \tag{2.1}$$

where for any $\ell \in (0,1)$ the functions $h_\ell \in H^2_{\boldsymbol{a},f}$ are given by

$$h_\ell(t) = \begin{cases} h^{*,(0,\ell)}_{(\boldsymbol{a},\boldsymbol{0})}(t) & , \ \textit{for } t \in [0,\ell), \\ 0 & , \ \textit{for } t \in [\ell,1]; \end{cases} \tag{2.2}$$

and the function $\overline{h} \in H^2_{\boldsymbol{a},f}$ is the linear function $\overline{h}(t) = a + \alpha t, t \in [0,1]$.

Note that $\overline{h}$ does not pick up reward for any boundary condition $\boldsymbol{a} \neq \boldsymbol{0}$ whereas for $\boldsymbol{a} = \boldsymbol{0}$ it takes the maximal reward. The function h_ℓ picks up the reward in $[\ell, 1]$, see Figure 1,2,3. This motivates the following definitions. For any $\tau \in \mathbb{R}$ and $\boldsymbol{a} \in \mathbb{R}^2$ we let

$$\mathcal{E}^\tau_{(\boldsymbol{a},\boldsymbol{0})}(\ell) = \mathcal{E}(h^{*,(0,\ell)}_{\boldsymbol{a},\boldsymbol{0}}) + \tau\ell, \tag{2.3}$$

and observe that for $\tau = \tau(\varepsilon)$

$$\Sigma^\varepsilon_f(h_\ell) = \mathcal{E}^{\tau(\varepsilon)}_{(\boldsymbol{a},\boldsymbol{0})}(\ell) - \tau(\varepsilon). \tag{2.4}$$

Henceforth minimiser of Σ^ε_f are given by functions of type h_ℓ only if ℓ is a minimiser of the function $\mathcal{E}^\tau_{(\boldsymbol{a},\boldsymbol{0})}$ in $[0, 1]$. We collect an analysis of the latter function in the next Proposition.

Proposition 2.2 (**Minimiser for $\mathcal{E}^\tau_{(\boldsymbol{a},\boldsymbol{0})}$**). *(a) For $\tau = 0$ the function $\mathcal{E}^0_{(\boldsymbol{a},\boldsymbol{0})}$ is strictly decreasing with $\lim_{\ell\to\infty} \mathcal{E}^0_{(\boldsymbol{a},\boldsymbol{0})}(\ell) = 0$.*

(b) For $\tau > 0$ the function $\mathcal{E}^\tau_{(a,0,\boldsymbol{0})}, a \neq 0$, has one local minimum at $\ell = \ell_1(\tau, a, 0) = \sqrt{|a|}(18/\tau)^{1/4}$, and the function $\mathcal{E}^\tau_{(0,\alpha,\boldsymbol{0})}, \alpha \neq 0$, has one local minimum at $\ell = \ell_1(\tau, 0, \alpha) = \sqrt{\frac{2}{\tau}}|\alpha|$. In both cases there exist $\tau_1(\boldsymbol{a})$ such that $\ell_1(\tau, \boldsymbol{a}) \leq 1$ for all $\tau \geq \tau_1(\boldsymbol{a})$.

(c) For $\tau > 0$ and $\boldsymbol{a} = (a, \alpha) \in \mathbb{R}^2$ with $w = |a|/|\alpha| \in (0, \infty)$ and $s = \operatorname{sign}(a\alpha)$ the function $\mathcal{E}^\tau_{(\boldsymbol{a},\boldsymbol{0})}$ has one local minimum at $\ell = \ell_1(\tau, \boldsymbol{a}) = \frac{1}{\sqrt{2\tau}}\big(|\alpha| + \sqrt{\alpha^2 + 6|a|\sqrt{2\tau}}\big)$ when $s = 1$, whereas for $s = -1$ the function $\mathcal{E}^\tau_{(\boldsymbol{a},\boldsymbol{0})}$ has two local minima at $\ell = \ell_1(\tau, \boldsymbol{a}) = \frac{1}{\sqrt{2\tau}}\big(-|\alpha| + \sqrt{\alpha^2 + 6|a|\sqrt{2\tau}}\big)$ and $\ell = \ell_2(\tau, \boldsymbol{a}) = \frac{1}{\sqrt{2\tau}}\big(|\alpha| + \sqrt{\alpha^2 - 6|a|\sqrt{2\tau}}\big)$, where ℓ_2 is a local minimum only if $\tau \leq \frac{\alpha^4}{72a^2}$. In all cases there are $\tau_i(\boldsymbol{a})$ such that $\ell_i(\tau, \boldsymbol{a}) \leq 1$ for all $\tau \geq \tau_i(\boldsymbol{a})$, $i = 1, 2$.

We shall study the zero sets of all minimiser, that is we need to check if h_ℓ has zeroes in $[0, \ell)$ before picking up the reward in $[\ell, 1]$.

Lemma 2.3. *Let $a > 0$, then the functions $h^{*,(0,\ell)}_{(a,\alpha,\boldsymbol{0})}$ with $\alpha > 0$, $h^{*,(0,\ell)}_{(0,\alpha,\boldsymbol{0})}$ with $\alpha \neq 0$, and $h^{*,(0,\ell)}_{(a,-\alpha,\boldsymbol{0})}$ with $\alpha\ell/a \in [0, 3)$ have no zeroes in $(0, \ell)$, whereas the functions $h^{*,(0,\ell)}_{(a,-\alpha,\boldsymbol{0})}$ with $\alpha\ell/a > 3$ have exactly one zero in $(0, \ell)$. Analogous statements hold for $a < 0$.*

There is a qualitative difference between the minimiser h_{ℓ_1} and h_{ℓ_2} as the latter one has a zero before picking up the reward on $[\ell_2, 1]$, see Figure 2.

In the following we write $\varepsilon_i(\boldsymbol{a})$ for the value of the reward with $\tau(\varepsilon_i(\boldsymbol{a})) = \tau_i(\boldsymbol{a})$ such that $\ell_i(\tau_i(\boldsymbol{a}), \boldsymbol{a}) \leq 1, i = 1, 2$.

Theorem 2.4 (**Minimiser for Σ^ε_f**). *(a) If $\boldsymbol{a} = (a, 0), a \neq 0$ or $\boldsymbol{a} = (0, \alpha), \alpha \neq 0$ or $w = |a|/|\alpha| \in (0, \infty)$ with $s = \operatorname{sign}(a\alpha) = 1$, there exists $\varepsilon^*(\boldsymbol{a}) > \varepsilon_1(\boldsymbol{a})$ such that*

$$\mathcal{M}^\varepsilon_f = \begin{cases} \{\overline{h}\} & , \text{ for } \varepsilon < \varepsilon^*(\boldsymbol{a}), \\ \{\overline{h}, h_{\ell_1}\} \text{ with } \Sigma^{\varepsilon^*}_f(\overline{h}) = \Sigma^{\varepsilon^*}_f(h_{\ell_1}) & , \text{ for } \varepsilon = \varepsilon^*(\boldsymbol{a}), \\ \{h_{\ell_1}\} & , \text{ for } \varepsilon > \varepsilon^*(\boldsymbol{a}). \end{cases}$$

(b) Assume $w = |a|/|\alpha| \in (0, \infty)$ and $s = \operatorname{sign}(a\alpha) = -1$. There are $\tau_0(\boldsymbol{a}) > 0$ and $\tau^*_1(\boldsymbol{a}) > 0$ such that the following statements hold.

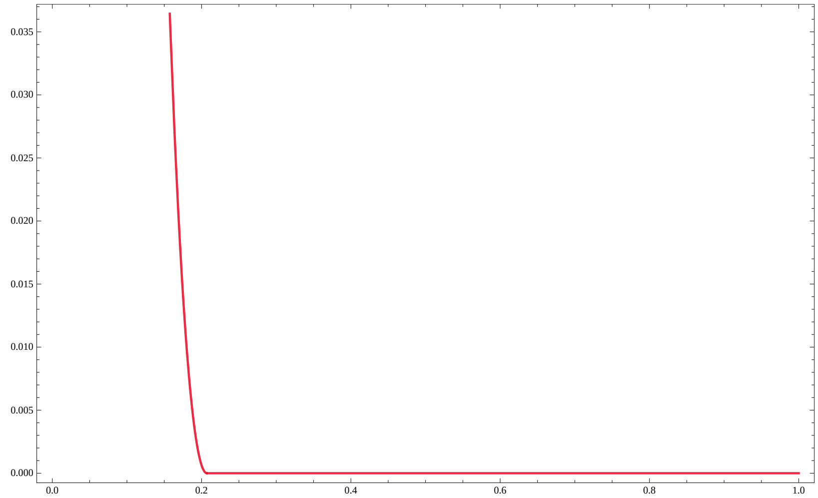

Figure 1: h_{ℓ_1} for $a = 1$ and $\alpha = -12, \tau = 288, \ell_1 = 1/2(\sqrt{2} - 1)$

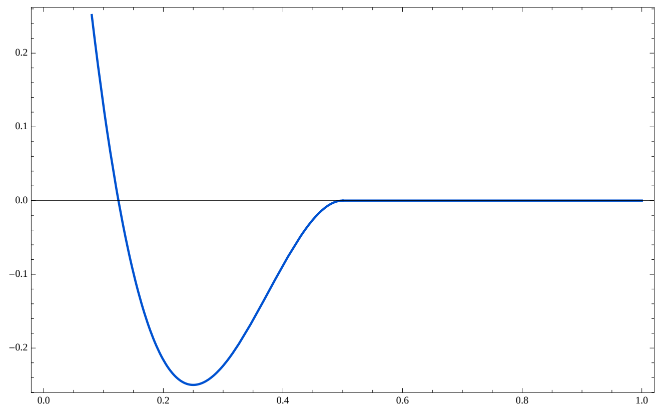

Figure 2: h_{ℓ_2} for $a = 1$ and $\alpha = -12, \tau = 288, \ell_2 = 1/2$

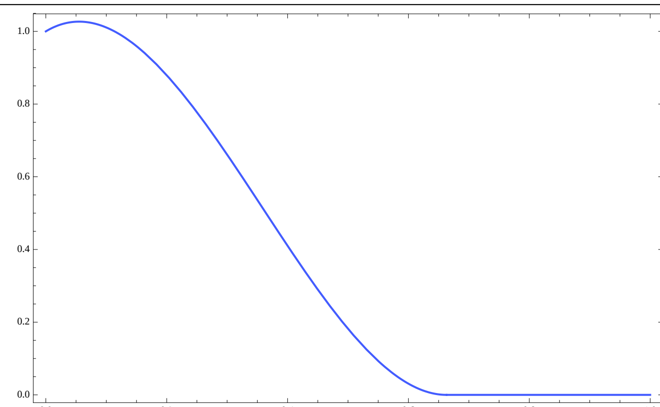

Figure 3: h_{ℓ_1} for $a = \alpha = 1$ and $\tau = 288, \ell_1 = 6/100(1 + \sqrt{101})$

(i) *Let* $\boldsymbol{a} \in D_1 := \{\boldsymbol{a} \in \mathbb{R}^2 \colon w \in (0, \infty)$ *and* $\tau_0(\boldsymbol{a}) > \tau_1^*(\boldsymbol{a})\}$. *Then there exist* $\varepsilon_{1,2}^*(\boldsymbol{a}) > 0$ *and* $\varepsilon_2^*(\boldsymbol{a}) > \varepsilon_2(\boldsymbol{a})$ *with* $\varepsilon_2^*(\boldsymbol{a}) < \varepsilon_{1,2}^*(\boldsymbol{a})$ *such that*

$$\mathcal{M}_f^\varepsilon = \begin{cases} \{\overline{h}\} & , \textit{ for } \varepsilon < \varepsilon_2^*(\boldsymbol{a}), \\ \{\overline{h}, h_{\ell_2}\} \textit{ with } \Sigma_f^{\varepsilon_2^*(\boldsymbol{a})}(\overline{h}) = \Sigma_f^{\varepsilon_2^*(\boldsymbol{a})}(h_{\ell_2}) & , \textit{ for } \varepsilon = \varepsilon_2^*(\boldsymbol{a}), \\ \{h_{\ell_2}\} & . \textit{ for } \varepsilon \in (\varepsilon_2^*(\boldsymbol{a}), \varepsilon_{1,2}^*(\boldsymbol{a})), \\ \{h_{\ell_1}, h_{\ell_2}\} \textit{ with } \Sigma_f^{\varepsilon_{1,2}^*(\boldsymbol{a})}(h_{\ell_1}) = \Sigma_f^{\varepsilon_{1,2}^*(\boldsymbol{a})}(h_{\ell_2}) & , \textit{ for } \varepsilon = \varepsilon_{1,2}^*(\boldsymbol{a}), \\ \{h_{\ell_1}\} & , \textit{ for } \varepsilon > \varepsilon_{1,2}^*(\boldsymbol{a}). \end{cases}$$

(ii) *Let* $\boldsymbol{a} \in D_2 := \{\boldsymbol{a} \in \mathbb{R}^2 \colon w \in (0,\infty)$ *and* $\tau_0(\boldsymbol{a}) = \tau_1^*(\boldsymbol{a}) = \tau_2^*(\boldsymbol{a})\}$. *Then for* $\varepsilon_{\rm c}^*(\boldsymbol{a}) > \tau_2(\boldsymbol{a})$ *with* $\tau(\varepsilon_{\rm c}^*(\boldsymbol{a})) = \tau_0(\boldsymbol{a})$,

$$\mathcal{M}_f^\varepsilon = \begin{cases} \{\overline{h}\} & , \text{ for } \varepsilon < \varepsilon_{\rm c}^*(\boldsymbol{a}), \\ \{\overline{h}, h_{\ell_1}, h_{\ell_2}\} \text{ with } \Sigma_f^{\varepsilon_{\rm c}^*(\boldsymbol{a})}(\overline{h}) = \Sigma_f^{\varepsilon_{\rm c}^*(\boldsymbol{a})}(h_{\ell_1}) = \Sigma_f^{\varepsilon_{\rm c}^*(\boldsymbol{a})}(h_{\ell_2}) & , \text{ for } \varepsilon = \varepsilon_{\rm c}^*(\boldsymbol{a}), \\ \{h_{\ell_1}\} & , \text{ for } \varepsilon > \varepsilon_{\rm c}^*(\boldsymbol{a}). \end{cases}$$

(iii) *Let* $\boldsymbol{a} \in D_3 := \{\boldsymbol{a} \in \mathbb{R}^2 \colon w \in (0,\infty)$ *and* $\tau_0(\boldsymbol{a}) < \tau_1^*(\boldsymbol{a})\}$. *Then for* $\varepsilon_1^*(\boldsymbol{a}) > 0$ *with* $\tau(\varepsilon_1^*(\boldsymbol{a})) = \tau_1^*(\boldsymbol{a})$,

$$\mathcal{M}_f^\varepsilon = \begin{cases} \{\overline{h}\} & , \text{ for } \varepsilon < \varepsilon_1^*(\boldsymbol{a}), \\ \{\overline{h}, h_{\ell_1}\} \text{ with } \Sigma_f^{\varepsilon_1^*(\boldsymbol{a})}(\overline{h}) = \Sigma_f^{\varepsilon_1^*(\boldsymbol{a})}(h_{\ell_1}) & , \text{ for } \varepsilon = \varepsilon_1^*(\boldsymbol{a}), \\ \{h_{\ell_1}\} & , \text{ for } \varepsilon > \varepsilon_1^*(\boldsymbol{a}). \end{cases}$$

Remark 2.5. *We have seen that the rate function* Σ_f^ε *can have up to three minimiser having the same value of the rate function. See Figure 1-2 for examples of these functions. The minimiser in Figure 1 has no single zero before picking up the reward. Note that the existence of the minimiser (see Figure 2) with a single zero before picking up a reward depends on the chosen boundary conditions. This minimiser only exist if the gradient at 0 has opposite sign of the value at zero. See Figure 3 for an example when the gradient has the same sign as the value of the function at zero. The minimiser* h_{ℓ_1} *is the global minimiser if the reward is increased unboundedly.*

2.2. **Dirichlet boundary.** We consider Dirichlet boundary conditions on both sides given by the vector $\boldsymbol{r} = (a,\alpha,b,\beta) = (\boldsymbol{a},\boldsymbol{b})$. In a similar way to Section 2.1 for free boundary conditions on the right hand side we define functions $h_{\ell,r} \in H^2_{\boldsymbol{r}}$ for any $\ell, r \geq 0$ with $\ell + r \leq 1$ by

$$h_{\ell,r}(t) = \begin{cases} h^{*,(0,\ell)}_{\boldsymbol{a},\boldsymbol{0}}(t) & ,t \in [0,\ell), \\ 0 & ,t \in [\ell, 1-r], \\ h^{*,(1-r,1)}_{\boldsymbol{0},\boldsymbol{b}}(t) & ,t \in (1-r,1]. \end{cases} \tag{2.5}$$

Furthermore, we define the following energy function depending only on ℓ and r,

$$E(\ell,r) = \mathcal{E}(h^{*,(0,\ell)}_{\boldsymbol{a},\boldsymbol{0}}) + \mathcal{E}(h^{*,(1-r,1)}_{\boldsymbol{0},\boldsymbol{b}}) - \tau(\varepsilon)(1-\ell-r), \tag{2.6}$$

and using (2.3) we get

$$E(\ell,r) = \mathcal{E}^{\tau(\varepsilon)}_{(\boldsymbol{a},\boldsymbol{0})}(\ell) + \mathcal{E}^{\tau(\varepsilon)}_{(\boldsymbol{0},b,-\beta)}(r) - \tau(\varepsilon) = \mathcal{E}^{\tau(\varepsilon)}_{(\boldsymbol{a},\boldsymbol{0})}(\ell) + \mathcal{E}^{\tau(\varepsilon)}_{(b,-\beta,\boldsymbol{0})}(r) - \tau(\varepsilon), \tag{2.7}$$

where β is replaced by $-\beta$ due to symmetry, that is, using that $h^{*,(1-r,1)}_{(\boldsymbol{0},\boldsymbol{b})}(t) = h^{*,(1-r,1)}_{(b,-\beta,\boldsymbol{0})}(2-r-t) = h^{*,(0,r)}_{(b,-\beta,\boldsymbol{0})}(1-t)$ for $t \in [1-r,1]$. Hence

$$\Sigma^\varepsilon(h_{\ell,r}) = \mathcal{E}(\ell,r).$$

For given boundary $\boldsymbol{r} = (a,\alpha,b,\beta)$ the function $h^*_{\boldsymbol{r}} \in H^2_{\boldsymbol{r}}$ given in Proposition A.1 does not pick up any reward in $[0,1]$.

Proposition 2.6. *For any Dirichlet boundary condition* $\boldsymbol{r} \in \mathbb{R}^4$ *the set* $\mathcal{M}^\varepsilon$ *of minimiser of the rate function* Σ^ε *in* $H^2_{\boldsymbol{r}}$ *is a subset of*

$$\{h_{\ell,r}, h^*_{\boldsymbol{r}} \colon \ell + r \leq 1\},$$

where ℓ *and* r *are minimiser of* $\mathcal{E}^{\tau(\varepsilon)}$ *in Proposition 2.2.*

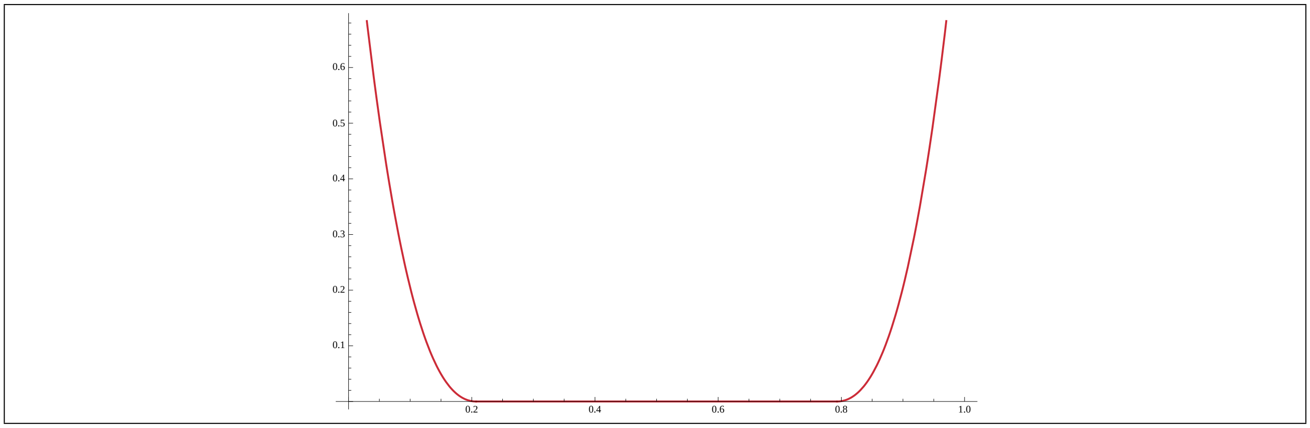

Figure 4: h_{ℓ_1} for $a=b=1$ and $\alpha=-12, \beta=12, \tau=288, \ell_1=1/2(\sqrt{2}-1)$

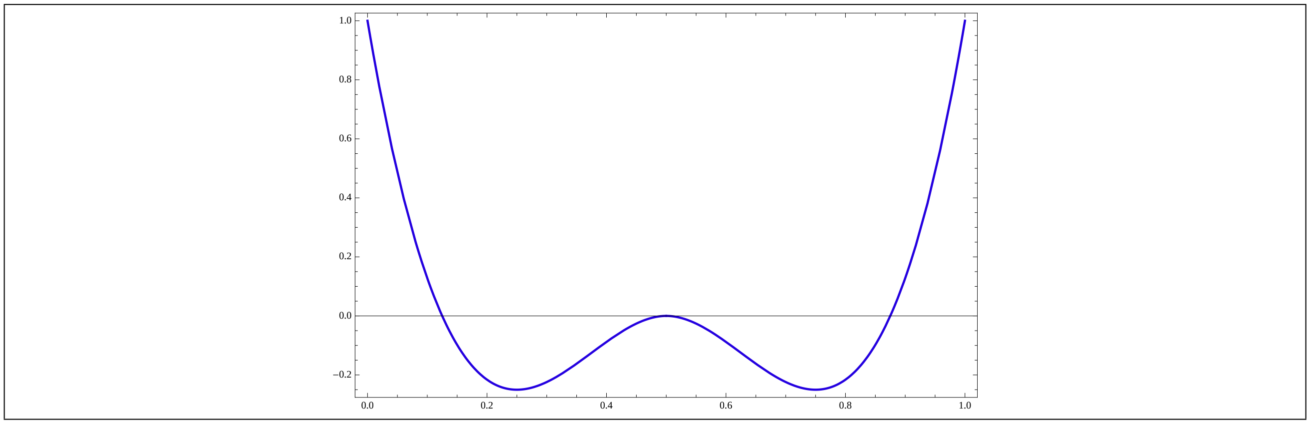

Figure 5: h_{ℓ_2} for $a=b=1$ and $\alpha=-12, \beta=12, \tau=288, \ell_2=1/2$

Proposition 2.6 allows to reduce the optimisation of the rate function Σ^ε to the minimisation of the function E defined in (2.7) for $0 \le \ell + r \le 1$. The general problem involves up to five parameters including the boundary conditions $\boldsymbol{r} \in \mathbb{R}^4$ and the pinning free energy $\tau(\varepsilon)$ for the reward ε. It involves studying several different sub cases and in order to demonstrate the key features of the whole minimisation problem we study only a special case in the following and only outline how any general case can be approached.

The *symmetric case* $\boldsymbol{r}=(a,\alpha,a,-\alpha)$: It is straightforward to see that

$$\Sigma^\varepsilon(h_{\ell_i,\ell_j}) = E(\ell_i,\ell_j) = \mathcal{E}^{\tau(\varepsilon)}_{(a,\alpha,\mathbf{0})}(\ell_i) + \mathcal{E}^{\tau(\varepsilon)}_{(a,\alpha,\mathbf{0})}(\ell_j) - \tau(\varepsilon), \quad i,j=1,2. \tag{2.8}$$

Clearly the unique minimiser $h^*_{\boldsymbol{r}}(t) = a+\alpha t-\alpha t^2$ of $\mathcal{E}$ has the symmetry $h^*_{\boldsymbol{r}}(1/2-t) = h^*_{\boldsymbol{r}}(1/2+t)$ for $t \in [0,1/2]$. The function E is not convex and thus we distinguish two different sets of parameter $(a,\tau(\varepsilon)) \in \mathbb{R}^3$ according to whether (i) $\ell_i(\tau(\varepsilon),\boldsymbol{a}) \le 1/2$ for $i=1,2$; or whether (ii) $\ell_2(\tau(\varepsilon),\boldsymbol{a}) > 1/2 > \ell_1(\tau(\varepsilon),\boldsymbol{a})$. There are no other cases for the parameter due to the condition $\ell_1+\ell_2 \le 1$ and the fact that $\ell_2(\tau(\varepsilon),\boldsymbol{a}) > \ell_1(\tau(\varepsilon),\boldsymbol{a})$.

Parameter regime (i):

$$\mathcal{D}_1 := \{(\boldsymbol{a},\tau) \in \mathbb{R}^3 \colon \ell_1(\tau,\boldsymbol{a}) \le 1/2 \,\wedge\, \ell_2(\tau,\boldsymbol{a}) \le 1/2 \text{ if } \ell_2(\tau,\boldsymbol{a}) \text{ is local minimum of } \mathcal{E}^\tau_{(\boldsymbol{a},\mathbf{0})}\}.$$

Parameter regime (ii):

$$\mathcal{D}_2 := \{(\boldsymbol{a},\tau) \in \mathbb{R}^3 \colon 1 \ge \ell_2(\tau(\varepsilon),\boldsymbol{a}) > 1/2 > \ell_1(\tau(\varepsilon),\boldsymbol{a}) > 0, \tau \le \frac{\alpha^4}{72a^2}\}.$$

We shall define the following values before stating our results.
There are $\varepsilon_i(\boldsymbol{a})$ such that $\ell_i(\tau(\varepsilon),\boldsymbol{a}) \le 1/2$ for all $\varepsilon \ge \varepsilon_i(\boldsymbol{a}), i = 1,2$. We denote by $\tau_1^*(\boldsymbol{a}) = \tau(\varepsilon_1^*(\boldsymbol{a}))$ the unique value of τ such that

$$\mathcal{E}^{\tau_1^*(\boldsymbol{a})}(\ell_1(\tau_1^*(\boldsymbol{a}),\boldsymbol{a})) - 1/2\tau_1^*(\boldsymbol{a}) = 1/2\mathcal{E}(h_{\boldsymbol{r}}^*). \tag{2.9}$$

Likewise, we denote $\tau_2^*(\boldsymbol{a})$ the unique value of τ such that $\mathcal{E}^{\tau_2^*(\boldsymbol{a})}(\ell_1(\tau_2^*(\boldsymbol{a}),\boldsymbol{a})) - 1/2\tau_2^*(\boldsymbol{a}) = 1/2\mathcal{E}(h_{\boldsymbol{r}}^*)$ when such a value exists in $\mathbb{R}$ otherwise we put $\tau_2^*(\boldsymbol{a}) = \infty$. We denote $\tau_0(\boldsymbol{a})$ the unique zero in Lemma 2.10 (a) of the difference $\Delta(\tau) = \mathcal{E}^\tau(\ell_1(\tau,\boldsymbol{a})) - \mathcal{E}^\tau(\ell_2(\tau,\boldsymbol{a}))$.

Theorem 2.7 (**Minimiser for Σ^ε, symmetric case**). *Let $\boldsymbol{r} = (a,\alpha,a,-\alpha)$.*

(a) If $\boldsymbol{a} = (a,0), a \neq 0$, or $\boldsymbol{a} = (0,\alpha), \alpha \neq 0$, or $w = |a|/|\alpha| \in (0,\infty)$ with $\operatorname{sign}(a\alpha) = 1$ and $\varepsilon \ge \varepsilon_1(\boldsymbol{a})$, then $(\boldsymbol{a},\tau(\varepsilon)) \in \mathcal{D}_1$ and there is $\varepsilon_1^*(\boldsymbol{a}) > \varepsilon_1(\boldsymbol{a})$ such that

$$\mathcal{M}^\varepsilon = \begin{cases} \{h_{\boldsymbol{r}}^*\} & , \text{ for } \varepsilon < \varepsilon_1^*(\boldsymbol{a}), \\ \{h_{\boldsymbol{r}}^*, h_{\ell_1,\ell_1}\} \text{ with } \Sigma^{\varepsilon_1^*(\boldsymbol{a})}(h_{\boldsymbol{r}}^*) = \Sigma^{\varepsilon_1^*(\boldsymbol{a})}(h_{\ell_1,\ell_1}) & , \text{ for } \varepsilon = \varepsilon_1^*(\boldsymbol{a}), \\ \{h_{\ell_1,\ell_1}\} & , \text{ for } \varepsilon > \varepsilon_1^*(\boldsymbol{a}). \end{cases}$$

(b) Assume $w = |a|/|\alpha| \in (0,\infty)$ and $s = \operatorname{sign}(a\alpha) = -1$. There are $\tau_0(\boldsymbol{a}) > 0$ and $\tau_1^*(\boldsymbol{a}) > 0$ such that the following statements hold.

(i) *Let $\boldsymbol{a} \in D_1 := \{\boldsymbol{a} \in \mathbb{R}^2 \colon w \in (0,\infty) \text{ and } \tau_0(\boldsymbol{a}) > \tau_1^*(\boldsymbol{a})\}$. Then there exists $\widetilde{\varepsilon}_{1,2}(\boldsymbol{a}) > 0$ such that $(\boldsymbol{a},\tau(\varepsilon)) \in \mathcal{D}_2$ for all $\varepsilon \in (\widetilde{\varepsilon}_{1,2}(\boldsymbol{a}), \varepsilon_2(\boldsymbol{a}))$ and $(a,\tau(\varepsilon)) \in \mathcal{D}_1$ for $\varepsilon \ge \varepsilon_2(\boldsymbol{a})$. Then there exist $\varepsilon_{1,2}^*(\boldsymbol{a}) > 0$ and $\varepsilon_2^*(\boldsymbol{a}) > 0$ with $\varepsilon_2^*(\boldsymbol{a}) < \varepsilon_{1,2}^*(\boldsymbol{a})$ such that*

$$\mathcal{M}^\varepsilon = \begin{cases} \{h_{\boldsymbol{r}}^*\} & , \text{ for } \varepsilon < \widetilde{\varepsilon}_{1,2}(\boldsymbol{a}), \\ \{h_{\boldsymbol{r}}^*, h_{\ell_2,\ell_1}\} \text{ with } \Sigma^\varepsilon(h_{\boldsymbol{r}}^*) \le \Sigma^\varepsilon(h_{\ell_2,\ell_1}) \vee \Sigma^\varepsilon(h_{\boldsymbol{r}}^*) > \Sigma^\varepsilon(h_{\ell_2,\ell_1}) & , \text{ for } \varepsilon \in (\widetilde{\varepsilon}_{1,2}(\boldsymbol{a}), \varepsilon_2(\boldsymbol{a})), \\ \{h_{\boldsymbol{r}}^*, h_{\ell_2,\ell_2}\} \text{ with } \Sigma^{\varepsilon_2^*(\boldsymbol{a})}(h_{\boldsymbol{r}}^*) = \Sigma^{\varepsilon_2^*(\boldsymbol{a})}(h_{\ell_2,\ell_2}) & , \text{ for } \varepsilon = \varepsilon_2^*(\boldsymbol{a}), \\ \{h_{\ell_2,\ell_2}\} & . \text{ for } \varepsilon \in (\varepsilon_2^*(\boldsymbol{a}), \varepsilon_{1,2}^*(\boldsymbol{a})), \\ \{h_{\ell_1,\ell_1}, h_{\ell_2,\ell_2}\} \text{ with } \Sigma^{\varepsilon_{1,2}^*(\boldsymbol{a})}(h_{\ell_1,\ell_1}) = \Sigma^{\varepsilon_{1,2}^*(\boldsymbol{a})}(h_{\ell_2,\ell_2}) & , \text{ for } \varepsilon = \varepsilon_{1,2}^*(\boldsymbol{a}), \\ \{h_{\ell_1,\ell_1}\} & , \text{ for } \varepsilon > \varepsilon_{1,2}^*(\boldsymbol{a}). \end{cases}$$

(ii) *Let $\boldsymbol{a} \in D_2 := \{\boldsymbol{a} \in \mathbb{R}^2 \colon w \in (0,\infty) \text{ and } \tau_0(\boldsymbol{a}) = \tau_1^*(\boldsymbol{a}) = \tau_2^*(\boldsymbol{a})\}$. Then there exists $\widetilde{\varepsilon}_{1,2}(\boldsymbol{a}) > 0$ such that $(\boldsymbol{a},\tau(\varepsilon)) \in \mathcal{D}_2$ for all $\varepsilon \in (\widetilde{\varepsilon}_{1,2}(\boldsymbol{a}), \varepsilon_2(\boldsymbol{a}))$ and $(a,\tau(\varepsilon)) \in \mathcal{D}_1$ for $\varepsilon \ge \varepsilon_2(\boldsymbol{a})$. Then there exists $\varepsilon_c^*(\boldsymbol{a}) > 0$ with $\tau(\varepsilon_c^*(\boldsymbol{a})) = \tau_0(\boldsymbol{a})$ and $\varepsilon_c^*(\boldsymbol{a}) \ge \varepsilon_2(\boldsymbol{a})$ such that*

$$\mathcal{M}^\varepsilon = \begin{cases} \{h_{\boldsymbol{r}}^*\} & , \text{ for } \varepsilon < \widetilde{\varepsilon}_{1,2}(\boldsymbol{a}), \\ \{h_{\boldsymbol{r}}^*, h_{\ell_2,\ell_1}\} \text{ with } \Sigma^\varepsilon(h_{\boldsymbol{r}}^*) \le \Sigma^\varepsilon(h_{\ell_2,\ell_1}) \vee \Sigma^\varepsilon(h_{\boldsymbol{r}}^*) > \Sigma^\varepsilon(h_{\ell_2,\ell_1}) & , \text{ for } \varepsilon \in (\widetilde{\varepsilon}_{1,2}(\boldsymbol{a}), \varepsilon_2(\boldsymbol{a})), \\ \{h_{\boldsymbol{r}}^*\} & , \text{ for } \varepsilon \in (\varepsilon_2(\boldsymbol{a}), \varepsilon_c^*(\boldsymbol{a})), \\ \{h_{\boldsymbol{r}}^*, h_{\ell_1,\ell_1}, h_{\ell_2,\ell_2}, h_{\ell_1,\ell_2}, h_{\ell_2,\ell_1}\} \text{ with } \Sigma^{\varepsilon_c^*(\boldsymbol{a})}(h_{\boldsymbol{r}}^*) = \Sigma^{\varepsilon_c^*(\boldsymbol{a})}(h_{\ell_1,\ell_1}) = & \\ \Sigma^{\varepsilon_c^*(\boldsymbol{a})}(h_{\ell_2,\ell_2}) = \Sigma^{\varepsilon_c^*(\boldsymbol{a})}(h_{\ell_1,\ell_2}) = \Sigma^{\varepsilon_c^*(\boldsymbol{a})}(h_{\ell_2,\ell_1}) & , \text{ for } \varepsilon = \varepsilon_c^*(\boldsymbol{a}), \\ \{h_{\ell_1,\ell_1}\} & , \text{ for } \varepsilon > \varepsilon_c^*(\boldsymbol{a}). \end{cases}$$

(iii) *Let $\boldsymbol{a} \in D_3 := \{\boldsymbol{a} \in \mathbb{R}^2 \colon w \in (0,\infty) \text{ and } \tau_0(\boldsymbol{a}) < \tau_1^*(\boldsymbol{a})\}$. Then there exists $\widetilde{\varepsilon}_{1,2}(\boldsymbol{a}) > 0$ such that $(\boldsymbol{a},\tau(\varepsilon)) \in \mathcal{D}_2$ for all $\varepsilon \in (\widetilde{\varepsilon}_{1,2}(\boldsymbol{a}), \varepsilon_1(\boldsymbol{a}))$ and $(a,\tau(\varepsilon)) \in \mathcal{D}_1$*

for $\varepsilon \geq \varepsilon_1(\boldsymbol{a})$. Then there exists $\varepsilon_1^*(\boldsymbol{a}) > \varepsilon_1(\boldsymbol{a})$ with $\tau(\varepsilon_1^*(\boldsymbol{a})) = \tau_1^*(\boldsymbol{a})$ such that

$$\mathcal{M}^\varepsilon = \begin{cases} \{h_{\boldsymbol{r}}^*\} & , \text{ for } \varepsilon < \widetilde{\varepsilon}_{1,2}(\boldsymbol{a}), \\ \{h_{\boldsymbol{r}}^*, h_{\ell_2,\ell_1}\} \text{ with } \Sigma^\varepsilon(h_{\boldsymbol{r}}^*) \leq \Sigma^\varepsilon(h_{\ell_2,\ell_1}) \vee \Sigma^\varepsilon(h_{\boldsymbol{r}}^*) > \Sigma^\varepsilon(h_{\ell_2,\ell_1}) & , \text{ for } \varepsilon \in (\widetilde{\varepsilon}_{1,2}(\boldsymbol{a}), \varepsilon_1^*(\boldsymbol{a})), \\ \{h_{\boldsymbol{r}}^*, h_{\ell_1,\ell_1}\} \text{ with } \Sigma^{\varepsilon_1^*(\boldsymbol{a})}(h_{\boldsymbol{r}}^*) = \Sigma^{\varepsilon_1^*(\boldsymbol{a})}(h_{\ell_1}) & , \text{ for } \varepsilon = \varepsilon_1^*(\boldsymbol{a}), \\ \{h_{\ell_1,\ell_1}\} & , \text{ for } \varepsilon > \varepsilon_1^*(\boldsymbol{a}). \end{cases}$$

Remark 2.8 (General boundary conditions). *For general boundary conditions $\boldsymbol{r} = (a, \alpha, b, \beta)$ one can apply the same techniques as for the symmetric case. Thus minimiser of Σ^ε are elements of*

$$\{h_{\boldsymbol{r}}^*, h_{\ell,\ell}, h_{r,r}, h_{\ell,r}, h_{r,\ell} \colon \ell + r \leq 1\}.$$

Remark 2.9 (Concentration of measures). *The large deviation principle in Theorem 1.5 immediately implies the concentration properties for $\gamma_N = \gamma_{N,\varepsilon}^{\boldsymbol{r}}$ and $\gamma_N = \gamma_{N,\varepsilon}^{\boldsymbol{a}}$:*

$$\lim_{N\to\infty} \gamma_N(\mathrm{dist}_\infty(h_N, \mathcal{M}^\varepsilon) \leq \delta) = 1, \tag{2.10}$$

for every $\delta > 0$, where $\mathcal{M}^\varepsilon = \{h^* \colon h^* \text{ minimiser of } I\}$ with $I = I^\varepsilon$ and $I = I_f^\varepsilon$, respectively, and dist_∞ *denotes the distance under $\|\cdot\|_\infty$. More precisely, for any $\delta > 0$ there exists $c(\delta) > 0$ such that*

$$\gamma_N(\mathrm{dist}_\infty(h_N, \mathcal{M}^\varepsilon) > \delta) \leq \mathrm{e}^{-c(\delta)N}$$

for large enough N. We say that two function $h_1, h_2 \in \mathcal{M}^\varepsilon$ coexist in the limit $N \to \infty$ under γ_N with probabilities $\lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = 1$ when

$$\lim_{N\to\infty} \gamma_N(\|h_N - h_i\|_\infty \leq \delta) = \lambda_i, \quad i = 1, 2,$$

hold for small enough $\delta > 0$. The same applies to the free boundary case on the right hand side and its set of minimiser $\mathcal{M}_f^\varepsilon$. For gradient models with quadratic interaction (Gaussian) the authors in [BFO09] *have investigated this concentration of measure problem and obtained statements depending on the dimension m of the underlying random walk (i.e. $(1+m)$-dimensional models). The authors are using finer estimates than one employs for the large deviation principle, in particular the make use of a renewal property of the partition functions. In our setting of Laplacian interaction the renewal structure of the partition functions is different and requires different type of estimates. In addition, the concentration of measure problem requires to study all cases of possible minimiser. We devote this study to future work.*

2.3. **Proofs: Variational analysis.**

2.3.1. *Free boundary condition.* PROOF OF PROPOSITION 2.1. Suppose that $h \in H_{\boldsymbol{a}}^2$ is not element of the set (2.1). It is easy to see that there is at least one function h^* in the set (2.1) with

$$\Sigma_f^\varepsilon(h^*) < \Sigma_f^\varepsilon(h). \tag{2.11}$$

For $\Sigma_f^\varepsilon(h) < \infty$, we distinguish two cases. If $|\mathcal{N}_h| = 0$, then $|\mathcal{N}_{\overline{h}}| = 0$ and we get

$$\Sigma_f^\varepsilon(h) = \mathcal{E}(h) > 0 = \mathcal{E}(\overline{h}) = \Sigma_f^\varepsilon(\overline{h})$$

by noting that $\overline{h}$ is the unique function with $\mathcal{E}(\overline{h}) = 0$. If $|\mathcal{N}_h| > 0$ we argue as follows. Let ℓ be the infimum and r be the supremum of the accumulation points of $\mathcal{N}_h$, and note that $\ell, r \in \mathcal{N}_h$. Since $|\mathcal{N}_h \cap [\ell, r]^{\mathrm{c}}| = 0$ we have

$$\Sigma_f^{\varepsilon}(h) = \mathcal{E}^{[0,\ell]}(h) + \mathcal{E}^{(\ell,r)}(h) - \tau(\varepsilon)|\{t \in (\ell, r) \colon h(t) = 0\}| + \mathcal{E}^{[r,1]}(h).$$

As $\ell, r \in \mathcal{N}_h$ we have that $\dot{h}(\ell) = \dot{h}(r) = 0$ as the differential quotient vanishes due to the fact that ℓ and r are accumulations points of $\mathcal{N}_h$. Thus the restrictions of h and $h^* = h_\ell$ to $[0, \ell]$ are elements of $H^2_{(\boldsymbol{a},\boldsymbol{0})}$. By the optimality of $h^{*,(0,\ell)}_{(\boldsymbol{a},\boldsymbol{0})}$ inequality (2.11) is satisfied for $h^* = h_\ell$. □

Proof of Proposition 2.2. The following scaling relations hold for $\ell > 0$ (in our cases $\ell \in (0,1)$) and $\boldsymbol{a} = (a, \alpha)$,

$$h^{*,(0,\ell)}_{(\boldsymbol{a},\boldsymbol{0})}(t) = h^{*,(0,1)}_{(a,\ell\alpha,\boldsymbol{0})}(t/\ell) \qquad \text{for } t \in [0, \ell]. \tag{2.12}$$

Using this and Proposition A.1 with $\boldsymbol{r} = (a, \ell\alpha, 0, 0)$ we obtain

$$\mathcal{E}^{(0,\ell)}(h^{*,(0,\ell)}_{(\boldsymbol{a},\boldsymbol{0})}) = \frac{1}{2\ell^3}\int_0^1 \big(\ddot{h}^*_{\boldsymbol{r}}(t)\big)^2\,\mathrm{d}t = \frac{1}{\ell^3}\big(6a^2 + 6a\alpha\ell + 2\alpha^2\ell^2\big),$$

and thus

$$\begin{aligned}
\mathcal{E}^{\tau}_{(\boldsymbol{a},\boldsymbol{0})}(\ell) &= \frac{1}{\ell^3}\big(6a^2 + 6a\alpha\ell + 2\alpha^2\ell^2\big) + \tau\ell,\\
\frac{\mathrm{d}}{\mathrm{d}\ell}\mathcal{E}^{\tau}_{(\boldsymbol{a},\boldsymbol{0})}(\ell) &= -\frac{18a^2}{\ell^4} - \frac{12a\alpha}{\ell^3} - \frac{2\alpha^2}{\ell^2} + \tau = -\frac{2}{\ell^4}(3a + \alpha\ell - \sqrt{\tau/2}\ell^2)(3a + \alpha\ell + \sqrt{\tau/2}),\\
\frac{\mathrm{d}^2}{\mathrm{d}\ell^2}\mathcal{E}^{\tau}_{(\boldsymbol{a},\boldsymbol{0})}(\ell) &= \frac{4}{\ell^5}(6a + \alpha\ell)(3a + \alpha\ell).
\end{aligned} \tag{2.13}$$

The derivative has the following zeroes

$$\begin{aligned}
\ell_{1/2} &= \frac{\alpha \pm \sqrt{\alpha^2 + 6a\sqrt{2\tau}}}{\sqrt{2\tau}},\\
\ell_{3/4} &= \frac{-\alpha \pm \sqrt{\alpha^2 - 6a\sqrt{2\tau}}}{\sqrt{2\tau}}.
\end{aligned}$$

(a) Our calculations (2.13) imply $\frac{\mathrm{d}}{\mathrm{d}\ell}\mathcal{E}_{(\boldsymbol{a},\boldsymbol{0})}(\ell) < 0$ for $\boldsymbol{a} \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\}$ and $\lim_{\ell\to\infty}\mathcal{E}_{(\boldsymbol{a},\boldsymbol{0})}(\ell) = 0$. If $a = \alpha = 0$, then $\mathcal{E}_{(0,0,\boldsymbol{0})}(\ell) = 0$ for all ℓ.

(b) If $a \neq 0$ and $\alpha = 0$ our calculations (2.13) imply that the function has local minimum at $\ell = \ell_1(\tau, \boldsymbol{a}) = \sqrt{|a|}\big(\frac{18}{\tau}\big)^{1/4}$, whereas for $a = 0$ and $\alpha \neq 0$ the function has local minimum at $\ell = \ell_1(\tau, \boldsymbol{a}) = \sqrt{\tau/2}|\alpha|$.

(c) Let $w = |a|/|\alpha| \in (0, \infty)$ and $s = 1$. Then (2.13) shows that the function has a local minimum at $\ell = \ell_1(\tau, \boldsymbol{a}) = \frac{1}{\sqrt{2\tau}}(|\alpha| + \sqrt{\alpha^2 + 6|a|\sqrt{2\tau}})$. If $s = -1$ we get a local minimum at $\ell = \ell_1(\tau, \boldsymbol{a}) = \frac{1}{\sqrt{2\tau}}(-|\alpha| + \sqrt{\alpha^2 + 6|a|\sqrt{2\tau}})$ and in case $\tau \leq \frac{\alpha^4}{72a^2}$ a second local minimum at $\ell = \ell_2(\tau, \boldsymbol{a}) = \frac{1}{\sqrt{2\tau}}(|\alpha| + \sqrt{\alpha^2 - 6|a|\sqrt{2\tau}})$. Note that $\ell_1(\tau, \boldsymbol{a}) < \ell_2(\tau, \boldsymbol{a})$ whenever $\ell_2(\tau, \boldsymbol{a})$ is local minimum. This follows immediately from the second derivative which is positive whenever $\ell_i(\tau, \boldsymbol{a}) \leq \frac{3a}{|\alpha|}$ or $\ell_i(\tau, \boldsymbol{a}) \geq \frac{6a}{|\alpha|}$ for $a > 0 > \alpha$ and $i = 1, 2$.

□

Proof of Lemma 2.3. We are using the scaling property

$$\begin{aligned} h^{*,(0,\ell)}_{(\boldsymbol{a},\boldsymbol{0})}(t) &= a h^{*,(0,1)}_{(1,sw^{-1}\ell,\boldsymbol{0})}(t/\ell) \quad \text{for } a \neq 0 \text{ and } t\in[0,\ell], s=\operatorname{sign}(a\alpha), w=|a|/|\alpha|,\\ h^{*,(0,\ell)}_{(\boldsymbol{a},\boldsymbol{0})}(t) &= h^{*,(0,1)}_{(0,\alpha\ell,\boldsymbol{0})}(t/\ell) \quad \text{for } a=0 \text{ and } t\in[0,\ell], \end{aligned} \tag{2.14}$$

and show the following equivalent statements, the functions $h^*_{(1,\ell,\boldsymbol{0})}$ with $\ell>0$, $h^*_{(1,-\ell,\boldsymbol{0})}$ with $\ell\in\mathbb{R}\setminus\{0\}$, and $h^*_{(0,\ell,\boldsymbol{0})}$ with $\ell\in[0,3)$ have no zeroes in $(0,1)$, whereas the functions $h^*_{(1,-\ell,\boldsymbol{0})}$ with $\ell>3$ have exactly one zero in $(0,1)$. Thus we study the unique minimiser of $\mathcal{E}$ given in Proposition A.1, that is, we consider first the functions $h^*_{(1,s\ell,\boldsymbol{0})}$ for $s\in\{-1,1\}$ and $\ell>0$. The function $h^*_{(1,s\ell,\boldsymbol{0})}$ has a zero in $(0,1)$ if and only if it has a local minimum at which it assumes a negative value. Its derivative has at most one zero in $(0,1)$ as by Proposition A.1 the derivative

$$\dot{h}^*_{\boldsymbol{r}}(t)=\ell s+2(-3-2\ell s)t+3(2+\ell s)t^2$$

is zero at $t=1$ for the boundary condition given by $\boldsymbol{r}=(1,s\ell,0,0)$. Now for $s=1$ the local extrema is a maximum as the function value at $t=0$ is greater than its value at $t=1$ and thus the derivative changes sign from positive to negative. For $s=-1$ and $\ell\le 3$ there is no local extrema as the first derivative is zero only at $t=1$ and has no second zero in $(0,1)$ and the second derivative $\ddot{h}^*_{(\boldsymbol{r},\boldsymbol{0})}(1)=6-2\ell$ at $t=1$ is strictly positive. Thus the derivative takes only negative values in $[0,1)$ and is zero at $t=0$. For $s=-1$ and $\ell>3$ there is a local minimum as the second derivative at $t=1$ is now strictly negative implying that the first derivative changes sign from negative to positive and thus has a zero at which the function value is negative. The functions $h^*_{(0,\ell,\boldsymbol{0})}$ have no zero in $(0,\ell)$ for $\ell\neq 0$ by definition as the only zeroes are $t=0$ and $t=1$. $\square$

Proof of Theorem 2.4. (a) (i) Let $\alpha=0$ and $a\neq 0$. Note that $\ell_1(\tau,\boldsymbol{a})\le 1$ if and only if $\tau\ge 18|a|^2$. Let $\varepsilon_1(\boldsymbol{a})$ be the maximum of $\varepsilon_{\rm c}$ and this lower bound. We write $\tau=\tau(\varepsilon)$. Now $\Sigma^\varepsilon_f(h_{\ell_1})=0$ if and only if $\tau=\tau^*$ with

$$\frac{6\sqrt{|a|}}{18^{3/4}}+18^{1/4}\sqrt{|a|}=(\tau^*)^{1/4},$$

and we easily see that $\Sigma^\varepsilon_f(h_{\ell_1(\tau,\boldsymbol{a})})<0$ for all $\tau>\tau^*$.

(ii) Now let $a=0$ and $\alpha\neq 0$. Note that $\ell_1(\tau,\boldsymbol{a})\le 1$ if and only if $\sqrt{\tau}\ge\sqrt{2}|\alpha|$ and thus let $\varepsilon_1(\boldsymbol{a})$ be the maximum of $\varepsilon_{\rm c}$ and $2|\alpha|^2$. Now $\Sigma^\varepsilon_f(h_{\ell_1})=0$ if and only if $\tau=\tau^*(\boldsymbol{a}):=8|\alpha|^2$, and $\Sigma^\varepsilon_f(h_{\ell_1(\tau,\boldsymbol{a})})<0$ as $\mathrm{d}/\mathrm{d}\tau(\Sigma^\tau_f(h_{\ell_1(\tau,\boldsymbol{a})})<0$ for all $\tau>\tau^*$.

(iii) Now let $s=\operatorname{sign}(a\alpha)=1$ and assume that $a,\alpha>0$ (the case $a,\alpha<0$ follows analogously). As $\ell_1(\tau,\boldsymbol{a})$ is decreasing in $\tau>0$ there is $\varepsilon_1(\boldsymbol{a})\ge\varepsilon_{\rm c}$ such that $\ell_1(\tau,\boldsymbol{a})\le 1$ for all $\tau\ge\tau_1(\boldsymbol{a})$. Lemma 2.10 (b) shows that there exists $\varepsilon_1^*(\boldsymbol{a})$ such that $\Sigma_f^{\varepsilon_1^*(\boldsymbol{a})}(h_{\ell_1(\tau_1^*,\boldsymbol{a})})=0$ and the uniqueness of that zero gives $\Sigma^\varepsilon_f(h_{\ell_1(\tau(\varepsilon),\boldsymbol{a})})<0$ for all $\varepsilon>\varepsilon_1^*(\boldsymbol{a})$.

(b) Let $s=\operatorname{sign}(a\alpha)=-1$ and assume $a>0>\alpha$ (the other case follows analogously). Clearly we have $\Sigma_f^{\varepsilon_i(\boldsymbol{a})}(h_{\ell_i(\tau(\varepsilon_i(\boldsymbol{a})),\boldsymbol{a})})>0$ as $\ell_i(\tau(\varepsilon_i(\boldsymbol{a})),\boldsymbol{a})=1$ for $i=1,2$, and for any $\varepsilon>\varepsilon_i(\boldsymbol{a})$ we have $\ell_i(\tau(\varepsilon),\boldsymbol{a})<1$ and thus

$$\frac{\mathrm{d}}{\mathrm{d}\tau}f_i(\tau)=\ell_i(\tau,\boldsymbol{a})-1<0 \text{ where we write } f_i(\tau)=\Sigma^\tau_f(h_{\ell_i(\tau,\boldsymbol{a})}),\quad i=1,2.$$

Furthermore, due to Lemma 2.10 there is a unique $\tau_0=\tau_0(\boldsymbol{a})$ such that

$$f_1(\tau)\ge f_2(\tau)\quad\text{for } \tau\le\tau_0 \text{ and } f_1(\tau)\le f_2(\tau) \text{ for } \tau\ge\tau_0 \text{ and } f_1(\tau_0)=f_2(\tau_0).$$

We thus know that f_1 is decreasing and that $f_1(\tau) \to -\infty$ for $\tau \to \infty$. As $f_1(\tau_1(\boldsymbol{a})) > 0$ there must be at least one zero which we denote $\tau_1^*(\boldsymbol{a})$ which we write as $\tau(\varepsilon_1^*(\boldsymbol{a}))$. The uniqueness of $\tau_1^*(\boldsymbol{a})$ is shown in Lemma 2.10 (b). Similarly, we denote by $\tau_2^*(\boldsymbol{a})$ the zero of f_2 when this zero exists (otherwise we set it equal to infinity), and one can show uniqueness of this zero in the same way as done for $\tau_1^*(\boldsymbol{a})$ in Lemma 2.10 (b). We can now distinguish three cases according to the sign of the functions f_1 and f_2 at the unique zero τ_0 of the difference $\Delta = f_1 - f_2$. That is, we distinguish whether $\tau_0(\boldsymbol{a})$ is greater, equal or less the unique zero $\tau_1^*(\boldsymbol{a})$ of f_1.

(i) Let $\boldsymbol{a} \in D_1 := \{\boldsymbol{a} \in \mathbb{R}^2 \colon w \in (0,\infty) \text{ and } \tau_0(\boldsymbol{a}) > \tau_1^*(\boldsymbol{a})\}$. Then $f_1(\tau_0(\boldsymbol{a})) = f_2(\tau_0(\boldsymbol{a})) < 0$ and thus $\tau_2^*(\boldsymbol{a})$ exists and satisfies $\tau_2^*(\boldsymbol{a}) < \tau_1^*(\boldsymbol{a})$. This implies immediately the statement by choosing $\varepsilon_{1,2}^*(\boldsymbol{a})$ and $\varepsilon_2^*(\boldsymbol{a})$ such that $\tau(\varepsilon_{1,2}^*(\boldsymbol{a})) = \tau_0(\boldsymbol{a})$ and $\tau(\varepsilon_2^*(\boldsymbol{a})) = \tau_2^*(\boldsymbol{a})$.

(ii) Let $\boldsymbol{a} \in D_2 := \{\boldsymbol{a} \in \mathbb{R}^2 \colon w \in (0,\infty) \text{ and } \tau_0(\boldsymbol{a}) = \tau_1^*(\boldsymbol{a}) = \tau_2^*(\boldsymbol{a})\}$. Then $f_1(\tau_0(\boldsymbol{a})) = f_2(\tau_0(\boldsymbol{a})) = 0$ and thus for $\varepsilon_{\rm c}^*(\boldsymbol{a})$ with $\tau(\varepsilon_{\rm c}^*(\boldsymbol{a})) = \tau_0(\boldsymbol{a})$ we get $\Sigma_f^{\varepsilon_{\rm c}^*(\boldsymbol{a})}(h_{\ell_1}) = \Sigma_f^{\varepsilon_{\rm c}^*(\boldsymbol{a})}(h_{\ell_2}) = \Sigma_f^{\varepsilon_{\rm c}^*(\boldsymbol{a})}(\overline{h}) = 0$. Then Lemma 2.10 (a) gives $\Sigma_f^\varepsilon(h_{\ell_1}) < \Sigma_f^\varepsilon(h_{\ell_2}) < 0$ for all $\varepsilon > \varepsilon_{\rm c}^*(\boldsymbol{a})$.

(iii) Let $\boldsymbol{a} \in D_3 := \{\boldsymbol{a} \in \mathbb{R}^2 \colon w \in (0,\infty) \text{ and } \tau_0(\boldsymbol{a}) < \tau_1^*(\boldsymbol{a})\}$. Then $f_1(\tau_0(\boldsymbol{a})) = f_2(\tau_0(\boldsymbol{a})) > 0$ and for $\varepsilon_1^*(\boldsymbol{a})$ with $\tau(\varepsilon_1^*(\boldsymbol{a})) = \tau_1^*(\boldsymbol{a})$ we get $\Sigma_f^{\varepsilon_1^*(\boldsymbol{a})}(h_{\ell_1}) = \Sigma_f^{\varepsilon_1^*(\boldsymbol{a})}(\overline{h}) = 0$ and $\Sigma_f^\varepsilon(h_{\ell_1}) < 0$ and $\Sigma_f^\varepsilon(h_{\ell_1}) < \Sigma_f^\varepsilon(h_{\ell_2})$ for $\varepsilon > \varepsilon_1^*(a)$.

□

Lemma 2.10. *(a) For any $\boldsymbol{a} \in \mathbb{R}^2$ with $w = |a|/|\alpha| \in (0,\infty)$ and $\tau \in (0, \frac{\alpha^4}{72a^2}]$ the function*

$$\Delta(\tau) := \mathcal{E}^\tau(\ell_1(\tau,\boldsymbol{a})) - \mathcal{E}^\tau(\ell_2(\tau,\boldsymbol{a}))$$

has a unique zero called τ_0, is strictly decreasing and strictly positive for $\tau < \tau_0$.

(b) For any $\boldsymbol{a} \in \mathbb{R}^2$ with $w \in (0,\infty)$ there is a unique solution of

$$\Sigma_f^\varepsilon(h_{\ell_1(\tau(\varepsilon),\boldsymbol{a})}) = 0, \qquad \tau = \tau(\varepsilon) \geq \tau_1(\boldsymbol{a}), \tag{2.15}$$

which we denote by $\tau_1^* = \tau(\varepsilon_1^*(\boldsymbol{a}))$.

Proof of Lemma 2.10. (a) The sign of the function Δ is positive for $\tau \to 0$ whereas the sign is negative if $\tau = \frac{\alpha^4}{72a^2}$. Hence, the continuous function Δ changes its sign and must have a zero. We obtain the uniqueness of this zero by showing that the function Δ is strictly decreasing. For fixed τ we have (Proposition 2.2)

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\mathcal{E}^\tau(\ell) = 0, \qquad \text{for } \ell = \ell_2(\tau,\boldsymbol{a}) \text{ or } \ell = \ell_2(\tau,\boldsymbol{a}).$$

The functions $\mathcal{E}^\tau(\ell_i(\tau,\boldsymbol{a}))$ are rational functions of $\ell_i(\tau,\boldsymbol{a})$ and depend explicitly on τ as well. Thus the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{E}^\tau(\ell_i(\tau)) = \ell_i(\tau). \qquad i = 1,2 \text{ and } \tau \in (0, \frac{\alpha^4}{72a^2}].$$

As $\ell_1(\tau,\boldsymbol{a}) < \ell_2(\tau,\boldsymbol{a})$ the first derivative of Δ is negative on $(0, \frac{\alpha^4}{72a^2}]$.

(b) We let $\tau_1^* = \tau(\varepsilon_1^*(\boldsymbol{a}))$ denote the solution of (2.15). As the rate function is strictly positive for vanishing τ and $\lim_{\tau\to\infty} \Sigma_f^\tau(\ell_1(\tau,\boldsymbol{a})) = -\infty$ we shall check whether there is a second solution to (2.15). Suppose there are $\tau(\varepsilon) > \tau(\varepsilon')$ solving (2.15) with

$$\Sigma_f^\varepsilon(h_{\ell_1(\tau(\varepsilon),\boldsymbol{a})}) = \Sigma_f^{\varepsilon'}(h_{\ell_1(\tau(\varepsilon'),\boldsymbol{a})}). \tag{2.16}$$

For fixed ℓ the function $\tau \mapsto \Sigma_f^\varepsilon(h_\ell) = \mathcal{E}^\tau_{(\boldsymbol{a},\boldsymbol{0})}(\ell) - \tau$ is strictly decreasing and thus

$$\Sigma_f^{\varepsilon'}(h_\ell) > \Sigma_f^\varepsilon(h_\ell) \qquad \text{for } \ell = \ell_1(\tau(\varepsilon')). \tag{2.17}$$

Now Proposition 2.2 gives

$$\Sigma_f^\varepsilon(h_{\ell_1(\tau(\varepsilon'),\boldsymbol{a})}) \geq \min_{\ell\in(0,1)} \Sigma_f^\varepsilon(h_\ell) = \Sigma_f^\varepsilon(h_{\ell_1(\tau(\varepsilon),\boldsymbol{a})}).$$

Combining (2.16) and (2.17) we arrive at a contradiction and thus the solution of (2.15) is unique. Hence $\Sigma_f^\varepsilon(h_{\ell_1(\tau(\varepsilon),\boldsymbol{a})}) < 0$ for all $\tau(\varepsilon) > \tau_1^* = \tau(\varepsilon_1^*(\boldsymbol{a}))$.

□

2.3.2. *Dirichlet boundary conditions.* Proof of Proposition 2.6. We argue as in our proof of Proposition 2.1 using (2.7) observing that for any $h \in H^2_{\boldsymbol{r}}$ with ℓ being the infimum of accumulation points of $\mathcal{N}_h$ and $1-r$ being the corresponding supremum,

$$\begin{aligned}\Sigma^\varepsilon(h) &= \mathcal{E}^{(0,\ell)}(h) - \tau(\varepsilon)(1-\ell-r) + \mathcal{E}^{(1-r,1)}(h) = \mathcal{E}^{\tau(\varepsilon)}_{(\boldsymbol{a},\boldsymbol{0})}(\ell) + \mathcal{E}^{\tau(\varepsilon)}_{b,-\beta,\boldsymbol{0})}(r) - \tau(\varepsilon),\\ &= E(\ell, r).\end{aligned}$$

The second statement follows from the Hessian of E being the product

$$\frac{\partial^2}{\partial \ell^2}\mathcal{E}^{\tau(\varepsilon)}_{(\boldsymbol{a},\boldsymbol{0})}(\ell)\frac{\partial^2}{\partial r^2}\mathcal{E}^{\tau(\varepsilon)}_{(b,-\beta,\boldsymbol{0})}(r)$$

of the second derivatives of the functions $\mathcal{E}^{\tau(\varepsilon)}$ (see Proposition 2.2). □

Proof of Theorem 2.7. (a): We first note that due to convexity of $\mathcal{E}$ the solutions $h^*_{\boldsymbol{r}}$ for boundary conditions $\boldsymbol{r} = (a, \alpha, a, -\alpha)$ are symmetric with respect to the $1/2-$ vertical line. Furthermore, in all three cases of (a) only $\ell_1(\tau, \boldsymbol{a})$ is a minimiser of $\mathcal{E}^\tau$ and thus of E due to symmetric boundary conditions and thus the Hessian (see above) of the energy function E (2.7) is positive implying convexity. Henceforth, when $\ell_1(\tau, \boldsymbol{a})$ is a minimiser of E the corresponding minimiser function (see Proposition 2.6) of the rate function has to be symmetric with respect to the $1/2-$ vertical line. These observations immediately give the proofs for all three cases in (a) of Theorem 2.7 because symmetric minimiser exist only if $\ell_1(\tau, \boldsymbol{a}) \leq 1/2$. Hence we conclude with Theorem 2.4 and $\ell_1(\tau, \boldsymbol{a}) \leq 1/2$ for $\varepsilon \geq \varepsilon_1(\boldsymbol{a})$ using the existence of $\tau_1^*(\boldsymbol{a})$ solving (2.9). The existence and uniqueness of $\tau_1^*(\boldsymbol{a})$ can be shown using an adaptation of Lemma 2.10 (b).

We are left to show all three sub cases (i)-(iii) of (b) in Theorem 2.7. In all these cases we argue differently depending on the parameter regime. If $(\boldsymbol{a}, \tau) \in \mathcal{D}_1$ we can argue as follows. If $\ell_1(\tau, \boldsymbol{a})$ and $\ell_2(\tau, \boldsymbol{a})$ both exist and are minimiser of the energy function E we obtain convexity as above (the mixed derivatives vanish due to the fact that E is a sum of functions of the single variables). Then we can argue as above and conclude with our statements for all there sub cases for parameter regime $\mathcal{D}_1$ with $\ell_i(\tau, \boldsymbol{a}) \leq 1/2, i = 1, 2$.

The only other case for the minimiser $\ell_1(\boldsymbol{a}, \tau(\varepsilon))$ of $\mathcal{E}^\tau$ is $1 \geq \ell_2(\tau(\varepsilon), \boldsymbol{a}) > 1/2 > \ell_1(\tau(\varepsilon), \boldsymbol{a}) > 0$ which gives a candidate for minimiser of Σ^ε which is not symmetric with respect to the $1/2-$ vertical line. It is clear that at the boundary of $\mathcal{D}_2$, namely $\ell_1 + \ell_2 = 1$, we get $\mathcal{E}(h^*_{\boldsymbol{r}}) < E(\ell_2(\tau(\varepsilon), \boldsymbol{a}), \ell_1(\tau(\varepsilon), \boldsymbol{a}))$. Depending on the values of the boundary conditions and the value of $\tau(\varepsilon)$ the minimiser can be either $h^*_{\boldsymbol{r}}$ or the non-symmetric function h_{ℓ_2,ℓ_1}, or both. As outlined in [Ant05] for elastic rods which pose similar variational problems there are no general statements about the minimiser in this regime, for any given values of the parameter one can check by computation which function has a lower numerical value. □

3. Proofs: Large deviation principles

This chapter delivers our proofs for the large deviation theorems. In Section 3.1 we prove the extension of Mogulskii's theorem to integrated random walks and integrated random walk bridges where in the Gaussian cases for the bridge measure case we use Gaussian calculus respectively an explicit representation of the bridge distribution in [GSV05]. In Section 1.2.3 we prove our main large deviation results in Theorem 1.5 for our models with pinning using novel techniques involving precise estimates for double zeros and singular zeros, expansion and Gaussian calculus.

3.1. **Sample path large deviation for integrated random walks and integrated random walk bridges.** We show Theorem 1.2 by using the contraction principle and an adaptation of Mogulskii's theorem ([DZ98, Chapter 5.1]).

(a) Recall the integrated random walk representation in Section 1.2.2 and define a family of random variables indexed by t as

$$\widetilde{Y}_N(t) = \frac{1}{N} Y_{\lfloor Nt \rfloor + 1}, \quad 0 \le t \le 1,$$

and let μ_N be the law of $\widetilde{Y}_N$ in $L_\infty([0,1])$. From Mogulskii's theorem [DZ98, Theorem 5.1.2] we obtain that μ_N satisfy in $L_\infty([0,1])$ the LDP with the good rate function

$$I^M(h) = \begin{cases} \int_0^1 \Lambda^*(\dot{h}(t))\,\mathrm{d}t & , \text{ if } h \in \mathcal{AC}, h(0) = \alpha, \\ \infty & \text{otherwise,} \end{cases}$$

where $\mathcal{AC}$ denotes the space of absolutely continuous functions. Clearly, the laws μ_N (as well the laws in Theorem 1.2) are supported on the space of functions continuous from the right and having left limits, of which our domains of all rate functions is a subset. Thus the LDP also holds in this space when equipped with the supremum norm topology. Furthermore, there are exponentially equivalent laws (in $L_\infty([0,1])$ fully supported on $\mathcal{C}([0,1];\mathbb{R})$ (see [DZ98, Chapter 5.1]). Then the LDPs for the equivalent laws hold in $\mathcal{C}([0,1];\mathbb{R})$ as well, and as $\mathcal{C}([0,1];\mathbb{R})$ is a closed subset of $L_\infty([01,1])$, the same LDP holds also in $L_\infty([0,1])$. The empirical profiles are functions of the integrated random walk $(Z_n)_{n\in\mathbb{N}_0}$, and

$$h_N(t) = \frac{1}{N^2} Z_{\lfloor Nt \rfloor} + \frac{1}{N^2} \int_{\frac{\lfloor Nt \rfloor}{N}}^t \left(Z_{\lfloor Ns \rfloor + 1} - Z_{\lfloor Ns \rfloor} \right) \mathrm{d}s = \frac{1}{N} \int_0^t \widetilde{Y}_N(s)\,\mathrm{d}s.$$

The contraction principle for the integral mapping gives immediately the rate function for the LDP for the empirical profiles h_N under above laws via the following infimum

$$J(h) = \inf_{g \in \mathcal{S}_h} I^M(g), \qquad \text{with } \mathcal{S}_h = \{ g \in L_\infty([0,1]) \colon \int_0^t g(s)\,\mathrm{d}s = h(t), t \in [0,1] \}.$$

If either $h(0) \neq \alpha$ or h is not differentiable, then $\mathcal{S}_h = \varnothing$. In the other cases one obtains $\mathcal{S}_h = \{\dot{h}\}$, and therefore $J \equiv \mathcal{E}_f$. This proves (a) of Theorem 1.2.

(b) In the Gaussian case the LDP can be shown by Gaussian calculus (e.g., [DS89]), or by employing the contraction principle for the Gaussian integrated random walk bridge. The explicit distribution of the Gaussian bridge leads to the follows mapping. We only sketch this approach for illustrations. For simplicity choose the boundary condition $\boldsymbol{r} = \mathbf{0}$ and $\boldsymbol{a} = (0,0)$. The cases for non-vanishing boundary conditions follow analogously. Then

$$\mathbb{P}^{\mathbf{0}} = \mathbb{P}^{(0,0)} \circ B_N^{-1},$$

where for $Z = (Z_1, \ldots, Z_{N+1})$,

$$B_N(Z)(x) = Z_x - A_N(x, Z_N, Z_{N+1} - Z_N), \quad x \in \{1, 2, \ldots, N+1\},$$

and

$$A_N(x,u,v) = \frac{1}{N(N+1)(N+2)}\Big(x^3(-2u+vN) + x^2(3uN+vN-vN^2) + x((2+3N)u - N^2 v\Big).$$

Clearly, $B_N(Z)(N) = B_N(Z)(N+1) = 0$.

3.2. **Sample path large deviation for pinning models.** In Section 3.2.1 we show the large deviation lower bound and in Section 3.2.2 the corresponding upper bound. It will be convenient to work in a slightly different normalisation. Instead of (1.10) we will show that

$$\liminf_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,\varepsilon}(\boldsymbol{r})}{Z_N(\mathbf{0})} \gamma_N(h_N \in \mathcal{O}) \geq -\inf_{h \in \mathcal{O}} \Sigma(h), \tag{3.1}$$

$$\limsup_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,\varepsilon}(\boldsymbol{r})}{Z_N(\mathbf{0})} \gamma_N(h_N \in \mathcal{K}) \leq -\inf_{h \in \mathcal{K}} \Sigma(h). \tag{3.2}$$

Where $Z_{N,\varepsilon}(\boldsymbol{r})$ is the partition function introduced in (1.2) with Dirichlet boundary condition given in (1.4) and $Z_N(\mathbf{0})$ is the partition function of the same model with pinning strength $\varepsilon = 0$ and Dirichlet boundary condition zero. Note for later use that precise asymptotics of the Gaussian partition function $Z_N(\mathbf{0})$ are presented in Appendix B. Once these bounds are established, they can be applied to the full space $\mathcal{C}([0,1];\mathbb{R})$ implying

$$\liminf_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,\varepsilon}(\boldsymbol{r})}{Z_N(\mathbf{0})} = -\inf_{h \in H} \Sigma(h),$$

so that (1.10) follows.

3.2.1. *Proof of the lower bound in Theorem 1.5.* Fix $g \in H$ and $\delta > 0$, and denote by $\mathcal{B} = \{h \in \mathcal{C} \colon \|h - g\|_\infty < \delta\}$. We establish the lower bound (3.1) in the form

$$\liminf_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,\varepsilon}(\boldsymbol{r})}{Z_N(\mathbf{0})} \gamma^{\boldsymbol{r}}_{N,\varepsilon}(h_N \in \mathcal{B}) \geq -\Sigma(g). \tag{3.3}$$

Reduction to "well behaved" g. Recall that by Sobolev embedding any $g \in H$ is automatically $\mathcal{C}^1([0,1])$ with $\frac{1}{2}$-Hölder continuous first derivative. We can write

$$\{t \in [0,1] \colon g(t) = 0\} = \bar{\mathcal{N}} \cup \mathcal{N},$$

where $\bar{\mathcal{N}}$ is the set of *isolated* zeros

$$\bar{\mathcal{N}} = \{t \in [0,1] \colon g(t) = 0 \text{ and } g \text{ has no further zeros in an open interval around } t\},$$

and where $\mathcal{N}$ is the set of all non isolated zeros. The set $\bar{\mathcal{N}}$ is at most countable, and therefore $|\bar{\mathcal{N}}| = 0$. These zeros do not contribute to the value of $\Sigma(g)$. The set $\mathcal{N}$ is closed.

Definition 3.1. *We say that $g \in H$ is* well behaved *if $\mathcal{N}$ is empty or the union of finitely many disjoint closed intervals, i.e.*

$$\mathcal{N} = \cup_{j=1}^{k} [\ell_j, r_j]$$

for $0 \leq \ell_1 < r_1 < \cdots < \ell_k < r_k \leq 1$.

Lemma 3.2. *For any $g \in H$ and $\delta > 0$ there exists a* well behaved *function $\hat{g} \in H$ such that $\|g - \hat{g}\|_\infty < \delta$ and $\Sigma(\hat{g}) \leq \Sigma(g)$.*

Proof. We start by observing that for $t \in \mathcal{N}$, we have $g'(t) = 0$. Indeed, by definition there exists a sequence (t_n) in $[0,1] \setminus \{t\}$ which converges to t and along which g vanishes. Hence

$$g'(t) = \lim_{n\to\infty} \frac{g(t_n) - g(t)}{t_n - t} = 0.$$

By uniform continuity of g there exists a δ' such that for $|t - t'| < \delta'$ we have $|g(t) - g(t')| < \delta$. We define recursively

$$\ell_1 = \inf \mathcal{N} \qquad r_1 = \inf\{t \in \mathcal{N} \colon (t, t+\delta') \cap \mathcal{N} = \varnothing\},$$
$$\ell_2 = \inf\{t \in \mathcal{N} \colon t > r_1\} \qquad r_2 = \inf\{t \in \mathcal{N} \colon t > \ell_2 \text{ and } (t, t+\delta') \cap \mathcal{N} = \varnothing\},$$

and so on. Then we set $\hat{g} = 0$ on the intervals $[\ell_j, r_j]$ and $\hat{g} = g$ elsewhere. The function $\hat{g}$ constructed in this way satisfies the desired properties. $\square$

Lemma 3.2 implies that it suffices to establish (3.3) for *well behaved* functions g and from now on we will assume that g is well behaved. Actually, we will first discuss the notationally simpler case where $\mathcal{N}$ consists of a *single* interval $[\ell, r]$ for $0 < \ell < r < 1$. We explain how to extend the argument to the general case in the last step.

Expansion and "good pinning sites". From now on we assume that there exist $0 < \ell < r < 1$ such that $g = 0$ on $\mathcal{N} = [\ell, r]$ and such that all zeros of g outside of $\mathcal{N}$ are isolated. Under these assumptions we will show that

$$\liminf_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,\varepsilon}(\boldsymbol{r})}{Z_N(\mathbf{0})} \gamma^{\boldsymbol{r}}_{N,\varepsilon}(h_N \in \mathcal{B}) \geq -\Big(\frac{1}{2}\int_0^\ell \ddot{g}^2(t)\,\mathrm{d}t - \tau(\varepsilon)(r-\ell) + \frac{1}{2}\int_r^1 \ddot{g}^2(t)\,\mathrm{d}t\Big). \quad (3.4)$$

The definition (1.2) of $\gamma^{\boldsymbol{r}}_{N,\varepsilon}$ can be rewritten as

$$Z_{N,\varepsilon}(\boldsymbol{r})\, \gamma^{\boldsymbol{r}}_{N,\varepsilon}(\mathrm{d}\phi) = \sum_{\mathcal{P}\subseteq\{1,\dots,N-1\}} \mathrm{e}^{-\mathcal{H}_{[-1,N+1]}(\phi)} \prod_{k\in\mathcal{P}} \varepsilon\delta_0(\mathrm{d}\phi_k) \prod_{k\in\{1,\dots,N-1\}\setminus\mathcal{P}} \mathrm{d}\phi_k \prod_{k\in\{-1,0,N,N+1\}} \delta_{\psi_k}(\mathrm{d}\phi_k). \quad (3.5)$$

The first crucial observation is that for certain choices of "pinning sites" $\mathcal{P}$ the right hand side of this expression becomes a product measure. Indeed, if $\mathcal{P}$ contains two adjacent sites $p, p+1$ we can write

$$\mathcal{H}_{[-1,N+1]}(\phi) = \mathcal{H}_{[-1,p]}(\phi) + \frac{1}{2}(\Delta\phi_p)^2 + \frac{1}{2}(\Delta\phi_{p+1})^2 + \mathcal{H}_{[p+1,N+1]}(\phi),$$

which turns into $\mathcal{H}_{[-1,p]}(\phi) + \phi_{p-1}^2 + \phi_{p+2}^2 + \mathcal{H}_{[p+1,N+1]}(\phi)$ if $\phi_p = \phi_{p+1} = 0$. This means that when ϕ_p and ϕ_{p+1} are pinned, the Hamiltonian decomposes into two independent contributions – one which depends only on (the *left* boundary conditions on $\phi(-1)$, $\phi(0)$ given in (1.4) and) $\phi_1, \dots, \phi_{p-1}$ and one which only depends on $\phi_{p+2}, \dots, \phi_{N-1}$ (and the *right* boundary conditions on $\phi(N), \phi(N+1)$). Then the term corresponding to this choice $\mathcal{P}$ in the expansion (3.5) factorises into two independent parts. We will now reduce ourselves to choices of pinning sites $\mathcal{P}$ which have this property.

Definition 3.3. *A subset $\mathcal{P} \subseteq \{1, \dots, N\}$ is a* very good *choice of pinning sites if*

- $\{0, \dots, p_* - 1\} \cap \mathcal{P} = \varnothing$ *and* $\{p^* + 2, \dots, N\} \cap \mathcal{P} = \varnothing$.
- $\{p_*,\ p_* + 1,\ p^*, p^* + 1\} \subseteq \mathcal{P}$,

Here we have set $p_* := \lfloor N\ell \rfloor$ and $p^* := \lfloor Nr \rfloor$ (we leave implicit the N dependence of p_* and p^*).

As all the terms in (3.5) are non-negative we can obtain a lower bound by reducing the sum to *very good* $\mathcal{P}$. In this way we get

$$\begin{aligned} Z_{N,\varepsilon}(\boldsymbol{r})\gamma^{\boldsymbol{r}}_{N,\varepsilon}(\mathcal{B}) \geq \sum_{\mathcal{P} \text{ very good}} \varepsilon^{|\mathcal{P}|}\, & Z_{[-1,p_*+1]}(\boldsymbol{r})\ \gamma^{\boldsymbol{r}}_{[-1,p_*+1]}\big(\sup_{0\le t\le \ell} |h_N(t) - g(t)| \le \delta\big) \\ & \times Z_{[p_*,p^*+1]\setminus\mathcal{P}}(\boldsymbol{0})\, \gamma^{\boldsymbol{0}}_{[p_*,p^*+1]\setminus\mathcal{P}}\big(\sup_{\ell\le t\le r} |h_N(t)| \le \delta\big) \\ & \times Z_{[p^*-1,N+1]}(\boldsymbol{r})\, \gamma^{\boldsymbol{r}}_{[p^*-1,N+1]}\big(\sup_{r\le t\le N} |h_N(t) - g(t)| \le \delta\big). \end{aligned} \tag{3.6}$$

The measures $\gamma^{\boldsymbol{r}}_{[-1,p_*+1]}$ and $\gamma^{\boldsymbol{r}}_{[p^*-1,N+1]}$ on the right hand side of this expression are defined as

$$\begin{aligned} \gamma^{\boldsymbol{r}}_{[-1,p_*+1]}(\mathrm{d}\phi) &= \frac{1}{Z_{[-1,p_*+1]}(\boldsymbol{r})} \mathrm{e}^{-\mathcal{H}_{[-1,p_*+1]}(\phi)} \prod_{k\in\{1,\dots,p_*-1\}} \mathrm{d}\phi_k \prod_{k\in\{-1,0,p_*,p_*+1\}} \delta_{\psi_k}(\mathrm{d}\phi_k) \\ \gamma^{\boldsymbol{r}}_{[p^*-1,N+1]}(\mathrm{d}\phi) &= \frac{1}{Z_{[p^*-1,N+1]}(\boldsymbol{r})} \mathrm{e}^{-\mathcal{H}_{[p^*,N+1]}(\phi)} \prod_{k\in\{p^*+2,\dots,N-1\}} \mathrm{d}\phi_k \prod_{k\in\{p^*,p^*+1,N,N+1\}} \delta_{\psi_k}(\mathrm{d}\phi_k), \end{aligned}$$

where we have adjusted the boundary conditions to $\psi(p_*) = \psi(p_*+1) = \psi(p^*) = \psi(p^*+1) = 0$. These measures do not depend on the specific choice $\mathcal{P}$ of very good pinning sites. The measure $\gamma^{\boldsymbol{0}}_{[p_*,p^*+1]\setminus\mathcal{P}}$ is defined as

$$\gamma^{\boldsymbol{0}}_{[p_*,p^*+1]\setminus\mathcal{P}}(\mathrm{d}\phi) = \frac{1}{Z_{[p_*,p^*+1]\setminus\mathcal{P}}(\boldsymbol{0})} \mathrm{e}^{-\mathcal{H}_{[p_*,p^*+1]}(\phi)} \prod_{k\in\mathcal{P}} \delta_0(\mathrm{d}\phi_k) \prod_{k\in\{p_*,\dots,p^*+1\}\setminus\mathcal{P}} \mathrm{d}\phi_k.$$

Note that none of these measures depends on the choice ε of the pinning strength, which only appears as a factor $\varepsilon^{|\mathcal{P}|}$ in each term in (3.6). Note furthermore, that all three measures $\gamma^{\boldsymbol{r}}_{[-1,p_*+1]}$, $\gamma^{\boldsymbol{r}}_{[p^*-1,N+1]}$ and $\gamma^{\boldsymbol{0}}_{[p_*,p^*+1]\setminus\mathcal{P}}$ are Gaussian.

Lemma 3.4. *For every $\epsilon > 0$ there exists an $N_* < \infty$ such that for all $N \geq N^*$ we have*

$$\gamma^{\boldsymbol{r}}_{[-1,p_*+1]}\big(\sup_{0\le t\le\ell} |h_N(t) - g(t)| \le \delta\big) \geq \exp\Big(- N\Big[\int_0^\ell \ddot{g}(t)^2\mathrm{d}t - \inf_h \int_0^\ell \ddot{h}(t)^2\mathrm{d}t + \epsilon\Big]\Big) \tag{3.7}$$

$$\gamma^{\boldsymbol{r}}_{[p^*-1,N+1]}\big(\sup_{r\le t\le 1} |h_N(t) - g(t)| \le \delta\big) \geq \exp\Big(- N\Big[\int_r^1 \ddot{g}(t)^2\mathrm{d}t - \inf_h \int_r^1 \ddot{h}_r(t)^2\,\mathrm{d}t + \epsilon\Big]\Big)\ , \tag{3.8}$$

where the infimum is taken over all $h\colon [0,\ell] \to \mathbb{R}$ and $h\colon [r,1] \to \mathbb{R}$ which satisfy the right boundary conditions, i.e. $h(0) = a$, $\dot{h}(0) = \alpha$, $h(\ell) = 0$, $\dot{h}(\ell) = 0$ for (3.7) and $h(r) = 0$, $\dot{h}(r) = 0$, $h(1) = b$, $\dot{h}(1) = \beta$ for (3.8).

Proof. This follows immediately from the Gaussian large deviation principle presented in Proposition 1.2. □

Lemma 3.5. *There exists an $N_* < \infty$ such that for $N \geq N_*$ and for all good $\mathcal{P} \subseteq \{1, \ldots, N-1\}$ we have*

$$\gamma^{\mathbf{0}}_{[p_*,p^*+1]\setminus\mathcal{P}}\big(\sup_{\ell \leq t \leq r} |h_N(t)| \leq \delta \big) \geq \frac{1}{2} \ .$$

Proof. By the definition of h_N we get

$$\begin{aligned}\gamma^{\mathbf{0}}_{[p_*,p^*+1]\setminus\mathcal{P}}\big(\sup_{\ell \leq t \leq r} |h_N(t)| > \delta \big) &\leq \gamma^{\mathbf{0}}_{[p_*,p^*+1]\setminus\mathcal{P}}\big(\sup_{p_* \leq k \leq p^*} |\phi(k)| > \delta N^2 \big) \\ &\leq \sum_{p_* \leq k \leq p^*} \gamma^{\mathbf{0}}_{[p_*,p^*+1]\setminus\mathcal{P}}\big(|\phi(k)| > \delta N^2 \big) \ .\end{aligned}$$

Recall that under $\gamma^{\mathbf{0}}_{[p_*,p^*+1]\setminus\mathcal{P}}$ all $\phi(k)$ are centred Gaussian random variables and that the sum on the right hand side goes over at most $p^* - p_* \leq N$ terms. Hence in order to conclude it is sufficient to prove that for all N and for all $\mathcal{P}$ and for all $k \in \{p_*, \ldots, p^*\}$ the variance of $\phi(k)$ under $\gamma^{\mathbf{0}}_{[p_*,p^*+1]\setminus\mathcal{P}}$ is bounded by N^3.

To see this, we recall a convenient representation of Gaussian variances: If C be the covariance matrix of a centred non-degenerate Gaussian measure on $\mathbb{R}^N$. Then we have for $k = 1, \ldots, N$,

$$C_{k,k} = \sup_{y \in \mathbb{R}^N \setminus \{0\}} \frac{y_k^2}{\langle y, C^{-1} y \rangle} \ ,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical scalar product on $\mathbb{R}^N$. This identity follows immediately from the Cauchy-Schwarz inequality. In our context, this implies that the variance of $\phi(k)$ under $\gamma^{\mathbf{0}}_{[p_*,p^*+1]\setminus\mathcal{P}}$ is given by

$$\sup_{\substack{\eta\colon \{p_*,\ldots,p^*+1\} \to \mathbb{R} \\ \eta(k)=0 \ \text{for} \ k \in \mathcal{P}}} \frac{\eta(k)^2}{2\,\mathcal{H}_{[p_*,p^*+1]}(\eta)} \leq \sup_{\substack{\eta\colon \{p_*,\ldots,p^*+1\} \to \mathbb{R} \\ \eta(k)=0 \ \text{for} \ k \in \{p_*,p_*+1,p^*,p^*+1\}}} \frac{\eta(k)^2}{2\,\mathcal{H}_{[p_*,p^*+1]}(\eta)} \ ,$$

where the inequality follows because the supremum is taken over a larger set.

The quantity on the right hand side can now be bounded easily. By homogeneity we can reduce the supremum to *test vectors* η that satisfy $\eta(k) = 1$. Invoking the homogeneous boundary conditions, for such η there must exist a $j \in \{p_*, \ldots, p^*\}$ such that $\eta(j+1) - \eta(j) \geq \frac{1}{N}$. Invoking the homogenous boundary conditions once more (this time for the difference $\eta(p_*+1) - \eta(p_*)$) we get

$$\begin{aligned}\frac{1}{N} &\leq \sum_{m=p_*+1}^{j} \big(\eta(m+1) - \eta(m)\big) - \big(\eta(m) - \eta(m-1)\big) = \sum_{m=p_*+1}^{j} \Delta\eta(m) \leq \sum_{m=p_*}^{p^*} \big|\Delta\eta(m)\big| \\ &\leq (p^* - p_*)^{\frac{1}{2}} \Big(\sum_{m=p_*}^{p^*} \big|\Delta\eta(m)\big|^2 \Big)^{\frac{1}{2}} \ .\end{aligned}$$

Using the bound $p^* - p_* \leq N$ we see that η must satisfy

$$\mathcal{H}_{[p_*,p^*]}(\eta) \geq \frac{1}{2N^3} \ ,$$

which implies the desired bound on the variance. □

The pinning potential. First of all, we observe that the minimal energy terms appearing in (3.7) and (3.8) can be absorbed into the boundary conditions. We obtain by Proposition A.1

in Appendix A as well as by identity (B.2) in conjunction with Proposition A.3 that for every $\epsilon > 0$ and for N large enough

$$\exp\big(N \inf_h \int_0^\ell \ddot{h}(t)^2 \mathrm{d}t\big) Z_{[-1,p_*+1]}(\boldsymbol{r}) \ge Z_{\mathcal{P},\ell}(\mathbf{0}) \exp(-\epsilon N)$$
$$\exp\big(N \inf_h \int_r^1 \ddot{h}(t)^2 \mathrm{d}t\big) Z_{[p^*-1,N+1]}(\boldsymbol{r}) \ge Z_{[p^*-1,N+1]}(\mathbf{0}) \exp(-\epsilon N)\,.$$

Therefore, combining (3.6) with Lemma 3.4 and Lemma 3.5 we obtain for any $\epsilon > 0$ and for N large enough

$$\begin{aligned}\frac{Z_{N,\varepsilon}(\boldsymbol{r})}{Z_N(\mathbf{0})}\gamma^{\boldsymbol{r}}_{N,\varepsilon}(\mathcal{B}) \ge \frac{1}{2}\exp\Big(-N\int_0^\ell \ddot{g}(t)^2\mathrm{d}t - N\int_r^1 \ddot{g}(t)^2 \mathrm{d}t - N\epsilon\Big)\\ \times \sum_{\mathcal{P}\text{ very good}} \varepsilon^{|\mathcal{P}|}\frac{Z_{[-1,p_*+1]}(\mathbf{0}) Z_{[p_*,p^*+1]\setminus\mathcal{P}}(\mathbf{0})\; Z_{[p^*-1,N+1]}(\mathbf{0})}{Z_N(\mathbf{0})}\,.\end{aligned}$$

Therefore, it remains to treat the sum of the partition functions on the right hand side. First of all, we observe that $Z_{[-1,p_*+1]}(\mathbf{0})$ and $Z_{[p^*-1,N+1]}(\mathbf{0})$ and $Z_N(\mathbf{0})$ do not depend on the choice of very good $\mathcal{P}$ so that they can be taken out of the sum, i.e. we can write

$$\begin{aligned}&\sum_{\mathcal{P}\text{ very good}} \varepsilon^{|\mathcal{P}|}\frac{Z_{[-1,p_*+1]}(\mathbf{0}) Z_{[p_*,p^*+1]\setminus\mathcal{P}}(\mathbf{0})\; Z_{[p^*-1,N+1]}(\mathbf{0})}{Z_N(\mathbf{0})}\\ &\quad= \frac{Z_{[-1,p_*+1]}(\mathbf{0}) Z_{p^*-p_*}(\mathbf{0})\; Z_{[p^*-1,N+1]}(\mathbf{0})}{Z_N(\mathbf{0})} \sum_{\mathcal{P}\text{ very good}} \varepsilon^{|\mathcal{P}|}\frac{Z_{[p_*,p^*+1]\setminus\mathcal{P}}(\mathbf{0})}{Z_{p^*-p_*}(\mathbf{0})}\,.\end{aligned}$$

Here we have multiplied and divided by the Gaussian partition function $Z_{p^*-p_*}(\mathbf{0})$ because it allows us to compare the sum on the right hand side to the limit (1.7) which defines $\tau(\varepsilon)$. More precisely we get

$$\sum_{\mathcal{P}\text{ very good}} \varepsilon^{|\mathcal{P}|}\frac{Z_{[p_*,p^*+1]\setminus\mathcal{P}}(\mathbf{0})}{Z_{p^*-p_*}(\mathbf{0})} = \frac{Z_{p^*-p_*,\varepsilon}}{Z_{p^*-p_*}} \ge \exp\big((r-\ell)\tau(\varepsilon) - N\epsilon\big)\,.$$

for N large enough (depending on ϵ).

Therefore, to conclude it only remains to observe that according to Appendix B the quotient

$$\frac{Z_{[-1,p_*+1]}(\mathbf{0}) Z_{p^*-p_*}(\mathbf{0})\; Z_{[p^*-1,N+1]}(\mathbf{0})}{Z_N(\mathbf{0})}$$

decays at most polynomially in N which implies that it disappears on an exponential scale. Therefore, (3.4) follows. For the case of Dirichlet boundary conditions on the left hand side and free boundary conditions on the right hand side we easily obtain a version of (3.4) when we replace $\boldsymbol{r}$ by $\boldsymbol{a}$.

We conclude with the lower bound of Theorem 1.5 once we have shown that the proof above can be easily extended to well behaved functions g which have a finite number of intervals $I \subset (0,1)$ on which g is zero. All the steps above can be extended to a finite number of zero intervals using the expansion and splitting. As the number of such zero intervals is finite we finally conclude with our lower bound.

3.2.2. *Proof of the upper bound of Theorem 1.5.* For the upper bound we need to show that

$$\limsup_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,\varepsilon}(\boldsymbol{r})}{Z_N(\mathbf{0})} \gamma^{\boldsymbol{r}}_{N,\varepsilon}(\mathcal{K}) \le -\inf_{h\in\mathcal{K}} \Sigma(h), \tag{3.9}$$

for all closed $\mathcal{K} \subset \mathcal{C}([0,1];\mathbb{R})$.

Reduction to a simpler statement. First of all we observe:

Lemma 3.6. *The family $\{\gamma^{\boldsymbol{r}}_{N,\varepsilon}\}_N$ is exponentially tight in $\mathcal{C}([0,1];\mathbb{R})$.*

The proof of this lemma can be found at the end of this section. Lemma 3.6 implies that it suffices to establish (3.9) for compact sets $\mathcal{K}$. Going further, it suffices to show that for any $g \in \mathcal{K}$ and any $\epsilon > 0$ there exists a $\delta = \delta(g,\epsilon) > 0$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,\varepsilon}(\boldsymbol{r})}{Z_N(\mathbf{0})} \gamma^{\boldsymbol{r}}_{N,\varepsilon}(B(g,\delta)) \le -\Sigma(g) + \epsilon. \tag{3.10}$$

Here $B(g,r) = \{h \in \mathcal{C} \colon \|h-g\|_\infty < r\}$ denotes the L^∞ ball of radius r around g.

We give the simple argument to show that (3.10) implies (3.9): For any compact set $\mathcal{K}$ and any $\epsilon > 0$ there exists a finite set $\{g_1,\ldots,g_M\} \subset \mathcal{K}$ such that $\mathcal{K} \subseteq \cup_{j=1}^M B(g_j, \delta(g_j,\epsilon))$. Then (3.10) yields

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,\varepsilon}(\boldsymbol{r})}{Z_N(\mathbf{0})} \gamma^{\boldsymbol{r}}_{N,\varepsilon}(\mathcal{K}) &\le \limsup_{N\to\infty} \frac{1}{N} \log \sum_{j=1}^M \frac{Z_{N,\varepsilon}(\boldsymbol{r})}{Z_N(\mathbf{0})} \gamma^{\boldsymbol{r}}_{N,\varepsilon}(B(g_j,\delta(g_j,\varepsilon))) \\
&\le \max_{j=1,\ldots,M} \limsup_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,\varepsilon}(\boldsymbol{r})}{Z_N(\mathbf{0})} \gamma^{\boldsymbol{r}}_{N,\varepsilon}(B(g_j,\delta(g_j,\epsilon))) \\
&\le -\min_{j=1,\ldots,M} \Sigma(g_j) + \epsilon \\
&\le -\inf_{h\in\mathcal{K}} \Sigma(h) + \epsilon,
\end{aligned}$$

so (3.9) follows because $\epsilon > 0$ can be chosen arbitrarily small.

For fixed g and ϵ the value of δ is determined by the following lemma.

Lemma 3.7. *For any $g \in \mathcal{C}([0,1];\mathbb{R})$ and all $\epsilon > 0$ there exists a $\bar\delta > 0$ and a closed set $\mathcal{I} \subset [0,1]$ such that the following hold:*

(1) $\mathcal{I}$ is the union of finitely many disjoint closed intervals, i.e.

$$\mathcal{I} = \cup_{j=1}^M [\ell_j, r_j] \tag{3.11}$$

for some finite M and $0 \le \ell_1 < r_1 < \ell_2 < r_2 < \ldots < r_M \le 1$.

(2) The level-set $\{t \in [0,1] \colon |g(t)| \le \bar\delta\}$ is contained in $\mathcal{I}$.

(3) The measure of $\mathcal{I}$ satisfies the bound

$$|\mathcal{I}| \le |\{t\in[0,1]\colon |g(t)| = 0\}| + \epsilon.$$

The proof of this lemma is also given at the end of this section.

Expansion and key lemmas. We will now proceed to prove that (3.10) holds for a fixed g and ϵ and a suitable $\delta \in (0,\bar\delta)$, where $\bar\delta$ is given by Lemma 3.7. For simplicity, we will assume that the set $\mathcal{I}$ constructed in this lemma consists of *a single* interval $[\ell, r]$. The argument for the case of a finite union of disjoint intervals is identical, only requiring slightly more complex notation, and will be omitted.

We write

$$\gamma^{\boldsymbol{r}}_{N,\varepsilon}(B(g,\delta)) = \frac{1}{Z_{N,\varepsilon}(\boldsymbol{r})} \sum_{\mathcal{P}} \varepsilon^{|\mathcal{P}|} \, Z_{[-1,N+1]\setminus\mathcal{P}} \, \gamma^{\boldsymbol{r}}_{[-1,N+1]\setminus\mathcal{P}}(B(g,\delta)), \tag{3.12}$$

where as above $\gamma^{\boldsymbol{r}}_{[-1,N+1]\setminus\mathcal{P}}$ denotes the Gaussian measure over $\{-1,\dots,N+1\}$ which is pinned at the sites in $\mathcal{P}$, i.e.

$$\gamma^{\boldsymbol{r}}_{[-1,N+1]\setminus\mathcal{P}}(\mathrm{d}\phi) = \frac{1}{Z_{[-1,N+1]\setminus\mathcal{P}}(\boldsymbol{r})} \mathrm{e}^{-\mathcal{H}_{[-1,p_*+1]}(\phi)} \prod_{k\in\{1,\dots,N-1\}\setminus\mathcal{P}} \mathrm{d}\phi_k \prod_{k\in\mathcal{P}\cup\{-1,0,N,N+1\}} \delta_{\psi_k}(\mathrm{d}\phi_k),$$

and $Z_{[-1,N+1]\setminus\mathcal{P}}(\boldsymbol{r})$ is the corresponding Gaussian normalisation constant. By definition $|g(t)| > \bar{\delta} > \delta$ for $t \in [0,1]\setminus\mathcal{I}$, so in (3.12) it suffices to sum over those sets of pinning sites $\mathcal{P} \subseteq N\mathcal{I}\cap\mathbb{Z}$. The next two lemmas simplify the expressions in the sum (3.12). For the moment we only deal with homogeneous boundary conditions $\boldsymbol{r} = 0$. We start by introducing some notation which will be used to simplify the partition functions. As above in Definition 3.3 we will be interested in sets of pinning sites $\mathcal{P}$ that allow to separate the Hamiltonian $\mathcal{H}$ into independent parts:

Definition 3.8. *Let $\mathcal{P} \subseteq \{1,\dots,N-1\}$ be non-empty and let $p_* = \min\mathcal{P}$ and $p^* = \max\mathcal{P}$. We will call $\mathcal{P}$ an* good *choice of pinning sites if $\{p_*+1, p^*-1\} \subseteq \mathcal{P}$. We will also call the empty set good.*

Note that the *very good* sets introduced in Definition 3.3 are indeed *good*. The difference between the two notions is that we do not prescribe the precise value of p_* and p^* for good sets. They will however always be confined to the interval $[\lfloor \ell N\rfloor, \lfloor rN\rfloor + 1]$. We also introduce the following operation of *correcting* a set to make it good.

Definition 3.9. *Let $\mathcal{P} \subseteq \{1,\dots,N-1\}$ be non-empty with $p_* = \min\mathcal{P}$ and $p^* = \max\mathcal{P}$. Then we define*

$$c(\mathcal{P}) = \mathcal{P} \cup \{p_*+1, p^*-1\}.$$

We also set $c(\varnothing) = \varnothing$.

For later use we remark that on the one hand the correction map c adds at most two points to a given set $\mathcal{P}$, and that on the other hand for a given good set $\mathcal{P}$ there are at most 4 distinct $\widetilde{\mathcal{P}}$ with $c(\widetilde{\mathcal{P}}) = \mathcal{P}$. The following Lemma permits to replace the partition function $Z_{[-1,N+1]\setminus\mathcal{P}}(\mathbf{0})$ in (3.12) by the partition function $Z_{[-1,N+1]\setminus c(\mathcal{P})}(\mathbf{0})$ with corrected choice of pinning sites.

Lemma 3.10. *For every non-empty $\mathcal{P} \subseteq \{1,\dots,N-1\}$ we have*

$$Z_{[-1,N+1]\setminus\mathcal{P}}(\mathbf{0}) \le (2\pi N) Z_{[-1,N+1]\setminus c(\mathcal{P})}(\mathbf{0}).$$

PROOF. For any $\mathcal{A} \subset \{1,\dots,N-1\}$ set

$$\gamma^{\mathbf{0}}_{\mathcal{A}}(\mathrm{d}\phi) = \frac{1}{Z_{\mathcal{A}}(\mathbf{0})} \mathrm{e}^{-\mathcal{H}_{[-1,N+1]}(\phi)} \prod_{k\in\mathcal{A}} \mathrm{d}\phi_k \prod_{k\in\{-1,\dots,N+1\}\setminus\mathcal{A}} \delta_0(\mathrm{d}\phi_k).$$

We derive an identity that links the Gaussian partition function $Z_{\mathcal{A}}(\mathbf{0})$ to $Z_{\mathcal{A}\setminus\{j\}}(\mathbf{0})$ for an arbitrary $\mathcal{A} \subseteq \{1, \ldots, N-1\}$ and $j \in \mathcal{A}$. We have

$$Z_{\mathcal{A}}(\mathbf{0}) = \int_{\mathbb{R}} \Big(\int \mathrm{e}^{-\mathcal{H}_{[-1,N+1]}(\phi)} \prod_{k \in \mathcal{A}\setminus\{j\}} \mathrm{d}\phi_k \prod_{k \in \{-1,\ldots,N+1\}\setminus\mathcal{A}} \delta_0(\mathrm{d}\phi_k) \Big)\, \mathrm{d}\phi_j. \tag{3.13}$$

Denote by ϕ^* the unique minimiser of $\mathcal{H}_{[-1,N+1]}$ subject to the constraints that $\phi^*(k) = 0$ for $k \in \{-1, \ldots, N+1\} \setminus \mathcal{A}$ and $\phi^*(j) = 1$. Then by homogeneity for any $y \in \mathbb{R}$ the function $y\phi^*(k)$ is the unique minimiser of $\mathcal{H}_{[-1,N+1]}$ constrained to be zero on the same set, but satisfying $y\phi^*(j) = y$. This implies that for any $\phi\colon \{-1, \ldots, N+1\} \to \mathbb{R}$ satisfying the same pinning constraint we have

$$\mathcal{H}_{[-1,N+1]}(\phi) = \mathcal{H}_{[-1,N+1]}(\phi - \phi(j)\phi^*) + \phi(j)^2 \mathcal{H}_{[-1,N+1]}(\phi^*).$$

As in the proof of Lemma 3.5 we can see that $\mathcal{H}_{[-1,N+1]}(\phi^*) = \frac{1}{2\mathrm{var}(\phi(j))}$ where $\mathrm{var}(\phi(j))$ denotes the variance of $\phi(j)$ under $\gamma^{\mathbf{0}}_{\mathcal{A}}$. This allows to rewrite (3.13) as

$$\begin{aligned} Z^{\mathbf{0}}_{\mathcal{A}} &= \int_{\mathbb{R}} \Big(\int \mathrm{e}^{-\mathcal{H}_{[-1,N+1]}(\phi - \phi(j)\phi^*)} \prod_{k \in \mathcal{A}\setminus\{j\}} \mathrm{d}\phi(k) \prod_{k \in \{-1,\ldots,N+1\}\setminus\mathcal{A}} \delta_0(\mathrm{d}\phi_k) \Big) \mathrm{e}^{-\frac{y^2}{2\mathrm{var}(\phi(j))}}\, \mathrm{d}y \\ &= Z^{\mathbf{0}}_{\mathcal{A}\setminus\{j\}} \int_{\mathbb{R}} \mathrm{e}^{\frac{-y^2}{2\mathrm{var}(\phi(j))}}\, \mathrm{d}y = Z^{\mathbf{0}}_{\mathcal{A}\setminus\{j\}} \sqrt{2\pi\, \mathrm{var}(\phi(j))}. \end{aligned}$$

As the correction map c adds at most two points to the pinned set it only remains to get an upper bound on the variance of $\phi(j)$ for $j = p_* + 1$ or $j = p^* - 1$ under $\gamma^{\mathbf{0}}_{[-1,N+1]\setminus\mathcal{P}}$, or equivalently a lower bound on $\mathcal{H}_{[-1,N+1]}(\phi^*)$; we show the argument for $p_* + 1$. It is very similar to the upper bound on the variance derived in Lemma 3.5, but this time we obtain a better bound using the fact that $p_* + 1$ is adjacent to a pinned sight. More precisely, using that $\phi^*(p_*) = 0$ and the fact that the homogenous boundary conditions imply $\phi^*(0) - \phi^*(-1) = 0$, we get

$$\begin{aligned} 1 = \phi^*(p_* + 1) - \phi^*(p_*) &= \sum_{j=0}^{p_*} \big(\phi^*(j+1) - \phi^*(j)\big) - \big(\phi^*(j) - \phi^*(j-1)\big) = \sum_{j=0}^{p_*} \Delta\phi^*(j) \\ &\leq (p_* + 1)^{\frac{1}{2}} \Big(\sum_{j=0}^{p_*} (\Delta\phi^*(j))^2 \Big)^{\frac{1}{2}} \leq N^{\frac{1}{2}} \big(\mathcal{H}_{[-1,N+1]}(\phi^*)\big)^{\frac{1}{2}}. \end{aligned}$$

This finishes the argument. □

The next Lemma provides an upper bound on the Gaussian probabilities appearing in (3.12) (still for homogeneous boundary conditions). Its proof makes use of a strong form of Gaussian large deviation bounds provided by the Gaussian isoperimetric inequality.

Lemma 3.11. *Let $g \in \mathcal{C}([0,1];\mathbb{R})$ and $\epsilon > 0$. Then there exists a $\delta \in (0, \bar{\delta})$ and an $N_0 > 0$ such that for all $N \geq N_0$ and all $\mathcal{P} \subseteq \{1, \ldots, N-1\}$*

$$\gamma^{\mathbf{0}}_{[-1,N+1]\setminus\mathcal{P}}(h_N \in B(g, \delta)) \leq \exp\big(- N(\mathcal{E}(g) - \epsilon)\big).$$

Proof. For this proof we use the notation

$$\mathcal{E}_N(h) = \frac{1}{2} \sum_{j=0}^{N} \frac{1}{N} N^4 \Big(h\Big(\frac{j+1}{N}\Big) + h\Big(\frac{j-1}{N}\Big) - 2h\Big(\frac{j}{N}\Big) \Big)^2$$

for the discrete approximations of $\mathcal{E}$. Note that according to the defining relation (1.3) between h_N and ϕ we have

$$\mathcal{H}_{[-1,N+1]}(\phi) = N\mathcal{E}_N(h_N).$$

We now recall the Gaussian isoperimetric inequality (see e.g. [Led96]) in the following convenient form: Let $N > 0$ and $\mathcal{A} \subseteq \{1, \dots, N-1\}$ and assume that $\delta > 0$ is large enough to ensure that

$$\gamma^{\mathbf{0}}_{\mathcal{A}}(h_N \in B(0, \tilde{\delta})) \geq \frac{1}{2}.$$

Then for any $r \geq 0$ we get

$$\gamma^{\mathbf{0}}_{\mathcal{A}}\Big(\Big\{h_N : \inf_{f \in \mathcal{S}_N(r^2/2)} \|h_N - f\| \geq \tilde{\delta}\Big\}\Big) \leq \Phi(r\sqrt{N}). \tag{3.14}$$

Here

$$\Phi(\sqrt{N}r) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{N}r}^{\infty} \mathrm{e}^{-\frac{x^2}{2}} dx \leq \frac{1}{\sqrt{2\pi}\sqrt{N}r} \mathrm{e}^{-\frac{Nr^2}{2}}$$

denotes the tail of the standard normal distribution and

$$\mathcal{S}_N(r^2/2) = \Big\{h \colon \mathcal{E}_N(h) \leq \frac{r^2}{2}\Big\},$$

is the sub-level set of $\mathcal{E}_N$.

The desired statement then follows if we can show that there exists a $\delta > 0$ such that on the one hand for N large enough

$$\gamma^{\mathbf{0}}_{[-1,N+1]\setminus} B(0, \delta) \tag{3.15}$$

uniformly over all $\mathcal{P}$ as well as

$$\inf_{f \in B(g, 2\delta)} \mathcal{E}_N(f) \geq \mathcal{E}(g) - \varepsilon. \tag{3.16}$$

Indeed, (3.16) implies that for all $h \in B(g, \delta)$ we have

$$\inf_{f \in \mathcal{S}_N(\mathcal{E}(g) - \varepsilon)} \|f - h\|_\infty \geq \delta$$

which permits to invoke (3.14).

The proof of the first statement (3.15) is identical to the argument in Lemma 3.5. For the second one we use Lemma A.2.

□

Conclusion. We now apply these two Lemmas to the terms appearing in the sum (3.12). For each $\mathcal{P} \neq \varnothing$ we can write

$$Z_{[-1,N+1]\setminus\mathcal{P}}\ \gamma^{\boldsymbol{r}}_{[-1,N+1]\setminus\mathcal{P}}(B(g,\delta)) = \mathrm{e}^{-\mathcal{H}_{\Lambda_N}(\phi^*_{\boldsymbol{r},N})} Z_{[-1,N+1]\setminus\mathcal{P}}(\mathbf{0})\ \gamma^{\mathbf{0}}_{[-1,N+1]\setminus\mathcal{P}}(B(g - h^*_{\boldsymbol{r},N}, \delta)),$$

where we have used (B.2) to include the boundary conditions into the Gaussian partition function. The function $\phi^*_{\boldsymbol{r},N}$ is the minimiser of $\mathcal{H}_{\Lambda_N}$ subject to the boundary conditions $\boldsymbol{r}$ and pinned on the sites in $\mathcal{P}$. The profile $h^*_{\boldsymbol{r},N}$ is the rescaled version of $\phi^*_{\boldsymbol{r},N}$. Using the previous two Lemmas as well as Lemma A.2, which permits to rewrite $\mathcal{H}_{\Lambda_N}(\phi^*_{\boldsymbol{r},N}) \geq \mathcal{E}(h^*_{\boldsymbol{r},N}) - \epsilon$, we bound the last expression by

$$\begin{aligned}
&\leq \exp\big(-N(\mathcal{E}(h^*_{\boldsymbol{r},N}) - \epsilon\big)(2\pi N) Z_{[-1,N+1]\setminus c(\mathcal{P})}(\mathbf{0}) \exp\big(-N(\mathcal{E}(g - h^*_{\boldsymbol{r},N}) - \epsilon)\big) \\
&\leq (2\pi N) Z_{[-1,N+1]\setminus c(\mathcal{P})}(\mathbf{0}) \exp\big(-N(\mathcal{E}(g) - 2\epsilon)\big).
\end{aligned}$$

Plugging this into (3.12) we obtain

$$\frac{Z_{N,\varepsilon}(\boldsymbol{r})}{Z_N(\mathbf{0})} \gamma^{\boldsymbol{r}}_{N,\varepsilon}(B(g,\delta)) \leq (2\pi N) \exp\big(-N(\mathcal{E}(g) - 2\epsilon)\big) \sum_{\mathcal{P}} \varepsilon^{|\mathcal{P}|} \frac{Z_{[-1,N+1]\setminus c(\mathcal{P})}(\mathbf{0})}{Z_N(\mathbf{0})}.$$

The sum appearing in this expression can be rewritten as

$$\begin{aligned}
\sum_{\mathcal{P}} \varepsilon^{|\mathcal{P}|} \frac{Z_{[-1,N+1]\setminus \mathcal{P}}(\mathbf{0})}{Z_N(\mathbf{0})} &\leq 4 \sum_{\mathcal{P}\text{ good}} \varepsilon^{|\mathcal{P}|} \frac{Z_{[-1,N+1]\setminus \mathcal{P}}(\mathbf{0})}{Z_N(\mathbf{0})} \leq 4 \sum_{\ell N \leq k_1 \leq k_2 \leq rN} \sum_{\substack{\mathcal{P}\text{ good} \\ p_* = k_1 \\ p^* = k_2}} \varepsilon^{|\mathcal{P}|} \frac{Z_{[-1,N+1]\setminus \mathcal{P}}(\mathbf{0})}{Z_N(\mathbf{0})} \\
&= 4 \sum_{\ell N \leq k_1 \leq k_2 \leq rN} \sum_{\substack{\mathcal{P}\text{ good} \\ p_* = k_1 \\ p^* = k_2}} \varepsilon^{|\mathcal{P}|} \frac{Z_{[-1,p_*+1]}(\mathbf{0}) Z_{[p_*,p^*]\setminus\mathcal{P}}(\mathbf{0})\; Z_{[p^*-1,N+1]}(\mathbf{0})}{Z_N(\mathbf{0})}.
\end{aligned}$$

As in the proof of the upper bound the inner sum can be rewritten as

$$\begin{aligned}
&\sum_{\substack{\mathcal{P}\text{ good} \\ p_* = k_1 \\ p^* = k_2}} \varepsilon^{|\mathcal{P}|} \frac{Z_{[-1,p_*+1]}(\mathbf{0}) Z_{[p_*,p^*]\setminus\mathcal{P}}(\mathbf{0})\; Z_{[p^*-1,N+1]}(\mathbf{0})}{Z_N(\mathbf{0})} \\
&\qquad = \frac{Z_{p^*-p_*,\varepsilon}}{Z_{p^*-p_*}} \frac{Z_{[-1,p_*+1]}(\mathbf{0}) Z_{p^*-p_*}(\mathbf{0})\; Z_{[p^*-1,N+1]}(\mathbf{0})}{Z_N(\mathbf{0})}.
\end{aligned}$$

As the number of terms in the outer sum is bounded by N^2, and as the Gaussian partition functions only grow polynomially (see Appendix B), we obtain the desired conclusion from (1.7) by employing uniformity which follows from the fact that we are summing over different members of the same converging sequence.

Proofs of Lemmas.

Proof of Lemma 3.6. Due to the Arzelà-Ascoli Theorem it suffices to show that

$$\limsup_{M\to\infty} \limsup_{N\to\infty} \frac{1}{N} \log\Big(\gamma^{\boldsymbol{r}}_{N,\varepsilon}\big(h \in \mathcal{C}([0,1];\mathbb{R}) \colon \|\dot{h}\|_\infty \geq M\big)\Big) = -\infty \tag{3.17}$$

As the measure $\gamma^{\boldsymbol{r}}_{N,\varepsilon}$ is a convex combination of the Gaussian measures $\gamma^{\boldsymbol{r}}_{\mathcal{A}}$ we start by establishing that each of these measures satisfies this bound. For any fixed choice of pinning sites $\mathcal{P} \subseteq \{0, \dots, N\}$ we have

$$\begin{aligned}
\gamma^{\boldsymbol{r}}_{\mathcal{A}}\big(h \in \mathcal{C} \colon \|\dot{h}\|_\infty \geq M\big) &= \gamma^{\mathcal{A}}_{\boldsymbol{r}}\big(\phi \colon |\phi(k+1) - \phi(k)| \geq NM \text{ for at least one } k \in \{0, \dots, N\}\big) \\
&\leq \sum_{k=0}^{N} \gamma^{\boldsymbol{r}}_{\mathcal{A}}\big(\phi \colon |\phi(k+1) - \phi(k)| \geq NM\big) \\
&\leq (N+1) \sup_{k=0}^{N} \exp\Big(-\frac{(NM)^2}{2\mathrm{var}(\phi_k)}\Big)
\end{aligned}$$

As the variances of the $\phi(k+1)-\phi(k)$ are bounded by N (as seen in Lemma 3.5) we can conclude. $\square$

Proof of Lemma 3.7. By sigma-additivity of the Lebesgue measure there exists a $\delta>0$ such that

$$|\{x\in[0,1]\colon |g(x)|<2\delta\}|\le|\{x\in[0,1]\colon g(x)=0\}|+\varepsilon.$$

For any $x\in\{x\in[0,1]\colon |g(x)|<2\delta\}$ there exists a $\rho_x>0$ such that the ball $B(x,\rho_x)\cap[0,1]$ is still contained in this set. By compactness there exists a finite set $\{x_1,\dots,x_{\tilde M}\}$ such that

$$\bigcup_{j=1}^{\tilde M}B(x_j,\rho_{x_j})\supseteq\{x\in[0,1]\colon |g(x)|\le\delta\}.$$

We then set $\mathcal{I}=\overline{\cup_{j=1}^{\tilde M}B(x_j,\rho_{x_j})}\cap[0,1]$ and claim that this set has the desired properties. Indeed, the union of finitely many open intervals can always be written as the union of a (potentially smaller number of) *disjoint* open intervals. The closure of such a set is the union of a finite (again, potentially smaller) number of disjoint closed intervals. The set $\mathcal{I}$ contains $\{x\in[0,1]\colon |g(x)|\le\delta\}$ by construction. Furthermore

$$\bigcup_{j=1}^{\tilde M}B(x_j,\rho_{x_j})\cap[0,1]\subseteq\{x\in[0,1]\colon |g(x)|<2\delta\}$$

which implies that the measure of this set is bounded by $|\{x\in[0,1]\colon g(x)=0\}|+\varepsilon$. Adding a finite number of boundary points does not change the Lebesgue measure, so that $\mathcal{I}$ satisfies the same bound. $\square$

Appendix

A. Energy minimiser

We outline the standard solution for the variational problem of minimising the *energy functional*

$$\mathcal{E}(h)=\frac{1}{2}\int_0^1\ddot h^2(t)\,\mathrm{d}t\qquad\text{for }h\in H^2_{\boldsymbol r},\tag{A.1}$$

where $\boldsymbol r=(a,\alpha,b,\beta)$.

Proposition A.1. *The variational problem, minimise $\mathcal{E}$ in $H^2_{\boldsymbol r}$, has a unique solution denoted by $h^*_{\boldsymbol r}\in H^2_{\boldsymbol r}$ and given as*

$$h^*_{\boldsymbol r}(t)=a+\alpha t+k(\boldsymbol r)t^2+c(\boldsymbol r)t^3,\quad t\in[0,1],$$

with

$$k(\boldsymbol r)=3(b-a)-2\alpha-\beta,\qquad\textit{and } c(\boldsymbol r)=(\alpha+\beta)-2(b-a).$$

Furthermore, $\mathcal{E}(h^*_{\boldsymbol r})=(2k(\boldsymbol r)^2+6k(\boldsymbol r)c(\boldsymbol r)+6c(\boldsymbol r)^2)$.

Proof. Let $D_{\boldsymbol r}=\{h\in L_2([0,1])\colon\text{ exists }f\in H^2_{\boldsymbol r}\text{ s.t. }\ddot f=h\}$. Then

$$\langle h,g-h\rangle_{L_2}=0\qquad\text{for all }g\in D_{\boldsymbol r}$$

follows for all h with $h^{(4)} \equiv 0$. For $f \in H^2_{\boldsymbol{r}}$ let $g \in D_{\boldsymbol{r}}$ be given by $g = \ddot{f}$. Then for any $h \in H^2_{\boldsymbol{r}}$ with $h^{(4)} \equiv 0$ we obtain,

$$\mathcal{E}(f) = \frac{1}{2}\langle g, g\rangle_{L_2} \geq \frac{1}{2}\langle \ddot{h}, \ddot{h}\rangle_{L_2} = \mathcal{E}(h).$$

We obtain the uniqueness by convexity and conclude with noting that $(h^*_{\boldsymbol{r}})^{(4)} \equiv 0$, see [Mit13] for an overview of bi-harmonic solutions. $\square$

We now consider the quadratic potential $V(\eta) = \frac{1}{2}\eta^2$ and we shall minimise the Hamiltonian (1.1) in $\Lambda_N := \{-1, 0, \ldots, N, N+1\}$ over the set

$$\Omega^N_{\boldsymbol{r}} = \{\phi \in \mathbb{R}^{\Lambda_N} \colon \phi(-1) = \psi^{(N)}(-1), \phi(0) = \psi^{(N)}(0), \phi(N) = \psi^{(N)}(N), \phi(N+1) = \psi^{(N)}(N+1)\}$$

of configurations with given boundary conditions.

Lemma A.2. *For any $N \in \mathbb{N}$ let $h_N \colon \big\{-\frac{1}{N}, 0, \ldots, 1, \frac{N+1}{N}\big\} \to \mathbb{R}$ be given with boundary values $\boldsymbol{r} = (a, \alpha, b, \beta)$ i.e.*

$$h_N(0) = a, \quad h_N(1) = b, \quad N\Big(h_N(0) - h_N\Big(-\frac{1}{N}\Big)\Big) = \alpha, \quad N\Big(h_N\Big(1 + \frac{1}{N}\Big) - h_N(1)\Big) = \beta.$$

We interpolate h_N linearly between the grid-points. Furthermore set

$$\mathcal{E}_N(h_N) = \frac{1}{2}\sum_{j=0}^{N} N^3\Big[h_N\Big(\frac{j+1}{N}\Big) - h_N\Big(\frac{j-1}{N}\Big) + 2h_N\Big(\frac{j}{N}\Big)\Big]^2.$$

Then if h_N converges uniformly over $[0,1]$ to a function h we have

$$\liminf_{N\to\infty} \mathcal{E}_N(h_N) \geq \mathcal{E}(h) = \frac{1}{2}\int_0^1 (h''(t))^2 \,\mathrm{d}t.$$

Proof. We fix a subsequence (N_k) along which $\mathcal{E}_N(h_N)$ converges to $\liminf_{N\to\infty}\mathcal{E}_N(h_N)$ which we can assume to be finite without loss of generality. Along this sequence $\mathcal{E}_N(h_N)$ is bounded. We drop the extra-index k and assume from now on that

$$\sup_N \mathcal{E}_N(h_N) = \bar{C} < \infty. \tag{A.2}$$

We will first consider discrete derivatives of h_N. For $j = -1, \ldots, N$, we set

$$g_N\Big(\frac{j}{N}\Big) = N\Big[h_N\Big(\frac{j+1}{N}\Big) - h_N\Big(\frac{j}{N}\Big)\Big], \tag{A.3}$$

and as before we interpret g_N as a function $[-\frac{1}{N}, 1] \to \mathbb{R}$ by linear interpolation between the grid-points. Note that h_N corresponds to the configuration $\phi_{h_N}(x) = N^2 h_N(x/N)$ for $x \in \Lambda_N$. Thus the Hamiltonian $\mathcal{H}_{\Lambda_N}(h_N)$ and therefore the functional $\mathcal{E}_N(h_N)$ can be re-expressed in terms of g_N as

$$\mathcal{H}_{\Lambda_N}(\phi_{h_N}) = \mathcal{E}_N(h_N) = \frac{1}{2}\sum_{j=0}^{N} N\Big[g_N\Big(\frac{j}{N}\Big) - g_N\Big(\frac{j-1}{N}\Big)\Big]^2 = \frac{1}{2}\int_{-\frac{1}{N}}^{1} g_N'(t)^2 \,\mathrm{d}t.$$

So, (A.2) immediately implies the uniform Hölder bound

$$|g_N(t) - g_N(s)| \leq \int_s^t |g_N'(r)| \,\mathrm{d}r \leq |t-s|^{\frac{1}{2}}\Big(\int_{-\frac{1}{N}}^{1} g_N'(r)^2 \,\mathrm{d}r\Big)^{\frac{1}{2}} \leq |t-s|^{\frac{1}{2}}\sqrt{2\bar{C}} \tag{A.4}$$

for $-\frac{1}{N} \leq s < t \leq 1$. We introduce the (slightly) rescaled function $\tilde{g}_N \colon [0,1] \to \mathbb{R}$ defined as

$$\tilde{g}_N(t) = g_N\Big(\frac{N}{N+1}\Big(t + \frac{1}{N}\Big)\Big)$$

and observe that

$$\int_0^1 \tilde{g}_N'(t)^2 dt = \frac{N}{N+1}\int_{-\frac{1}{N}}^1 g_N'(t)^2\,\mathrm{d}t \leq 2\bar{C}.$$

Observing that $\tilde{g}_N(1) = \beta$ we can conclude that there is a subsequence N_k along which $\tilde{g}_{N_k}$ converges weakly in $H^1([0,1])$ to a function g which satisfies

$$\int_0^1 g'(t)^2\,\mathrm{d}t \leq \liminf_{N\to\infty} \int \tilde{g}_{N_k}'(t)^2\,\mathrm{d}t = \liminf_{N\to\infty} \frac{N}{N+1}\int g_{N_k}'(t)^2\,\mathrm{d}t = \liminf_{N\to\infty} \mathcal{E}(h_N).$$

Thus, the desired statement follows as soon as we have established that for all $x \in [0,1]$,

$$h(x) = a + \int_0^x g(t)\,\mathrm{d}t,$$

because then we get $\frac{1}{2}\int_0^1 h''(t)^2\,\mathrm{d}t = \frac{1}{2}\int_0^1 g'(t)^2\,\mathrm{d}t$. To see this we rewrite the defining relation (A.3) of g_N for any N and any $x \in [\frac{j}{N}, \frac{j+1}{N}]$, $j \geq 0$, as

$$h_N(x) = a + \sum_{k=0}^{j-1}\frac{1}{N}g_N(k) + \frac{(Nx-j)}{N}g_N(j) = a + \int_0^x \tilde{g}_N(t)\,\mathrm{d}t + E_N, \tag{A.5}$$

where the error term E_N satisfies

$$E_N \leq \Big|\int_0^x g_N(t)\,\mathrm{d}t - \int_0^x \tilde{g}_N(t)\,\mathrm{d}t\Big| + \Big|\int_0^x \Big(g_N(t) - g_N\Big(\frac{\lfloor tN\rfloor}{N}\Big)\Big)\,\mathrm{d}t\Big|.$$

The definition of $\tilde{g}_N$ together with a uniform boundedness of g_N in $L^1([0,1])$ imply that the first term converges to zero as $N \to \infty$ while the second term can be seen to go to zero by the uniform Hölder bound (A.4). We can then conclude by going back to (A.5) and noting that on the one hand $h_N(x)$ converges to $h(x)$ by assumption and that on the other hand the weak convergence of $\tilde{g}_N$ in $H^1([0,1])$ implies that $\int_0^x \tilde{g}_N(t)\,\mathrm{d}t$ converges to $\int_0^x g(t)\,\mathrm{d}t$.

$\square$

Proposition A.3. *For any $N \in \mathbb{N}, N > 2$, the variational problem, minimise $\mathcal{H}_{\Lambda_N}$ in $\Omega_{\boldsymbol{r}}^N$ has a unique bi-harmonic solution $\phi_{\boldsymbol{r},N}^* \in \Omega_{\boldsymbol{r}}^N$ satisfying*

$$\begin{cases} \Delta^2\phi_{\boldsymbol{r},N}^*(x) = 0 & \text{for } x \in \{0,1,\dots,N\}, \\ \phi_{\boldsymbol{r},N}^*(x) = \psi^{(N)}(x) & \text{for } x \in \partial\Lambda_N = \{-1,0,N,N+1\}. \end{cases} \tag{A.6}$$

Let $h_{\boldsymbol{r},N}^*(\xi) := \frac{1}{N^2}\phi_{\boldsymbol{r},N}^*(\xi N)$ for $\xi \in \{-\frac{1}{N}, 0, \frac{1}{N}, \dots, 1, \frac{N+1}{N}\}$, then $h_{\boldsymbol{r},N}^*$ Γ-converges to $h_{\boldsymbol{r}}^*$ as $N \to \infty$. Moreover,

$$\frac{1}{N}\mathcal{H}_{\Lambda_N}(\phi_{\boldsymbol{r},N}^*) \longrightarrow \frac{1}{2}\mathcal{E}(h_{\boldsymbol{r}}^*) \quad \text{as } N \to \infty.$$

Proof. Clearly, the Euler-Lagrange equations are equivalent with (A.6) and thus

$$\mathcal{H}_{\Lambda_N}(\phi) \geq \langle \phi_{\boldsymbol{r},N}^*, \phi_{\boldsymbol{r},N}^*\rangle_{\mathbb{R}^{\Lambda_N}}.$$

Similar to Proposition A.1 one can show that the unique minimiser $\phi^*_{\boldsymbol{r},N} \in \Omega^N_{\boldsymbol{r}}$ is a polynomial of order three such that

$$h^*_{\boldsymbol{r},N}(t) = a_N + \alpha_N t + k_N(\boldsymbol{r})t^2 + c_N(\boldsymbol{r})t^3, \qquad t \in \{-\frac{1}{N}, 0, \frac{1}{N}, \dots, 1, \frac{N+1}{N}\},$$

with

$$\begin{aligned} a_N &= a; \quad \alpha_N = \frac{2b - a(2+3N) + N(3b + \alpha(N+1) - \beta)}{(N+1)(N+2)}; \\ k_N(\boldsymbol{r}) &= N\frac{(-\alpha + \beta + N(3(b-a) - 2\alpha - \beta))}{(N+1)(N+2)}; \\ c_N(\boldsymbol{r}) &= N^2\frac{(2(a-b) + \alpha + \beta)}{(N+1)(N+2)}. \end{aligned}$$

This implies the convergence of $h^*\boldsymbol{r}, N$ to $h^*_{\boldsymbol{r}}$. Thus we have established the limit for a recovery sequence and the statement for the Γ-convergence follows in conjunction with Lemma A.2. To see the convergence of the minimal energies note that for any N the function $h^*_{\boldsymbol{r},N}$ is the unique minimiser for the functional $\mathcal{E}_N$, see proof of Lemma A.2.

□

B. Partition function

We collect some known results about the partition function for the case with no pinning (see [Bor10] and [BS99]). The partition function with zero boundary condition $\boldsymbol{r} = \boldsymbol{0}$ is

$$\begin{aligned} Z_N(\boldsymbol{0}) = \int \mathrm{e}^{-\mathcal{H}_{[-1,N+1]}(\phi)} \prod_{k=1}^{N-1} \mathrm{d}\phi_k \prod_{k\in\{-1,0,N,N+1\}} \delta_0(\mathrm{d}\phi_k) &= \int_{\mathbb{R}^{N-1}} \mathrm{e}^{-\frac{1}{2}\langle w, B_{N-1}w\rangle} \prod_{i=1}^{N-1} \mathrm{d}w_i \\ = \Big(\frac{(2\pi)^{N-1}}{\det(B_{N-1})}\Big)^{1/2} = \frac{1}{12}(2 + (N-1))^2(3 + 4(N-1) + (N-1)^2), \end{aligned} \tag{B.1}$$

where the matrix B_{N-1} reads as

$$\begin{pmatrix} 6 & -4 & 1 & 0 & \cdots & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ 0 & 1 & -4 & 6 & -4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 1 & -4 & 6 \end{pmatrix}.$$

We can easily obtain the following relation for the partition functions with given boundary $\boldsymbol{r}$ (via $\psi^{(N)}$) and zero boundary condition $\boldsymbol{r} = \boldsymbol{0}$ for models without pinning.

$$Z_N(\boldsymbol{r}) = \exp\big(-\mathcal{H}_{\Lambda_N}(\phi^*_{\boldsymbol{r},N})\big) Z_N(\boldsymbol{0}). \tag{B.2}$$

Acknowledgments

S. Adams thanks J.D. Deuschel for discussions on this problem as well T. Funaki and H. Sakagawa for discussions on Laplacian models and the warm hospitality during his visit in July 2015.

References

[Ant05] S. Antman, *Nonlinear Problems in elasticity*, Applied Mathematical Sciences **107**, 2nd ed., Springer (2005).

[BFO09] E. Bolthausen, T. Funaki and T. Otobe, Concentration under scaling limits for weakly pinned Gaussian random walks, Probab. Theory Relt. Fields **143**:441-480 (2009).

[BCF14] E. Bolthausen, T. Chiyonobu and T. Funaki, Scaling limits for weakly pinned Gaussian random fields under the presence of two possible candidates, ArXiv: 1406.7766v1 (preprint) (2014).

[Bor10] M. Borecki, Pinning nad Wetting Models for Polymers with $(\nabla + \Delta)$-Interaction, PhD thesis TU-Berlin, (2010).

[BLL00] R. Bundschuh, M. Lässig and R. Lipowsky, Semiflexible polymers with attractive Interactions, Eur. Phys. J. E **3**, 295-306 (2000).

[BS99] A. Böttcher and B. Silbermann, *Introduction to large truncated Toeplitz matrices*, Springer-Verlag, (1999).

[CD08] F. Caravenna and J.D. Deuschel, Pinning and Wetting Transition for $(1+1)$-dimensional Fields with Laplacian interaction, Annals Probab. **36**, No. 6, 2388-2433 (2008).

[DS89] J.-D. Deuschel and D. Stroock, *Large Deviations*, American Mathematical Society, (1989).

[CD09] F. Caravenna and J.D. Deuschel, Scaling limits of $(1 + 1)$-dimensional pinning models with Laplacian interaction, Annals Probab. **37**, No. 3, 903–945 (2009).

[DZ98] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, 2nd edition, Springer, Berlin (1998).

[F] T. Funaki, Stochastic Interface Models, in: J. Picard (ed.), Lectures on probability theory and statistics, Lecture Notes in Mathematics, vol. **1869**, Springer-Verlag, Berlin (2005).

[FO10] T. Funaki and T. Otobe, Scaling limits for weakly pinned random walks with two large deviation minimizers, J. Math. Soc. Japan Vol. **62**, No. 3, 1005-1041 (2010).

[FS04] T. Funaki and H. Sakagawa, Large deviations for $\nabla\varphi$ Interface models and derivation of free boundary problems, Advanced Studies in Pure Mathematics **39**, 173–211 (2004).

[GSV05] D. Gasbarra, T. Sottinen and E. Valkeila, in Stochastic Analysis and Applications: The Abel Symposium 2005 edited by Fred Espen Benth, Giulia Di Nunno, Tom Lindstrom, Bernt Øksendal, Tusheng Zhang, 361–382, Springer (2007).

[HV09] O. Hryniv and Y. Velenik, Some Rigorous Results about Semiflexible Polymers I. Free and Confined Polymers, Stoch. Proc. Appl. **119**, 3081-3100 (2009).

[Led96] M. Ledoux, Isoperimetry and Gaussian analysis, Lectures on probability theory and statistics, Springer, 165-294 (1996).

[Mit13] D. Mitrea, *Distributions, Partial Differential Equations, and Harmonic analysis*, Springer Universitext (2013).

[Mog76] A.A. Mogul'skii, Large deviations for trajectories of multi-dimensional random walks, Theory Probab. Appl., **21**, 300-315 (1976).

[Sin92] Y.G. Sinai, Distribution of some functionals of the integral of a random walk, Teoret. Mat. Fiz. **90**, 323–353 (1992).

Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

E-mail address: S.Adams@warwick.ac.uk