Glauber dynamics of 2D Kac-Blume-Capel model and their stochastic PDE limits

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Abstract

We study the Glauber dynamics of a two dimensional Blume-Capel model (or dilute Ising model) with Kac potential parametrized by (β, θ) - the "inverse temperature" and the "chemical potential". We prove that the locally averaged spin field rescales to the solution of the dynamical Φ^4 equation near a curve in the (β, θ) plane and to the solution of the dynamical Φ^6 equation near one point on this curve. Our proof relies on a discrete implementation of Da Prato-Debussche method [DPD03] as in [MW16] but an additional coupling argument is needed to show convergence of the linearized dynamics.

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1 Introduction

The theory of singular stochastic partial differential equations (SPDEs) has witnessed enormous progress in the last years. Most prominently, Hairer's work on regularity structures [Hai14] allowed to develop a stable notion of solution for a large class of SPDEs which satisfy a scaling condition called *subcriticality*. Roughly speaking, a semi-linear SPDE equation is subcritical (or super-renormalizable), if the behaviour of solutions on small scales is dominated by the evolution of the linearized Gaussian dynamics. The class of subcritical equations includes, for example, the KPZ equation in one spatial dimension, as well as reaction diffusion equations with polynomial nonlinearities

$$dX = (\Delta X + \sum_{k=1}^{n} \mathfrak{a}_{2k-1} X^{2k-1})\, dt + dW \qquad \mathfrak{a}_{2n-1} < 0 \tag{1.1}$$

driven by a space time white noise dW, if the space dimension d satisfies $d < \frac{2n}{n-1}$ (of course strictly speaking the dimension d has to be an integer but one could emulate fractional dimensions by adjusting the linear operator or the covariances of the noise). In particular, for $d = 3$, equation (1.1) is only subcritical for the exponent $2n - 1 = 3$ while for $d = 2$, equation (1.1) is subcritical for all n. We will refer to these equations as dynamical Φ^4_3 and Φ^{2n}_2 equations. Note that even in the subcritical case the expression (1.1) has to be interpreted with caution: for $d \geq 2$ a renormalization procedure which amounts to subtracting one or several infinite terms has to be performed. The fact that these solutions behave like the linearized dynamics on small scales but very nontrivially on large scales is related with the role they play in the description of crossover regimes between universality classes in statistical physics. For example, the KPZ equation describes the crossover regime between the Edwards-Wilkinson (Gaussian) fixed point and the "KPZ fixed point", while the dynamical Φ^4 equation describes such a crossover mechanism between the Gaussian and the "Wilson-Fisher fixed point". In two space dimensions the existence of infinitely many fixed points was predicted by conformal field theory, and the Φ^{2n}_2 equations should describe the crossover regimes between the Gaussian and this family of fixed points ([FFS92, Fig. 4.3]).

One key interest when studying these SPDEs is to understand how they arise as scaling limits of various microscopic stochastic systems. Here it is important to note that the equations are *not* scale invariant themselves (this is immediate from subcriticality). However, they arise as scaling limits of systems with tunable model parameters that are modified as the system is rescaled. Starting with Bertini and Giacomin's famous result [BG97] on the convergence of the weakly asymmetric simple exclusion process to the KPZ equation, by now many results in this direction have been obtained for the KPZ equation (for example [ACQ11, DT16, CT15, CST16, Lab16] based on the Cole-Hopf transform, [GJ14, GJ16, DGP16] based on the notion of energy solution, and [HQ15, HS15] based on regularity structures). Connections between the stationary Φ^4_2 theory and Ising-like models were already observed in the seventies; early references include [SG73] where the equilibrium Φ^4_2 theory was obtained from an Ising-like model by a two-step limiting procedure. The dynamical equation (1.1) in one dimension was obtained as a scaling limit for a dynamic Ising model with Kac interation in the nineties [BPRS93, FR95]. More precisely, the Kac Ising model is a spin model taking values in the $\{\pm 1\}$ valued configurations over a graph ($\mathbb{Z}$ or a subinterval of $\mathbb{Z}$ in the case of [BPRS93, FR95]). The static equilibrium model is given as the Gibbs measures associated to the Hamiltonian

$$\mathscr{H}_\gamma(\sigma) = -\frac{1}{2} \sum_{k,j} \kappa_\gamma(k-j)\, \sigma(j)\, \sigma(k), \tag{1.2}$$

where κ_γ is a non-negative interaction kernel parametrised by $\gamma > 0$ which determines the interaction range between spins. In [BPRS93, FR95] the Glauber dynamics for this model were considered and it was shown that the locally averaged field $h_\gamma = \sigma * \kappa_\gamma$ converges in law to a solution to the Φ^4_1 equation when suitably rescaled. Similar results in higher dimensions $d = 2, 3$ were conjectured in [GLP99] but a complete proof in the two dimensional case was given only recently [MW16]. A similar convergence result is expected to hold in three dimensions, though a complete proof has not been established yet; however in [HX16, SX16] it was shown that a class of continuous phase coexistence models rescale to Φ^4_3. [1]

[1] In [HX16] also different limits such as a dynamical Φ^3_3 theory, which may blow up in finite time where obtained, but in order to achieve this the $\sigma \mapsto -\sigma$ symmetry in the model had to be broken.

The tunable parameter in all of the results on convergence of variants of the asymmetric simple exclusion process to KPZ, is the asymmetry of the exclusion process: making it smaller and smaller corresponds to making the model locally more "Gaussian" which in turn corresponds to the fact that the dynamics on small scales are dominated by solutions of the linear equation. In the Kac-Ising case this tunable parameter is the range of the interaction kernel κ_γ. As the system is observed on larger and larger scales locally more and more particles interact i.e. locally the system is closer to mean field.

In order to obtain the scaling limit to Φ^4_2 in [MW16] five parameters had to be chosen in a certain way: three "scaling parameters" namely the space scaling, the time scaling, the rescaling of the field as well as two "model parameters", the range of the Kac interaction and the temperature. It turns out that in order to obtain a non-linear scaling SPDE as scaling limit, one has to choose the temperature close to the mean field critical value, although in two dimensions there is a small shift which corresponds to the renormalization procedure for the limiting equation, and a similar effect is expected in three dimensions. The remaining parameters have to be tuned in exactly the right way to balance all terms in the equation. It is natural to expect that in two space dimensions introducing additional parameters should allow to balance even more terms leading to higher order terms in the equation. In this work we show that this is indeed the case. We allow for microscopic spin to take values in $\{\pm 1, 0\}$ i.e. we add the possibility of a spin value 0. The Hamiltonian thus becomes:

$$\mathscr{H}_\gamma(\sigma) = -\frac{1}{2}\sum_{k,j} \kappa_\gamma(k-j)\,\sigma(j)\,\sigma(k) - \tilde{\theta}\sum_j \sigma(j)^2, \tag{1.3}$$

where the extra parameter $\tilde{\theta}$ plays a role of chemical potential which describes a ratio of the number of "magnetized" spins ($\sigma(j) \neq 0$) over the number of "neutral" spins ($\sigma(j) = 0$). In the limit $\tilde{\theta} \to \infty$ we recover the original Kac-Ising model.

This model is the (Kac version) of the Blume-Capel model (initially proposed by [Blu66, Cap66]). This Blume-Capel model as well as the closely related (but slightly more complex) "Blume-Emery-Griffiths" (BEG) model [BEG71] have been widely used to describe "multi-critical" phenomena in equilibrium physics. Physicists also studied phase transitions for the Glauber type dynamics of mean field BEG model [CDK06]. Mathematically, the mean field model in equilibrium was studied by in series of papers [EOT05, CEO07, EMO10] (see more references therein), analyzed the phase diagrams and proved that the suitably rescaled total spin converges to a random variable which is distributed with density Ce^{-cx^2}, Ce^{-cx^4} or Ce^{-cx^6} in different regimes. Also, the work [EM14] obtained the rates of these convergences. Regarding the dynamics, mixing theorems are also proved, see [KOT11, EKLV14]. The Blume-Capel model is also often referred as the (site) dilute Ising model (c.f. for instance the physics book [FMS12, Section 7.4.3] or on the mathematical side [HSS00, CKS95] and references therein): one considers the site percolation of the square lattice with percolation probability p and the usual Ising model on the percolation clusters. The joint measure of the percolation and Ising model is then the Gibbs measure with Hamiltonian (1.3) if one identifies $e^{\beta\tilde{\theta}} = (1-p)^{-1} - 1$. The Glauber dynamics are then defined on both percolation and Ising configurations. The results of this article can then be stated as convergence to the SPDEs by suitable tuning the Ising temperature and percolation probability.

Our main result, Theorem 2.5, shows that for a one parameter family of parameters we obtain the Φ^4_2 equation in the scaling limit. This family ends at a "tricritical point" where (after different rescaling) we get the Φ^6_2 equation (see Figure 1). Our equation for this curve of parameters and the value of the tricritical point coincide with the mean field results

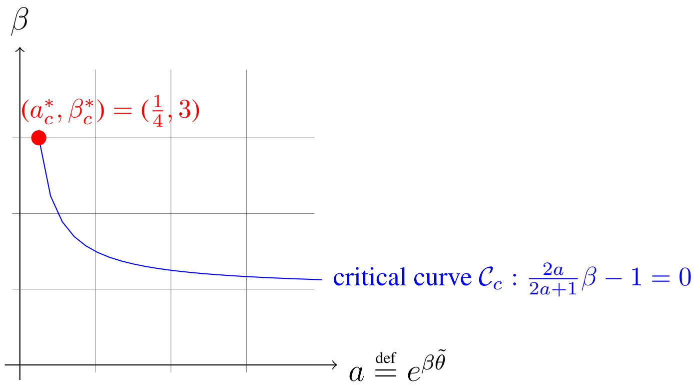

Figure 1: The Glauber dynamic of Blume-Capel model rescales to the Φ^4_2 equation for a curve of parameters in the $(\tilde{\theta}, \beta)$ plane, parametrized here in terms of $(a = e^{\beta\tilde{\theta}}, \beta)$. The leading coefficient of the non-linearity in the limiting equation changes along the curve and vanishes at the tricritical point (a_c^*, β_c^*). Close to this point a different rescaling leads to the Φ^6_2 equation. Following the curve beyond this point would lead to a change of sign in the leading order term resulting in finite time blowup of the corresponding SPDE.

in [BEG71], but as in the [MW16] logarithmic corrections to these mean field values are necessary to obtain the convergence results. These logarithmic corrections correspond exactly to the "logarithmic infinities" that appear in the renormalization procedures for the limiting equation.

Meta-theorem 1.1 *Let $h_\gamma = \kappa_\gamma * \sigma$ be the locally averaged spin field of the Glauber dynamic of Kac-Blume-Capel model. There exist a one parameter family of "critical values" and one "tri-critical value", such that when (β, θ) approaches a critical value at a suitable rate (which reflects the renormalization procedure for the limiting equation), $X_\gamma(t, x) = \gamma^{-1} h_\gamma(t/\gamma^2, x/\gamma^2)$ converges to the solution of the dynamical Φ^4 equation, and when (β, θ) approaches the tri-critical value at a suitable rate, $X_\gamma(t, x) = \gamma^{-1} h_\gamma(t/\gamma^4, x/\gamma^3)$ converges to the solution of the dynamical Φ^6 equation.*

It seems natural to conjecture that if one makes the model more complex (e.g. by allowing even more general spins and extra interaction terms in the Hamiltonian) any Φ^{2n}_2 model could be obtained.

On a technical level just as [MW16] our method relies on a discretization of Da Prato-Debussche's solution theory for (1.1) in two dimensions [DPD03]. A main step is to prove convergence in law (with respect to the right topology) for the linearized dynamics as well as suitably defined "Wick powers" of these linearizations. In a second step this is then put into discretization of the "remainder equation" and tools from harmonic analysis are used to control the error. The most striking difference in the present work with respect to the technique in [MW16] is a difficulty to describe the fluctuation characteristics. In [MW16] the quadratic variation of the martingale M_γ (see (2.11) below for its definition) is equal to a deterministic constant up to an small error which can be controlled with a soft method. In the framework of the present paper this is not true anymore, and the quadratic variation has to be averaged over large temporal and spatial scales to characterize the noise in the limiting equation as white noise. We implement this averaging by coupling the spin field

$\sigma(t,k)$ to a much simpler field $\tilde{\sigma}(t,k)$ which can be analyzed directly. This auxiliary process lacks the subtle large scale effects of σ captured in our main result, but it has similar local jump dynamics and it turns out that $\sigma(t,k)$ coincides with $\tilde{\sigma}(t,k)$ for many t and k which is enough.

The structure of the paper is as follows. In Section 2 we discuss the two scaling regimes of our model and formally derive the limiting equation in each regime. Section 3 is mainly aimed to show the convergence of the linearized equation. It is here that we present the coupling argument used to show the averaging of the martingale fluctuation. Section 4 contains the rest of argument (the discrete Da Prato-Debussche method etc.). This part of the argument is close to [MW16], but one difference with respect to [MW16] is the replacement of the L^∞ norm used there by an L^p norm which becomes necessary because of an error term which arises in the coupling argument and which is only controlled in L^p.

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2 Model, formal derivations and main result

The (Kac-)Blume-Capel model in equilibrium is defined as a Gibbs measure λ_γ on the configuration space $\Sigma_N = \{-1, 0, +1\}^{\Lambda_N}$ with $\Lambda_N = \mathbb{Z}^2/(2N+1)\mathbb{Z}^2$ being the two-dimensional discrete torus of size $2N+1$. More precisely

$$\lambda_\gamma(\sigma) \stackrel{\text{def}}{=} \frac{1}{\mathscr{Z}_\gamma} \exp\Big(-\beta \mathscr{H}_\gamma(\sigma)\Big)\,,$$

where $\beta > 0$ is the inverse temperature, and $\mathscr{Z}_\gamma$ denotes the normalization constant that is equal to the sum of the exponential weights over all configurations $\sigma \in \Sigma_N$. The Hamiltonian $\mathscr{H}_\gamma$ of the model is defined via

$$\mathscr{H}_\gamma(\sigma) \stackrel{\text{def}}{=} -\frac{1}{2} \sum_{k,j\in\Lambda_N} \kappa_\gamma(k-j)\,\sigma(j)\,\sigma(k) - \tilde{\theta} \sum_{j\in\Lambda_N} \sigma(j)^2 \tag{2.1}$$

where $\tilde{\theta}$ is a real parameter, $\sigma \in \Sigma_N$, and κ_γ is the interaction kernel which has support size $O(\gamma^{-1})$, which is constructed as follows: Let $\mathfrak{K}\colon \mathbb{R}^2 \to [0,1]$ be a rotation invariant $\mathcal{C}^2$ function with support contained in the ball of radius 3 around the origin, such that

$$\int_{\mathbb{R}^2} \mathfrak{K}(x)\,dx = 1, \qquad \int_{\mathbb{R}^2} \mathfrak{K}(x)\,|x|^2\,dx = 4\,. \tag{2.2}$$

Then, for $0 < \gamma < \frac{1}{3}$, $\kappa_\gamma\colon \Lambda_N \to [0,\infty)$ is defined as $\kappa_\gamma(0) = 0$ and

$$\kappa_\gamma(k) = \frac{\gamma^2\,\mathfrak{K}(\gamma k)}{\sum_{k\in\Lambda_N\setminus\{0\}} \gamma^2\,\mathfrak{K}(\gamma k)} \qquad k \neq 0\,. \tag{2.3}$$

We are interested in the following Glauber dynamics, a natural Markov process on $(\Sigma_N, \lambda_\gamma)$ which is reversible for λ_γ. This process is defined in terms of the jump rates

$c_\gamma(\sigma;\sigma(j) \to \bar\sigma(j))$ for a configuration σ, to change its spin $\sigma(j)$ at position $j \in \Lambda_N$ to $\bar\sigma(j) \in \{\pm 1, 0\}$. This rate only depends on the final value $\bar\sigma(j)$ and is given by

$$\begin{aligned} c_\gamma(\sigma, j, -1) &\stackrel{\text{def}}{=} c_\gamma(\sigma; \sigma(j) \to -1) = e^{-\beta h_\gamma(\sigma,j)+\theta}/\mathcal{N}_{\beta,\theta}(h_\gamma(\sigma,j)) \;, \\ c_\gamma(\sigma, j, 0) &\stackrel{\text{def}}{=} c_\gamma(\sigma; \sigma(j) \to 0) = 1/\mathcal{N}_{\beta,\theta}(h_\gamma(\sigma,j)) \;, \\ c_\gamma(\sigma, j, 1) &\stackrel{\text{def}}{=} c_\gamma(\sigma; \sigma(j) \to +1) = e^{\beta h_\gamma(\sigma,j)+\theta}/\mathcal{N}_{\beta,\theta}(h_\gamma(\sigma,j)) \end{aligned}$$

where $\theta \stackrel{\text{def}}{=} \tilde\theta\beta$ and h_γ is the locally averaged field

$$h_\gamma(\sigma, k) \stackrel{\text{def}}{=} \sum_{j\in\Lambda_N} \kappa_\gamma(k-j)\,\sigma(j) =: \kappa_\gamma \star \sigma(k) \;, \tag{2.4}$$

and $\mathcal{N}_{\beta,\theta}(h_\gamma(\sigma,j))$ is a normalization factor

$$\mathcal{N}_{\beta,\theta}(h_\gamma(\sigma,j)) \stackrel{\text{def}}{=} e^{-\beta h_\gamma(\sigma,j)+\theta} + 1 + e^{\beta h_\gamma(\sigma,j)+\theta} \;.$$

This can be written in a streamlined way

$$c_\gamma(\sigma, j, \bar\sigma(j)) = e^{\bar\sigma(j)\beta h_\gamma(\sigma,j)+\bar\sigma(j)^2\theta}/\mathcal{N}_{\beta,\theta}(h_\gamma(\sigma,j)) \;. \tag{2.5}$$

The generator of the Markov process is then given by

$$\mathscr{L}_\gamma f(\sigma) = \sum_{j\in\Lambda_N} \sum_{\bar\sigma(j)\in\{0,\pm1\}} c_\gamma(\sigma, j, \bar\sigma(j))\,(f(\bar\sigma) - f(\sigma)) \tag{2.6}$$

where $f : \Sigma_N \to \mathbb{R}$ and $\bar\sigma$ is the new spin configuration obtained by flipping the spin $\sigma(j)$ in the configuration σ to $\bar\sigma(j)$. Let

$$h_\gamma(t, k) \stackrel{\text{def}}{=} h_\gamma(\sigma(t), k)$$

then one has

$$h_\gamma(t,k) = h_\gamma(0,k) + \int_0^t \mathscr{L}_\gamma\, h_\gamma(s,k)\,ds + m_\gamma(t,k) \;, \tag{2.7}$$

where the process $m_\gamma(\cdot, k)$ is a martingale, whose explicit form (quadratic variation etc.) will be discussed in Section 3. For the moment we focus on the drift term $\mathscr{L}_\gamma\, h_\gamma(s,k)$. Since σ and $\bar\sigma$ can only differ in their spin values at site j, one has

$$h_\gamma(\bar\sigma, k) - h_\gamma(\sigma, k) = \kappa_\gamma(k-j)\,(\bar\sigma(j) - \sigma(j)) \;,$$

and pluggin this into (2.6) yields

$$\mathscr{L}_\gamma h_\gamma(\sigma,k) = \sum_{j\in\Lambda_N} \sum_{\bar\sigma(j)\in\{\pm1,0\}} \kappa_\gamma(j-k)\,(\bar\sigma(j) - \sigma(j))\,c_\gamma(\sigma, j, \bar\sigma(j)) \;.$$

Using the fact that $\sum_{\bar\sigma(j)\in\{\pm1,0\}} c_\gamma(\sigma, j, \bar\sigma(j)) = 1$, one can alternatively write

$$\mathscr{L}_\gamma h_\gamma(\sigma,k) = \sum_{j\in\Lambda_N} \kappa_\gamma(j-k)\Big(-\sigma(j) + \sum_{\bar\sigma(j)\in\{\pm1,0\}} \bar\sigma(j)\,c_\gamma(\sigma, j, \bar\sigma(j))\Big) \;.$$

The Taylor expansion of $c_\gamma(\sigma, j, \bar\sigma(j))$ in $\beta h_\gamma(\sigma, j)$ gives

$$c_\gamma(\sigma, j, \bar\sigma(j)) = \sum_{n=0}^{\infty} c_n\, \beta^n h_\gamma(\sigma, j)^n \tag{2.8}$$

where the coefficients c_n are given by (we only list the ones we will use):

$$c_1 = \frac{\bar\sigma(j)e^{\bar\sigma(j)^2\theta}}{1+2e^\theta}\,, \qquad c_3 = \frac{\bar\sigma(j)e^{\bar\sigma(j)^2\theta}\Big(\bar\sigma(j)^2 + 2\,(\bar\sigma(j)^2-3)\,e^\theta\Big)}{6(1+2e^\theta)^2}\,,$$

$$c_5 = \frac{\bar\sigma(j)e^{\bar\sigma(j)^2\theta}\Big(4\,(\bar\sigma(j)^2-5)^2\,e^{2\theta} - 2\,(8\bar\sigma(j)^2+5)\,e^\theta + \bar\sigma(j)^2\Big)}{120(1+2e^\theta)^3}\,.$$

Therefore one has

$$\begin{aligned}\mathscr{L}_\gamma h_\gamma(\sigma, k) = \Big(\kappa_\gamma \star h_\gamma(\sigma, k) - h_\gamma(\sigma, k)\Big) + A_{\beta,\theta}\,\kappa_\gamma \star h_\gamma(\sigma, k)\\ + B_{\beta,\theta}\,\kappa_\gamma \star h_\gamma^3(\sigma, k) + C_{\beta,\theta}\,\kappa_\gamma \star h_\gamma^5(\sigma, k) + \dots\end{aligned}$$

where the remaining terms denoted by "$\cdots$" are terms of the form $\kappa_\gamma \star h_\gamma^n$ with n odd and $n > 5$, and

$$\begin{aligned}&A_{\beta,\theta} \stackrel{\text{def}}{=} \frac{2a}{2a+1}\beta - 1\,, \qquad B_{\beta,\theta} \stackrel{\text{def}}{=} -\frac{a(4a-1)}{3(2a+1)^2}\beta^3\,,\\ &C_{\beta,\theta} \stackrel{\text{def}}{=} \frac{a(64a^2-26a+1)}{60(1+2a)^3}\beta^5 \qquad (a \stackrel{\text{def}}{=} e^\theta = e^{\beta\tilde\theta})\,.\end{aligned} \tag{2.9}$$

Note that all the terms $\kappa_\gamma \star h_\gamma^n$ with even powers n vanish, because $c_\gamma(\sigma, j, \bar\sigma(j))$ remains unchanged under $(h_\gamma(\sigma, j), \bar\sigma(j)) \mapsto (-h_\gamma(\sigma, j), -\bar\sigma(j))$, thus the coefficients c_n in (2.8) for n even must be even functions in $\bar\sigma(j)$. Multiplying this coefficient by $\bar\sigma(j)$ and summing over $\bar\sigma(j) \in \{\pm 1, 0\}$ necessarily yields zero.

Remark 2.1 As mentioned in Section 1, letting $\theta \to \infty$ in the Hamiltonian (2.1) one recovers the Kac-Ising model. Here in the above expansion for $\mathscr{L}_\gamma h_\gamma$, if we send $\theta \to \infty$, we obtain the same coefficients in the corresponding expansion [MW16, Eq. (2.10)] for the Ising case.

We set $\varepsilon = \frac{2}{2N+1}$. Now every *microscopic* point $k \in \Lambda_N$ can be identified with $x = \varepsilon k \in \Lambda_\varepsilon = \{x = (x_1, x_2) \in \varepsilon\mathbb{Z}^2\colon\ x_1, x_2 \in (-1, 1)\}$. We view Λ_ε as a discretization of the continuous torus $\mathbb{T}^2$ identified with $[-1, 1]^2$. We define the scaled field

$$X_\gamma(t, x) = \delta^{-1} h_\gamma(t/\alpha, x/\varepsilon)\,, \tag{2.10}$$

so that

$$\begin{aligned}dX_\gamma(t, x) = \Big(\frac{\varepsilon^2}{\gamma^2}\frac{1}{\alpha}\widetilde\Delta_\gamma X_\gamma(t, x) + \frac{A_{\beta,\theta}}{\alpha} K_\gamma \star_\varepsilon X_\gamma(t, x) + \frac{B_{\beta,\theta}\delta^2}{\alpha} K_\gamma \star_\varepsilon X_\gamma^3(t, x)\\ + \frac{C_{\beta,\theta}\delta^4}{\alpha} K_\gamma \star_\varepsilon X_\gamma^5(t, x) + K_\gamma \star_\varepsilon E_\gamma(t, x)\Big)\, dt + dM_\gamma(t, x)\,,\end{aligned} \tag{2.11}$$

where the martingale M_γ is defined by $M_\gamma(t, x) = \delta^{-1} m_\gamma(t/\alpha, x/\varepsilon)$ and has an explicit quadratic variation of order $\varepsilon^2/(\delta^2\alpha)$ (see (3.8) below); the function $K_\gamma(x) \stackrel{\text{def}}{=} \varepsilon^{-2}\kappa_\gamma(\varepsilon^{-1}x)$

is scaled to approximate the Dirac distribution; the convolution $\star_\varepsilon$ on Λ_ε is defined through $X \star_\varepsilon Y(x) = \sum_{z\in\Lambda_\varepsilon} \varepsilon^2 X(x-z)Y(z)$; and $\widetilde{\Delta}_\gamma X = \frac{\gamma^2}{\varepsilon^2}(K_\gamma \star_\varepsilon X - X)$, so that $\widetilde{\Delta}_\gamma$ scales like the continuous Laplacian. The error term E_γ is given by

$$E_\gamma = \frac{1}{\delta\alpha}\Big(\frac{\sum_{\bar\sigma\in\{\pm1,0\}} \bar\sigma\, e^{\bar\sigma\beta\delta X_\gamma + \bar\sigma^2\theta}}{\sum_{\bar\sigma\in\{\pm1,0\}} e^{\bar\sigma\beta\delta X_\gamma+\bar\sigma^2\theta}} - \frac{2a}{2a+1}\beta\delta X_\gamma - B_{\beta,\theta}\delta^3 X_\gamma^3 - C_{\beta,\theta}\delta^5 X_\gamma^5\Big)\,. \quad (2.12)$$

Now formally:

- By choosing $A_{\beta,\theta}/\alpha = O(1)$ (which means that one tunes β,θ close to a curve in the $\beta-\theta$ plane given by $A_{\beta,\theta}=0$) and the scaling of $\varepsilon,\alpha,\delta$ such that the Laplacian, martingale and cubic terms are all of $O(1)$, namely

$$\varepsilon \approx \gamma^2, \qquad \alpha = \gamma^2, \qquad \delta = \gamma\,, \quad (2.13)$$

  one formally obtains the Φ^4 equation, as long as $B_{\beta,\theta}\delta^2/\alpha$ is strictly positive.

- However, if (β,θ) is tuned to be close to a special point $(\beta_c^*,\theta_c^*) = (3, -\ln 4)$ (which is a mean field value of a "tricritical" point given by $A_{\beta,\theta} = B_{\beta,\theta} = 0$) on the aforementioned curve, then under the scaling (2.13), the coefficient $B_{\beta,\theta}\delta^2/\alpha$ vanishes, which would formally result in an Ornstein-Uhlenbeck process. To observe a nontrivial limit we have to consider a different scale. In fact by imposing that both $A_{\beta,\theta}/\alpha = O(1)$ and $B_{\beta,\theta}\delta^2/\alpha = O(1)$ and that the Laplacian, martingale and quintic terms are all of $O(1)$, namely

$$\varepsilon \approx \gamma^3, \qquad \alpha = \gamma^4, \qquad \delta = \gamma\,, \quad (2.14)$$

  one formally obtains the Φ^6 equation.

We will refer to the above two cases as "the first (scaling) regime" and "the second (scaling) regime". The curve in the $\beta-\theta$ plane was shown in Fig. 1 Note that at (β_c,θ_c) the coefficient in front of X^5 is negative ($C_{\beta_c,\theta_c} = -9/20$) as desired for long time existence of solution.

Here, since the domain Λ_N has integer size, we can only choose our space rescaling as $\varepsilon = \frac{2}{2N+1}$, and $N = \lfloor\gamma^{-2}\rfloor$ in the first regime or $N = \lfloor\gamma^{-3}\rfloor$ in the second regime. This is why we wrote $\approx$ above. Write

$$\Delta_\gamma = c_{\gamma,2}^2 \widetilde{\Delta}_\gamma = \frac{\varepsilon^2}{\gamma^2\alpha}\widetilde{\Delta}_\gamma \quad (2.15)$$

where the coefficient $c_{\gamma,2} = \frac{\varepsilon}{\gamma^2}$ in the first regime (2.13) or $c_{\gamma,2} = \frac{\varepsilon}{\gamma^3}$ in the second regime (2.14) and is close to 1 up to an error $O(\gamma^2)$.

Remark 2.2 In d space dimensions, the only difference in the above scaling arguments is that the rescaled martingale $M_\gamma(t,x)$ has an explicit quadratic variation of order $\varepsilon^d/(\delta^2\alpha)$, so the condition of retaining Laplacian, martingale and quintic terms becomes

$$\varepsilon \approx \gamma^{\frac{6}{6-2d}}\,, \qquad \alpha = \gamma^{\frac{2d}{3-d}}\,, \qquad \delta = \gamma^{\frac{d}{6-2d}}\,,$$

It is manifest now that if $d=3$ the above relation cannot be satisfied, which corresponds exactly to the fact that the subcriticality condition for the Φ^6_d model is $d<3$. This may be compared with the scaling for the Φ^4_d model in [MW16, Remark 2.2] as following.

$$\varepsilon \approx \gamma^{\frac{4}{4-d}}, \qquad \alpha = \gamma^{\frac{2d}{4-d}}, \qquad \delta = \gamma^{\frac{d}{4-d}}\,.$$

As discussed in [MW16], the above formal derivation is not correct. Instead, in the first regime, fixing a point (a_c, β_c) on the curve $\mathcal{C}_c$, one should write the linear and cubic terms as

$$K_\gamma \star_\varepsilon \Big(\frac{B_{\beta,\theta}\delta^2}{\alpha}(X_\gamma^3 - 3\mathfrak{c}_\gamma X_\gamma) + \frac{A_{\beta,\theta} + 3\mathfrak{c}_\gamma B_{\beta,\theta}\delta^2}{\alpha} X_\gamma\Big) \tag{2.16}$$

where $\mathfrak{c}_\gamma$ is a logarithmically divergent renormalization constant, and tune (a, β) such that $(A_{\beta,\theta} + 3\mathfrak{c}_\gamma B_{\beta,\theta}\delta^2)/\alpha = \mathfrak{a}_1 + c_1(\gamma)$ where $\mathfrak{a}_1 \in \mathbb{R}$ is a fixed constant, and $c_1(\gamma)$ is a quantity vanishing as $\gamma \to 0$ which will give us certain freedom, namely,

$$\frac{2a}{2a+1}\beta - 1 = \gamma^2\Big(\mathfrak{c}_\gamma \, \frac{a(4a-1)}{(2a+1)^2}\beta^3 + \mathfrak{a}_1 + c_1(\gamma)\Big) \; .$$

The precise value of $\mathfrak{c}_\gamma$ will be given below (Eq. (2.36)); the difference between $\beta_c \mathfrak{c}_\gamma$ and

$$\sum_{\substack{\omega \in \mathbb{Z}^2 \\ 0<|\omega|<\gamma^{-1}}} \frac{1}{4\pi^2|\omega|^2}$$

remains bounded as γ goes to 0. One could well take $c_1(\gamma) = 0$; but the above tuning is not very transparent because there are two parameters (a, β) and the right hand side also involves a, β. To make the tuning (2.17) more explicit, we can for instance first choose $a = a(\gamma)$ to be any sequence such that $|a - a_c| = O(\gamma^2)$, and then replace the quantity $\frac{a(4a-1)}{(2a+1)^2}\beta^3$ by $\frac{a_c(4a_c-1)}{(2a_c+1)^2}\beta_c^3$ with an error of $o(\gamma)$. We then choose $c_1(\gamma)$ to exactly cancel this error, and tune β according to

$$\frac{2a}{2a+1}\beta - 1 = \gamma^2\Big(\frac{a_c(4a_c-1)}{(2a_c+1)^2}\beta_c^3 \, \mathfrak{c}_\gamma + \mathfrak{a}_1\Big) \; , \tag{2.17}$$

where a stands for the sequence $a(\gamma)$ chosen above that converges to a_c. Note that if $a \to \infty$ we recover from (2.17) the choice of β in [MW16, Eq (2.18)].

In the second regime, recall that the fifth Hermite polynomial is $x^5 - 10x^3 + 15x$. One should write the linear, cubic and quintic terms as

$$\begin{aligned} K_\gamma \star_\varepsilon &\Big(\frac{A_{\beta,\theta}}{\alpha}X_\gamma + \frac{B_{\beta,\theta}\delta^2}{\alpha}X_\gamma^3 + \frac{C_{\beta,\theta}\delta^4}{\alpha}X_\gamma^5\Big) \\ = K_\gamma \star_\varepsilon &\Big(\frac{C_{\beta,\theta}\delta^4}{\alpha}(X_\gamma^5 - 10\mathfrak{c}_\gamma X_\gamma^3 + 15\mathfrak{c}_\gamma^2 X_\gamma) + \frac{B_{\beta,\theta}\delta^2 + 10C_{\beta,\theta}\delta^4\mathfrak{c}_\gamma}{\alpha}(X_\gamma^3 - 3\mathfrak{c}_\gamma X_\gamma) \\ &+ \big(\frac{A_{\beta,\theta}}{\alpha} + 3\mathfrak{c}_\gamma \frac{B_{\beta,\theta}\delta^2 + 5C_{\beta,\theta}\delta^4\mathfrak{c}_\gamma}{\alpha}\big)X_\gamma\Big) \end{aligned} \tag{2.18}$$

So one should tune (a, β) such that the coefficient in front of $(X_\gamma^3 - 3\mathfrak{c}_\gamma X_\gamma)$ is equal to $\mathfrak{a}_3 + c_3(\gamma)$ where $\mathfrak{a}_3 \in \mathbb{R}$ is a fixed constant; noting that $C_{\beta,\theta} = C_{\beta_c,\theta_c} + o(\gamma) = -9/20 + o(\gamma)$, one can replace $C_{\beta,\theta}$ by $-9/20$ and suitably choose $c_3(\gamma)$ to cancel this error, and thus obtain

$$-\frac{a(4a-1)}{3(2a+1)^2}\beta^3 = \gamma^2\Big(\frac{9}{2}\,\mathfrak{c}_\gamma + \mathfrak{a}_3\Big) \; . \tag{2.19}$$

One should furthermore impose that the coefficient in front of X_γ in (2.18) is equal to $\mathfrak{a}_1 + c_1(\gamma)$ where $\mathfrak{a}_1 \in \mathbb{R}$ is a fixed constant, and suitably choose $c_1(\gamma)$ to get

$$\frac{2a}{2a+1}\beta - 1 = \gamma^4\Big(-3\mathfrak{c}_\gamma\mathfrak{a}_3 - \frac{27}{4}\mathfrak{c}_\gamma^2 + \mathfrak{a}_1\Big) \; . \tag{2.20}$$

Combining the above two conditions we see that the correct tuning of the parameters ($\beta, a = e^{\theta}$) is

$$\begin{aligned} a &= \frac{1}{4} - \gamma^2\Big(\frac{9}{8}\,\mathfrak{c}_\gamma + \frac{\mathfrak{a}_3}{4}\Big)\,, \\ \beta &= 3 - \gamma^4\Big(9\mathfrak{c}_\gamma\mathfrak{a}_3 + \frac{81}{4}\mathfrak{c}_\gamma^2 - 3\mathfrak{a}_1\Big)\,. \end{aligned} \tag{2.21}$$

The limiting SPDEs

We briefly review the well-posedness theory for the Φ^{2n} equation

$$dX = (\Delta X + \sum_{k=1}^{n} \mathfrak{a}_{2k-1} X^{2k-1})\,dt + \sqrt{2/\beta_c}dW \qquad X(0) = X^0 \tag{2.22}$$

in two space dimensions with $\mathfrak{a}_{2n-1} < 0$, and the parameter $\beta_c > 0$ will correspond to a critical value of β described above. In order to interpret the solution to the above equation, let $W_\varepsilon(t,x) = \frac{1}{4}\sum_{|\omega|<\varepsilon^{-1}} e^{i\pi\omega\cdot x}\,\hat{W}(t,\omega)$ be a spatially regularized cylindrical Wiener process, and consider the *renormalized* equation

$$dX_\varepsilon = \Big(\Delta X_\varepsilon + \sum_{k=0}^{n} \mathfrak{a}_{2k-1} H_{2k-1}(X_\varepsilon, \mathfrak{c}_\varepsilon)\Big)\,dt + \sqrt{2/\beta_c}\,dW_\varepsilon, \tag{2.23}$$

where $H_m = H_m(x,c)$ are Hermite polynomials defined recursively by setting $H_0 = 1$ and $H_m = xH_{m-1} - c\,\partial_x H_{m-1}$ so that $H_1 = x$, $H_2 = x^2 - c$, $H_3 = x^3 - 3cx$, etc. The constant $\mathfrak{c}_\varepsilon$ is given by

$$\mathfrak{c}_\varepsilon = {\beta_c}^{-1} \sum_{0<|\omega|<\varepsilon^{-1}} \frac{1}{4\pi^2|\omega|^2}\,. \tag{2.24}$$

In particular, the constants $\mathfrak{c}_\varepsilon$ diverge logarithmically as $\varepsilon \to 0$. Then, [DPD03] shows that X_ε converges to nontrivial limit.

More precisely, let

$$X_\varepsilon(t) = Z_\varepsilon(t) + P_t X^0 + v_\varepsilon(t)$$

where $P_t = e^{t\Delta}$ is the solution operator of the heat equation on the torus $\mathbb{T}^2$, and

$$Z_\varepsilon(t,\cdot) = \sqrt{2/\beta_c}\int_0^t P_{t-s}\,dW_\varepsilon(s,\cdot)$$

is the solution to the linear equation with zero initial data. Letting

$$Z_\varepsilon^{:m:}(t,x) \stackrel{\text{def}}{=} H_m(Z_\varepsilon(t,x), \mathfrak{c}_\varepsilon(t)) \tag{2.25}$$

for

$$\begin{aligned} \mathfrak{c}_\varepsilon(t) = \mathbb{E}[Z_\varepsilon(t,0)^2] &= \frac{1}{2\beta_c}\sum_{|\omega|<\varepsilon^{-1}}\int_0^t \exp\left(-2r\pi^2|\omega|^2\right)\,dr \\ &= \frac{t}{2\beta_c} + \frac{1}{\beta_c}\sum_{0<|\omega|<\varepsilon^{-1}}\frac{1}{4\pi^2|\omega|^2}\Big(1 - \exp\big(-2t\pi^2\,|\omega|^2\big)\Big)\,, \end{aligned} \tag{2.26}$$

then $Z_\varepsilon^{:m:}$ converge almost surely and in every stochastic L^p space with respect to the metric of $\mathcal{C}([0,T],\mathcal{C}^{-\nu})$ - this is essentially [DPD03, Lemma 3.2]. We denote the limiting processes by $Z^{:m:}$. Note that $\mathfrak{c}_\varepsilon = \lim_{t\to\infty}(\mathfrak{c}_\varepsilon(t) - \frac{t}{2\beta_c})$, where the term $\frac{t}{2\beta_c}$ comes from the summand for $\omega = 0$ in (2.26) which does not converge as $t \to \infty$. Furthermore, for every fixed $t > 0$ the difference $|\mathfrak{c}_\varepsilon - \mathfrak{c}_\varepsilon(t)|$ is uniformly bounded in ε. This replacement of $\mathfrak{c}_\varepsilon$ by $\mathfrak{c}_\varepsilon(t)$ amounts to rewriting (2.23) as (2.30) below. Define $\mathfrak{a}_{2k-1}^{(\varepsilon)}(t)$ as time dependent coefficients such that

$$\sum_{k=1}^n \mathfrak{a}_{2k-1} H_{2k-1}(x,\mathfrak{c}_\varepsilon) = \sum_{k=1}^n \mathfrak{a}_{2k-1}^{(\varepsilon)}(t) H_{2k-1}(x,\mathfrak{c}_\varepsilon(t)) \; . \tag{2.27}$$

This is well-defined since the left hand side is an odd polynomial of degree $2n-1$ which can be uniquely expressed as a linear combination of odd Hermite polynomials $H_{2k-1}(x,\mathfrak{c}_\varepsilon(t))$. Note that the leading coefficients always satisfy $\mathfrak{a}_{2n-1} = \mathfrak{a}_{2n-1}^{(\varepsilon)}(t)$. For the other coefficients, for instance, when $n = 2$ one has $\mathfrak{a}_1^{(\varepsilon)}(t) = 3\mathfrak{a}_3(\mathfrak{c}_\varepsilon(t) - \mathfrak{c}_\varepsilon) + \mathfrak{a}_1$; when $n = 3$ one has

$$\begin{aligned} \mathfrak{a}_3^{(\varepsilon)}(t) &= 10\mathfrak{a}_5(\mathfrak{c}_\varepsilon(t) - \mathfrak{c}_\varepsilon) + \mathfrak{a}_3 \; , \\ \mathfrak{a}_1^{(\varepsilon)}(t) &= -15\mathfrak{a}_5(\mathfrak{c}_\varepsilon(t)^2 - \mathfrak{c}_\varepsilon^2) + 3(\mathfrak{c}_\varepsilon(t)\mathfrak{a}_3^{(\varepsilon)}(t) - \mathfrak{c}_\varepsilon \mathfrak{a}_3) + \mathfrak{a}_1 \; . \end{aligned} \tag{2.28}$$

In fact, plugging the first relation into the second, one has

$$\mathfrak{a}_1^{(\varepsilon)}(t) = 3\mathfrak{a}_3(\mathfrak{c}_\varepsilon(t) - \mathfrak{c}_\varepsilon) + 15\mathfrak{a}_5(\mathfrak{c}_\varepsilon(t) - \mathfrak{c}_\varepsilon)^2 + \mathfrak{a}_1 \; . \tag{2.29}$$

Then (2.23) can be rewritten as

$$dX_\varepsilon = \Big(\Delta X_\varepsilon + \sum_{k=1}^n \mathfrak{a}_{2k-1}^{(\varepsilon)}(t) H_{2k-1}(X_\varepsilon,\mathfrak{c}_\varepsilon(t))\Big)\, dt + \sqrt{2/\beta_c}\, dW_\varepsilon \; . \tag{2.30}$$

To proceed one needs the following simple fact, which generalizes (2.29).

Lemma 2.3 *For every $k = 1,\dots,n$, the difference $\mathfrak{a}_{2k-1} - \mathfrak{a}_{2k-1}^{(\varepsilon)}(t)$ is a polynomial of $\mathfrak{c}_\varepsilon - \mathfrak{c}_\varepsilon(t)$ without zero order term, with coefficients only depending on $\mathfrak{a}_1,\cdots,\mathfrak{a}_{2n-1}$. This difference is uniformly bounded in ε for every $t > 0$ and diverges logarithmically in t as $t \to 0$.*

Proof. By the differential operator representation of Hermite polymonials $H_m(x,c) = e^{-c\Delta/2}x^m$, where Δ is Laplacian in x and the exponential is understood as power series without convergence problem when acting on polynomials. So we have

$$\begin{aligned} H_{2k-1}(x,\mathfrak{c}_\varepsilon) &= e^{-\mathfrak{c}_\varepsilon\Delta/2}x^{2k-1} = e^{-\mathfrak{c}_\varepsilon(t)\Delta/2}e^{-(\mathfrak{c}_\varepsilon-\mathfrak{c}_\varepsilon(t))\Delta/2}x^{2k-1} \\ &= e^{-\mathfrak{c}_\varepsilon(t)\Delta/2}H_{2k-1}(x,\mathfrak{c}_\varepsilon - \mathfrak{c}_\varepsilon(t)) \; . \end{aligned}$$

The operator $e^{-\mathfrak{c}_\varepsilon(t)\Delta/2}$ replaces every monomial term x^m in the polymonial $H_{2k-1}(x,\mathfrak{c}_\varepsilon - \mathfrak{c}_\varepsilon(t))$ by $H_m(x,\mathfrak{c}_\varepsilon(t))$, which means that when re-expanding $H_{2k-1}(x,\mathfrak{c}_\varepsilon)$ on the left hand side of (2.27) w.r.t. the basis $H_m(x,\mathfrak{c}_\varepsilon(t))$ the coefficients only depend on $\mathfrak{c}_\varepsilon, \mathfrak{c}_\varepsilon(t)$ via $\mathfrak{c}_\varepsilon - \mathfrak{c}_\varepsilon(t)$. After this re-expansion we then compare the coefficients on the two sides of (2.27), noting that if $\mathfrak{c}_\varepsilon - \mathfrak{c}_\varepsilon(t) = 0$ then $\mathfrak{a}_{2k-1}^{(\varepsilon)} = \mathfrak{a}_{2k-1}$, and we obtain the first statement of the lemma. Note that

$$\lim_{\varepsilon\to 0}(\mathfrak{c}_\varepsilon - \mathfrak{c}_\varepsilon(t)) = -\frac{t}{2\beta_c} + \sum_{\omega\in\mathbb{Z}^2\setminus\{0\}} \frac{e^{-2t\pi^2|\omega|^2}}{4\beta\pi^2|\omega|^2} \; . \tag{2.31}$$

It is then obvious that the second statement of the lemma also holds. □

By this lemma the limiting coefficient $\lim_{\varepsilon\to 0}\mathfrak{a}^{(\varepsilon)}_{2k-1}(t)$ is integrable in t at $t=0$.

As a convenient way to deal with the initial data X^0, we further define $\tilde Z(t) = Z(t)+P_tX^0$ and

$$\tilde Z^{:m:}(t) = \sum_{k=0}^{m}\binom{m}{k}(P_tX^0)^{m-k}Z^{:k:}(t) \tag{2.32}$$

The following theorem, essentially [MW15, Theorem 3.2], states that the equation

$$\partial_t v = \Delta v + \sum_{k=1}^{n}\mathfrak{a}^{(\varepsilon)}_{2k-1}(t)\sum_{\ell=1}^{2k-1}\binom{2k-1}{\ell}\tilde Z^{:2k-1-\ell:}v^\ell \tag{2.33}$$

which is derived from (2.30), or equivalently

$$\partial_t v = \Delta v + \sum_{\ell=1}^{2n-1}\Big(\sum_{k\in\mathbb{Z}\cap[\frac{\ell+1}{2},n]}\mathfrak{a}^{(\varepsilon)}_{2k-1}(t)\binom{2k-1}{\ell}\tilde Z^{:2k-1-\ell:}\Big)v^\ell \tag{2.34}$$

with zero initial condition $v(0)=0$ is globally well-posed. The solution v is the limit of v_ε.

Theorem 2.4 *For $\nu>0$ small enough, fix an initial datum $X^0\in\mathcal{C}^{-\nu}$. For*

$$(Z, Z^{:2:},\dots,Z^{:2n-1:}) \in (L^\infty([0,T],\mathcal{C}^{-\nu}))^{2n-1}\ ,$$

let $(\tilde Z,\widetilde Z^{:2:},\cdots,\widetilde Z^{:2n-1:})$ be defined as in (2.32). Let $\mathcal{S}_T(Z,Z^{:2:},\dots,Z^{:2n-1:})$ denote the solution v on $[0,T]$ of the PDE (2.34). Then for any $\kappa>0$, the mapping

$$\mathcal{S}_T : (L^\infty([0,T],\mathcal{C}^{-\nu}))^{2n-1} \to \mathcal{C}([0,T],\mathcal{C}^{2-\nu-\kappa}(\mathbb{T}^2))$$

is Lipschitz continuous on bounded sets .

With the solution v given by this theorem we call $X(t)=Z(t)+P_tX^0+v(t)$ the solution to the dynamical Φ^{2n} equation (2.22) with initial data $X^0\in\mathcal{C}^{-\nu}$. (Due to the above theorem, Eq. (2.22) is sometimes written with each term X^{2k-1} replaced by $:X^{2k-1}:$ but we refrain from using this notation.)

Main result

As in [MW16], for any function $Y\colon\Lambda_\varepsilon\to\mathbb{R}$, we define its smooth extension to a function $\mathbb{T}^2\to\mathbb{R}$ which is denoted by $\mathsf{Ext}Y$ (but sometimes still written as Y) in the following way:

$$\mathsf{Ext}Y(x) = \frac14\sum_{\omega\in\{-N,\dots,N\}^2}\sum_{y\in\Lambda_\varepsilon}\varepsilon^2\, e^{i\pi\omega\cdot(x-y)}\,Y(y) \qquad (x\in\mathbb{T}^2) \tag{2.35}$$

which is the unique trigonometric polynomial of degree $\le N$ that coincides with Y on Λ_ε.

For any metric space $\mathcal{S}$, we denote by $\mathcal{D}(\mathbb{R}_+,\mathcal{S})$ the space of $\mathcal{S}$ valued cadlag function endowed with the Skorokhod topology. For any $\nu>0$ we denote by $\mathcal{C}^{-\nu}$ the Besov space $B^{-\nu}_{\infty,\infty}$ (see [MW16, Appendix A] for such spaces).

Assume that for $\gamma>0$, the spin configuration at time 0 is given by $\sigma_\gamma(0,k),\ k\in\Lambda_N$, and define for $x\in\Lambda_\varepsilon$

$$X^0_\gamma(x) = \delta^{-1}\sum_{y\in\Lambda_\varepsilon}\varepsilon^2K_\gamma(x-y)\,\sigma_\gamma(0,\varepsilon^{-1}y)\ .$$

We smoothly extend X_γ^0 (in the way described above) to $\mathbb{T}^2$ which is still denoted by X_γ^0. Let $X_\gamma(t,x)$, $t \geq 0$, $x \in \Lambda_\varepsilon^2$ be defined by (2.10) and extend $X_\gamma(t,\cdot)$ to $\mathbb{T}^2$, still denoted by X_γ.

Define

$$\mathfrak{c}_\gamma \stackrel{\text{def}}{=} \frac{1}{4\beta_c} \sum_{\substack{\omega\in\{-N,\dots,N\}^2\\ \omega\neq 0}} \frac{|\hat{K}_\gamma(\omega)|^2}{\gamma^{-b}(1-\hat{K}_\gamma(\omega))} , \tag{2.36}$$

where $b=2$ in the first regime and $b=4$ in the second regime.

The main result of this article is the following.

Theorem 2.5 *Suppose that the precise value of $\mathfrak{c}_\gamma$ is given by* (2.36)*, and that X_γ^0 converges to X^0 in $\mathcal{C}^{-\nu}$ for $\nu > 0$ small enough and that X^0, X_γ^0 are uniformly bounded in $\mathcal{C}^{-\nu+\kappa}$ for an arbitrarily small $\kappa > 0$.*

(1) Assume that the scaling exponents $\varepsilon, \alpha, \delta$ satisfy (2.13) *and the parameters $a = e^\theta$, β satisfy* (2.17) *for some (a_c, β_c) and $\mathfrak{a}_1 \in \mathbb{R}$ such that*

$$\frac{2a_c}{2a_c+1}\beta_c - 1 = 0 . \tag{2.37}$$

If $a_c > \frac{1}{4}$, then X_γ converges in law to the solution of the following dynamical Φ^4 equation:

$$dX = (\Delta X + \mathfrak{a}_1 X - \frac{a_c(4a_c-1)\beta_c^3}{3(2a_c+1)^2} X^3)\, dt + \sqrt{2/\beta_c}\, dW \qquad X(0) = X^0 .$$

(2) Under the same assumption in (1), if $a_c = \frac{1}{4}$, then X_γ converges in law to the linear equation:

$$dX = (\Delta X + \mathfrak{a}_1 X)\, dt + \sqrt{2/3}\, dW \qquad X(0) = X^0 .$$

(3) Assume that the scaling exponents $\varepsilon, \alpha, \delta$ satisfy (2.14) *and the parameters $a = e^\theta$, β satisfy* (2.21) *for some $\mathfrak{a}_1, \mathfrak{a}_3 \in \mathbb{R}$ and in particular*

$$(a, \beta) \to (1/4, 3) \qquad \text{as } \gamma \to 0 . \tag{2.38}$$

Then as $\gamma \to 0$, X_γ converges in law to the solution of a dynamical Φ^6 equation:

$$dX = (\Delta X + \mathfrak{a}_1 X + \mathfrak{a}_3 X^3 - \frac{9}{20} X^5)\, dt + \sqrt{2/3}\, dW \qquad X(0) = X^0 .$$

All the above convergences are with respect to the topology of $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})$.

Remark 2.6 Note that the coefficient $\sqrt{2/\beta_c}$ in front of the white noise in the limiting equations makes the interpretation of β as "inverse temperature" more meaningful. This means that the quadratic variation of our martingale should behaves like $2/\beta_c$ times the Dirac distribution. The quadratic variation will depend on the spin configuration σ and in the following proofs we will approximate σ by an i.i.d. spin system $\tilde{\sigma}$ so that at each site $\mathbb{P}(\tilde{\sigma} = \pm 1) = e^{\theta_c}/\mathcal{N}_c$ and $\mathbb{P}(\tilde{\sigma} = 0) = 1/\mathcal{N}_c$ where $\mathcal{N}_c = 1 + 2e^{\theta_c}$. (Recall that θ has the interpretation of "chemical potential" i.e. the "ratio" between ± 1 and 0 spins.) On average (over $\tilde{\sigma} \in \{-1, 0, +1\}$) the quadratic variation will then be shown as equal to (see (3.13))

$$\frac{4e^{\theta_c}}{1+2e^{\theta_c}} = \frac{2}{\beta_c}$$

where the last equality is by (2.37) or (2.38).

Remark 2.7 The limiting equations in the theorem are globally well-posed, see the paper [MW15], especially Remark 1.5 there. Actually, in case (1), if $a_c < \frac{1}{4}$, one can still prove that X_γ converges to a Φ^4 equation, but with a plus sign in front of X^3, which may blow up in finite time.

3 Convergence of the linearized equation

To prove the convergence result Theorem 2.5 we rewrite our discrete evolution in the Duhamel's form:

$$\begin{aligned} X_\gamma(t,\cdot) =& P_t^\gamma X_\gamma^0 + \int_0^t P_{t-s}^\gamma K_\gamma \star \Big(C_{\beta,\theta} X_\gamma^5(s,\cdot) + B_{\beta,\theta} X_\gamma^3(s,\cdot) \\ & + A_{\beta,\theta} X_\gamma(s,\cdot) + E_\gamma(s,\cdot)\Big)\, ds + \int_{s=0}^t P_{t-s}^\gamma \, dM_\gamma(s,\cdot) \qquad \text{on } \Lambda_\varepsilon \end{aligned} \tag{3.1}$$

where the coefficients are defined in (2.9), and P_t^γ is the heat operator associated with Δ_γ. Recall that the martingale m_γ was defined above in (2.7) and the rescaled martingales $M_\gamma(t,z) = \frac{1}{\delta} m_\gamma(\frac{t}{\alpha}, \frac{z}{\varepsilon})$ are defined on a rescaled grid $\Lambda_\varepsilon \subseteq [-1,1]^2$. An important step of proving convergence of (3.1) is to show convergence of the linearized system. For $x \in \Lambda_\varepsilon$, we denote by

$$Z_\gamma(t,x) \stackrel{\text{def}}{=} \int_{r=0}^t P_{t-r}^\gamma \, dM_\gamma(r,x) \tag{3.2}$$

the stochastic convolution appearing as the last term of (3.1). The process Z_γ is the solution to the linear stochastic equation

$$\begin{aligned} dZ_\gamma(t,x) &= \Delta_\gamma Z_\gamma(t,x) dt + dM_\gamma(t,x) \\ Z_\gamma(0,x) &= 0\,, \end{aligned} \tag{3.3}$$

for $x \in \Lambda_\varepsilon,\ t \geq 0$. As discussed in (2.35), we extend Z_γ to the entire torus $\mathbb{T}^2$ and still denote it by Z_γ. The tightness of the family Z_γ with respect to the topology of $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})$ is established below in Prop. 4.4. In this section we assume this result and prove the convergence in law of Z_γ to the solution of the stochastic heat equation.

The predictable quadratic covariations of the martingales $m_\gamma(\cdot,k)$ are given by

$$\begin{aligned} &\langle m_\gamma(\cdot,k), m_\gamma(\cdot,j)\rangle_t \\ &= \int_0^t \sum_{\ell\in\Lambda_N} \kappa_\gamma(k-\ell)\kappa_\gamma(j-\ell) \sum_{\bar\sigma\in\{\pm1,0\}} (\bar\sigma - \sigma(s,\ell))^2 c_\gamma(\sigma(s),\ell,\bar\sigma) ds. \end{aligned} \tag{3.4}$$

Following the reasoning from [MW16] we first construct a modified version of the martingales M_γ and the approximate stochastic convolution Z_γ for which we have a better control on this quadratic variation. To this end, we first define the stopping time $\tau_{\gamma,\mathfrak{m}}$ for a fixed $\nu \in (0,\frac{1}{2})$, any $\mathfrak{m} > 1$ and $0 < \gamma < 1$,

$$\tau_{\gamma,\mathfrak{m}} \stackrel{\text{def}}{=} \inf\{t \geq 0 : \|X_\gamma(t,\cdot)\|_{\mathcal{C}^{-\nu}} \geq \mathfrak{m}\}\,. \tag{3.5}$$

For $k \in \Lambda_N$ and for $t \geq 0$, define

$$\sigma_{\gamma,\mathfrak{m}}(t,k) \stackrel{\text{def}}{=} \begin{cases} \sigma(t,k) & \text{if } t < \frac{\tau_{\gamma,\mathfrak{m}}}{\alpha}, \\ \sigma'_{\gamma,\mathfrak{m}}(t,k) & \text{otherwise}\,. \end{cases}$$

Here $\sigma'_{\gamma,\mathfrak{m}}$ is a spin system with $\sigma'_{\gamma,\mathfrak{m}}(\tau_{\gamma,\mathfrak{m}}/\alpha, k) = \sigma(\tau^-_{\gamma,\mathfrak{m}}/\alpha, k)$, and for every $t > \tau_{\gamma,\mathfrak{m}}/\alpha$ and every $k \in \Lambda_N$ the jumps to spin values $+1, 0, -1$ at rates $\frac{e^{\theta_c}}{\mathcal{N}_c}, \frac{1}{\mathcal{N}_c}, \frac{e^{\theta_c}}{\mathcal{N}_c}$ respectively, independently from σ, with $\mathcal{N}_c = 1 + 2e^{\theta_c}$. (Recall that θ_c is a critical value of θ as in Section 2.) In other words, the rate function c_γ is replaced by

$$c^s_{\gamma,\mathfrak{m}}(\sigma(s), k, \bar\sigma) = \begin{cases} c_\gamma(\sigma(s), k, \bar\sigma) & \text{if } s < \frac{\tau_{\gamma,\mathfrak{m}}}{\alpha}, \\ (\frac{e^{\theta_c}}{\mathcal{N}_c}, \frac{1}{\mathcal{N}_c}, \frac{e^{\theta_c}}{\mathcal{N}_c}) & \text{otherwise} \end{cases} \tag{3.6}$$

where in the second case, $c^s_{\gamma,\mathfrak{m}}(\sigma(s), k, \bar\sigma)$ is independent of the configuration $\sigma(s)$ and the site k and thus only depends on $\bar\sigma$; so we only defined its values on the three points $\bar\sigma = 1, 0, -1$. We now construct processes $M_{\gamma,\mathfrak{m}}$ and $Z_{\gamma,\mathfrak{m}}$ following exactly the construction of M_γ and Z_γ with σ_γ replaced by $\sigma_{\gamma,\mathfrak{m}}$.

Define the rescaled rate function

$$C_{\gamma,\mathfrak{m}}(s, z, \bar\sigma) \stackrel{\text{def}}{=} c^{s/\alpha}_{\gamma,\mathfrak{m}}(\sigma_{\gamma,\mathfrak{m}}(s/\alpha), z/\varepsilon, \bar\sigma) \tag{3.7}$$

for every $s \geq 0$, $z \in \Lambda_\varepsilon$ and $\bar\sigma \in \{+1, 0, -1\}$. Of course $C_{\gamma,\mathfrak{m}}(s, z, \bar\sigma)$ still depends on the configuration $\sigma_{\gamma,\mathfrak{m}}$ but we suppress this dependence in the notation now. For the martingales $M_{\gamma,\mathfrak{m}}(t, z)$, Eq. (3.4) turns into

$$\begin{aligned} &\langle M_{\gamma,\mathfrak{m}}(\cdot, x), M_{\gamma,\mathfrak{m}}(\cdot, y)\rangle_t \\ &= \frac{\varepsilon^2}{\delta^2\alpha}\int_0^t \sum_{z\in\Lambda_\varepsilon} \varepsilon^2 K_\gamma(x-z)\, K_\gamma(y-z) \sum_{\bar\sigma\in\{\pm1,0\}} (\bar\sigma - \sigma(s, \varepsilon^{-1}z))^2 C_{\gamma,\mathfrak{m}}(s, z, \bar\sigma)\, ds. \end{aligned} \tag{3.8}$$

Recall that the kernel $K_\gamma(x) = \varepsilon^{-2}\kappa_\gamma(\varepsilon^{-1}x)$ is a rescaled version of κ_γ that behaves like an approximation of Dirac distribution δ; thus we obtain ε^4 when rescaling the two factors κ_γ but have moved an ε^2 into the sum to anticipate that the sum over z approximates $\delta(x-y)$, possibly times a constant. Since $\delta = \gamma$ in both "scaling regimes", we can also write the coefficient in front of the integral as $c^2_{\gamma,2} = \frac{\varepsilon^2}{\gamma^2\alpha}$ which was defined in (2.15). The constant $c_{\gamma,2}$ is close to 1.

Lemma 3.1 *The rates $C_{\gamma,\mathfrak{m}}$ defined in* (3.7) *satisfy*

$$C_{\gamma,\mathfrak{m}}(s, z, \pm1) = \frac{e^{\theta_c}}{\mathcal{N}_c} + E_\gamma$$
$$C_{\gamma,\mathfrak{m}}(s, z, 0) = \frac{1}{\mathcal{N}_c} + E'_\gamma$$

for every $s \geq 0$, $z \in \Lambda_\varepsilon$, where $\mathcal{N}_c = 1 + 2e^{\theta_c}$ and the random terms E_γ, E'_γ which depend on s, z are deterministically bounded by $C\gamma^{1-3\nu}$ with constant C depending linearly on $\mathfrak{m}$. The un-rescaled rates $c^s_{\gamma,\mathfrak{m}}(\sigma(s), k, \bar\sigma)$ satisfy the same estimates for every $s \geq 0$, $k \in \Lambda_N$ and $\bar\sigma \in \{\pm1, 0\}$.

Proof. By (3.7) it suffices to prove the stated estimates for $C_{\gamma,\mathfrak{m}}$ and that for $c_{\gamma,\mathfrak{m}}$ immediately follow. For $t > \tau_{\gamma,\mathfrak{m}}$, we have $E_\gamma = E'_\gamma = 0$ by definition. For $t \leq \tau_{\gamma,\mathfrak{m}}$, first of all, we note that $(\frac{e^\theta}{\mathcal{N}}, \frac{1}{\mathcal{N}}, \frac{e^\theta}{\mathcal{N}})$ with $\mathcal{N} = 1 + 2e^\theta$ are nothing but the values of c_γ defined in (2.5) for $\beta h_\gamma = 0$ at the three points $\bar\sigma = 1, 0, -1$. Since the derivatives of the functions $\frac{x}{1+2x}$ and $\frac{1}{1+2x}$ are both bounded by 2, the error caused by replacing $(\frac{e^\theta}{\mathcal{N}}, \frac{1}{\mathcal{N}}, \frac{e^\theta}{\mathcal{N}})$ by $(\frac{e^{\theta_c}}{\mathcal{N}_c}, \frac{1}{\mathcal{N}_c}, \frac{e^{\theta_c}}{\mathcal{N}_c})$ is bounded by $2|e^{\theta}-e^{\theta_c}|$; by the discussion above (2.17) (for the first scaling regime) or (2.21) (for the second scaling regime), this error is bounded by $C\gamma^{1-2\nu}$.

Furthermore, it is easy to check by (2.5) that for any $\bar{\sigma}(j)\in\{\pm1,0\}$ and any $\theta\in\mathbb{R}$, the rate c_γ viewed as a function of βh_γ has the derivative:

$$\frac{e^{\bar{\sigma}(j)\beta h_\gamma+\bar{\sigma}(j)^2\theta}\Big(\bar{\sigma}(j)(e^{-\beta h_\gamma+\theta}+1+e^{\beta h_\gamma+\theta})+e^{-\beta h_\gamma+\theta}-e^{\beta h_\gamma+\theta}\Big)}{(e^{-\beta h_\gamma+\theta}+1+e^{\beta h_\gamma+\theta})^2}\;,$$

which is bounded by 2. Therefore for $t<\tau_{\gamma,\mathfrak{m}}$,

$$\begin{aligned}|E_\gamma|\vee|E'_\gamma| &\le 2\beta|h_\gamma(\sigma(t/\alpha),z/\varepsilon)|+C\gamma^{1-2\kappa}=2\beta\delta|X_\gamma(t,z)|+C\gamma^{1-2\kappa}\\ &\le C(\nu)\gamma^{1-3\nu}(\|X_\gamma(t)\|_{\mathcal{C}^{-\nu}}+1)\;.\end{aligned}\tag{3.9}$$

In the last step of (3.9) we used the fact that $\delta=\gamma$ in both scaling regimes; $\beta\le 4$ for γ sufficiently small since in all three cases of Theorem 2.5 $\beta_c\le 3$; and the fact that since the Fourier coefficients of X_γ with frequency larger than γ^{-2} (resp. γ^{-3}) vanish, by [MW16, Lemma A.3], $\|X_\gamma(t)\|_{L^\infty}\le C\gamma^{-b\nu}\|X_\gamma(t)\|_{\mathcal{C}^{-\nu}}$ with $b=2$ in the first regime (resp. $b=3$ in the second regime). □

This lemma allows to rewrite the last terms appearing in (3.8) as

$$\sum_{\bar{\sigma}\in\{\pm1,0\}}(\bar{\sigma}-\sigma(\varepsilon^{-1}z))^2 C_{\gamma,\mathfrak{m}}(s,z,\bar{\sigma})=A(\sigma(\varepsilon^{-1}z))+E''_\gamma,\tag{3.10}$$

where the error E''_γ is again deterministically bounded by $C\gamma^{1-3\nu}$ (for a constant C which depends on $\mathfrak{m}$) and A is a function defined on three points $\{+1,0,-1\}$ as following

$$A(\sigma)=\begin{cases}2e^{\theta_c}/\mathcal{N}_c & \text{for } \sigma=0\\ 4e^{\theta_c}/\mathcal{N}_c+1/\mathcal{N}_c & \text{for } \sigma=\pm1\end{cases}\tag{3.11}$$

where $\mathcal{N}_c=1+2e^{\theta_c}$ as before. The main ingredient in the proof of Theorem 3.3 below is to show that the dependence on the microscopic configuration $\sigma(t,x)$ in this expression becomes irrelevant when averaging over long time intervals, and that A may be replaced by its average.

Before stating Theorem 3.3, we define a coupling between the microscopic spin process $\sigma(s,k)$ with an extremely simple auxiliary spin process $\tilde{\sigma}(s,k)$. For every given site $k\in\Lambda_N$ the spin $\tilde{\sigma}(\cdot,k)$ gets updated at the same random times as the original process $\sigma(\cdot,k)$ but the update is determined according to a fixed probability distribution $\tilde{P}$ on $\{\pm1,0\}$ independently of the values of both σ and $\tilde{\sigma}$ and independently of other sites, which motivated by Lemma 3.1 is given by

$$\tilde{P}=\begin{pmatrix}e^{\theta_c}/\mathcal{N}_c\\ 1/\mathcal{N}_c\\ e^{\theta_c}/\mathcal{N}_c\end{pmatrix}.\tag{3.12}$$

This process $\tilde{\sigma}$ does not capture any of the subtle large scale non-linear effects of the field σ described in our main result, but for any given site it coincides with σ for many times which allows to replace σ with $\tilde{\sigma}$ below (see e.g. (3.17)). The advantage of this replacement is that one can then average over $\tilde{\sigma} \in \{-1, 0, +1\}$: indeed, note that by (2.37) and (2.38) and the definition (3.11) for A

$$\mathbb{E}A(\tilde{\sigma}(r,k)) = \frac{e^{\theta_c}}{\mathcal{N}_c}A(-1) + \frac{e^{\theta_c}}{\mathcal{N}_c}A(1) + \frac{1}{\mathcal{N}_c}A(0) = \frac{4e^{\theta_c}}{1+2e^{\theta_c}} = \frac{2}{\beta_c}\,. \tag{3.13}$$

This is essentially the reason why the pre-factor $\sqrt{2/\beta_c}$ in front of the noise of the limiting equation shows up (see Remark 2.6). In the proof of Theorem 3.3 we only make use of the averaging in time over $\tilde{\sigma}$. The proof of Proposition 3.4 below then relies on the same construction and we will make use of the spatial averaging as well.

We now proceed to the construction of this coupling. By definition, for any fixed site $k \in \Lambda_N$ the process $\sigma(s,k)$ is a pure jump processes on $\{\pm 1, 0\}$. The joint law of all of these processes can be constructed as follows:

- For each site there is an independent Poisson clock, running at rate 1.
- At each jump of the Poisson clock the spin changes according to the transition probabilities given in the vector

  $$P(s,k) = \begin{pmatrix} c^s_{\gamma,\mathrm{m}}(\sigma_{\gamma,\mathrm{m}}(s),k,1) \\ c^s_{\gamma,\mathrm{m}}(\sigma_{\gamma,\mathrm{m}}(s),k,0) \\ c^s_{\gamma,\mathrm{m}}(\sigma_{\gamma,\mathrm{m}}(s),k,-1) \end{pmatrix}.$$

  Of course this vector depends on the configuration of the neighboring particles at time s.

The transition probabilities of the auxiliary processes $\tilde{\sigma}(s,k),\ k \in \Lambda_N$ are fixed and given by (3.12). In order to construct the coupling, we note that according to Lemma 3.1 there exists a number q satisfying

$$1 \geq q \geq 1 - C\gamma^{1-3\nu}\,,$$

such that $q\tilde{P} \leq P$ where the inequality of the two vectors is to be understood entry by entry. Therefore, we can write

$$P(s,k) = q\tilde{P} + (1-q)R(s,k),$$

where R is normalized to be a probability measure. The coupling is now the following:

- At the initial time each of the $\tilde{\sigma}(0,k)$ is distributed according to $\tilde{P}$ and the realizations for different sites $k \neq k'$ are independent.
- At each jump of the Poisson clock at site k, $\tilde{\sigma}(s,k)$ is updated according to $\tilde{P}$. This update is independent from the updates at other sites as well as the jump times.
- To determine the updated spin for $\sigma(s,k)$ after the same jump of the Poisson clock, the vector $R(s,k)$ are evaluated. It depends on the environment at the given time s.
- Toss a coin which yields 1 with probability q and 0 with probability $1-q$. If the outcome of this toss is 1 the spin $\sigma(s,k)$ is updated to the same value as $\tilde{\sigma}(s,k)$. If the outcome is 0 then $\sigma(s,k)$ is updated according to $R(s,k)$ independently of the update for $\tilde{\sigma}$.

It is clear that the process $\tilde{\sigma}$ constructed in this way is a jump Markov chain jumping according to $\tilde{P}$ and that the processes for different sites are independent. This construction is consistent with the jumping rule of σ (in particular σ jumps according to P). Furthermore, for every $k \in \Lambda_N$, after each jump the probability that $\tilde{\sigma}(s,k) \neq \sigma(s,k)$ is bounded by $C\gamma^{1-3\nu}$, where the constant C obtained from (3.9) does not depend on the location k and the jump-time.

To lighten the notation in the following calculation we introduce the centered random field $\bar{A}(\tilde{\sigma}(r,k)) = A(\tilde{\sigma}(r,k)) - \frac{2}{\beta_c}$ where A was defined in (3.11).

Lemma 3.2 *For every $r, r' \geq 0$ and $k, k' \in \Lambda_N$ we have*

$$\mathbb{E}\bar{A}(\tilde{\sigma}(r,k))\bar{A}(\tilde{\sigma}(r',k')) \leq C\mathbf{1}_{k=k'}e^{-|r-r'|}.$$

Proof. Recall from the construction that for $k \neq k'$ the random variables $\tilde{\sigma}(r,k)$ and $\tilde{\sigma}(r',k')$ are independent and that therefore for these $k \neq k'$ we have

$$\mathbb{E}\bar{A}(\tilde{\sigma}(r,k))\bar{A}(\tilde{\sigma}(r',k')) = 0.$$

To get bounds in the temporal correlations for $\tilde{\sigma}(\cdot,k)$ for a fixed site k we fix times $r' < r$ and denote by τ the first instance of the Poisson clock for site z after r'. Recall from the construction of $\tilde{\sigma}$ that if $r < \tau$ the spin values of $\tilde{\sigma}(r,k)$ and $\tilde{\sigma}(r,k)$ are identical. The value after τ becomes independent of the value before τ. With this discussion in mind we write

$$\begin{aligned} &\mathbb{E}\bar{A}(\tilde{\sigma}(r,k))\,\bar{A}(\tilde{\sigma}(r',k)) \\ &= \mathbb{E}\bar{A}(\tilde{\sigma}(r,k))^2\mathbf{1}_{\tau>r} + \mathbb{E}\bar{A}(\tilde{\sigma}(r,k))\bar{A}(\tilde{\sigma}(r',k))\mathbf{1}_{\tau\leq r}\;. \end{aligned}$$

The first term on the right hand side is bounded by

$$\mathbb{E}\bar{A}(\tilde{\sigma}(r,k))^2\mathbf{1}_{\tau<r} \leq \sup_{\bar{\sigma}\in\{\pm1,0\}} |A(\bar{\sigma})|\;\mathbb{P}(\tau > r) \leq Ce^{-|r-r'|}.$$

For the second term we write

$$\begin{aligned} &\mathbb{E}\bar{A}(\tilde{\sigma}(r,k))\bar{A}(\tilde{\sigma}(r',k))\mathbf{1}_{\tau\leq r} \\ &= \mathbb{E}\bar{A}(\tilde{\sigma}(r',k))\mathbf{1}_{\tau\leq r}\mathbb{E}(\bar{A}(\tilde{\sigma}(r,k))|\mathcal{F}_\tau) = 0, \end{aligned}$$

where $\mathcal{F}_\tau$ is the sigma algebra generated by $\tilde{\sigma}(\cdot,k)$ up to the stopping time τ. □

Theorem 3.3 (Convergence of Z_γ) *Let $\nu \in (0,1/2)$ and $\mathfrak{m} > 1$. As γ tends to 0, the processes $Z_{\gamma,\mathfrak{m}}$ converge in law to Z with respect to the Skorokhod topology on $\mathcal{D}(\mathbb{R}_+,\mathcal{C}^{-\nu})$, where Z is defined as*

$$Z(t,\cdot) \stackrel{\text{def}}{=} \sqrt{2/\beta_c}\int_0^t P_{t-s}\,dW(s,\cdot)\;.$$

Proof. Proposition 4.4 below for the case $n = 1$ shows that the family $\{Z_{\gamma,\mathfrak{m}}, \gamma \in (0,\frac{1}{3})\}$ is tight on $\mathcal{D}(\mathbb{R}_+,\mathcal{C}^{-\nu})$ and any weak limit is supported on $\mathcal{C}(\mathbb{R}_+,\mathcal{C}^{-\nu})$. Given this tightness result, we aim to show that any weak accumulation point $\bar{Z}$ solves the martingale problem discussed in Theorem 6.1 and Appendix D of [MW16]. The argument for the "drift" part of the martingale problem, namely establishing that

$$\mathscr{M}_{\bar{Z},\phi}(t) \stackrel{\text{def}}{=} (\bar{Z}(t),\phi) - \int_0^t (\bar{Z}(s),\Delta\phi)\,ds$$

is a local martingale for any test function $\phi \in \mathcal{C}^\infty$ is identical to [MW16]. Indeed, the claim we need to establish is that there exists a sequence of stopping times T_n with $T_n \uparrow \infty$ a.s. as $n \to \infty$ such that for all $s < t$ and all random variables F which are bounded and measurable with respect to the σ-algebra over $\mathcal{D}([0,s], \mathcal{C}^{-\nu})$ we have

$$\mathbb{E}\Big((\mathscr{M}_{\bar{Z},\phi}(t \wedge T_n) - \mathscr{M}_{\bar{Z},\phi}(s \wedge T_n))F \Big) = 0. \tag{3.14}$$

For any $\mathcal{C}^\infty$ function ϕ

$$\mathcal{M}_{\gamma,\phi}(t) = (Z_{\gamma,\mathfrak{m}}(t), \phi) - \int_0^t (Z_{\gamma,\mathfrak{m}}(s), \Delta_\gamma \phi)\, ds\ , \tag{3.15}$$

is a martingale by assumption and therefore the formula (3.14) with $\mathscr{M}_{\bar{Z},\phi}$ replaced by $\mathcal{M}_{\gamma,\phi}$ holds irrespective of the choice of stopping time T_n. Just as in [MW16, Eq. (6.6)] it follows that the approximate Laplacian Δ_γ appearing in expression (3.15) can be replaced by the full Laplacian Δ up to an error which is controlled by $C(\phi)\gamma^{2-2\kappa}$ in both the "first regime" and the "second regime". Furthermore, by assumption the processes $Z_{\gamma,\mathfrak{m}}$ converge in law to $\bar{Z}$ and as the functional that maps $Z_{\gamma,\mathfrak{m}}$ to $(Z_{\gamma,\mathfrak{m}}(t), \phi) - \int_0^t (Z_{\gamma,\mathfrak{m}}(s), \Delta\phi)\, ds$ is continuous with respect to the topology of $\mathcal{D}([0,t], \mathcal{C}^\nu)$ (recall that ϕ is smooth) we can pass to the limit as soon as we have some control over the uniform integrability of these random variables. This is precisely the role of the stopping times - if we set $T_{L,\gamma} = \inf\{t \geq 0 : \|Z_{\gamma,\mathfrak{m}}(t)\|_{\mathcal{C}^{-\nu}} > L\}$ then it follows just as in [MW16, Proof of Thm. 6.1] that (outside of a hypothetical countable set of values L) the processes $Z_{\gamma,\mathfrak{m}}(s \wedge T_{\gamma,N})$ also converge in law and furthermore for fixed L, s, t the random variables

$$(Z_{\gamma,\mathfrak{m}}(t \wedge T_{\gamma,N}), \phi) - \int_0^{t \wedge T_{\gamma,N}} (Z_{\gamma,\mathfrak{m}}(s \wedge T_{\gamma,N}), \Delta_\gamma \phi)\, ds\ , \tag{3.16}$$

are uniformly bounded as $\gamma \to 0$ which permits to pass to the limit and establishes (3.14).

The more interesting part concerns the quadratic variation. More precisely, we need to show that

$$\big(\mathscr{M}_{\bar{Z},\phi}(t)\big)^2 - \frac{2t}{\beta_c}\|\phi\|_{L^2}^2$$

is a local martingale; recall that the factor $2/\beta_c$ naturally appears from (3.13).

This follows if we can establish that for any trigonometric polynomial ϕ of degree $\leq \gamma^{-2}$ (or γ^{-3} depending on the regime), which automatically satisfies the identity

$$(M_{\gamma,\mathfrak{m}}(t), \phi) = \sum_{x \in \Lambda_\varepsilon} \varepsilon^2 M_{\gamma,\mathfrak{m}}(t,x)\phi(x)$$

we have

$$\begin{aligned} \langle (M_{\gamma,\mathfrak{m}}(t), \phi) \rangle &= c_{\gamma,2}^2 \sum_{x,y \in \Lambda_\varepsilon} \varepsilon^4 \phi(x)\phi(y) \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 K_\gamma(x-z)\, K_\gamma(y-z) \\ &\quad \times \int_0^t \sum_{\bar{\sigma} \in \{\pm 1, 0\}} (\bar{\sigma} - \sigma(\alpha^{-1}s, \varepsilon^{-1}z))^2 C_{\gamma,\mathfrak{m}}(s,z,\bar{\sigma})\, ds \\ &= \frac{2t}{\beta_c}\|\phi\|_{L^2}^2 + E_\gamma'''(t), \end{aligned}$$

for an error $E'''_\gamma(t)$ for which $\mathbb{E}|E'''_\gamma(t)| \to 0$ as $\gamma \to 0$. For this statement in turn (3.10) and (3.11) show that it is sufficient to prove that for every $z \in \Lambda_\varepsilon$ we have

$$\int_0^t A(\sigma(\alpha^{-1}s, \varepsilon^{-1}z))ds = \frac{2t}{\beta_c} + E''''_\gamma, \tag{3.17}$$

with a good control on E''''_γ. Indeed, one has $|c^2_{\gamma,2} - 1| \le O(\gamma^2)$ and by (2.2), (2.3) and $K_\gamma(x) = \varepsilon^{-2}\kappa_\gamma(\varepsilon^{-1}x)$,

$$\sum_{x,y\in\Lambda_\varepsilon} \varepsilon^4 \phi(x)\phi(y) \sum_{z\in\Lambda_\varepsilon} \varepsilon^2 K_\gamma(x-z)\, K_\gamma(y-z) \to \|\phi\|^2_{L^2} \ ,$$

independently of the scaling relation between ε and γ (thus it holds for both scaling regimes). Although we have assumed that ϕ is a trigonometric polynomial, by [MW16, Remark C.4], this is sufficient to characterize the law of $\overline{Z}$.

While the error terms $E_\gamma, E'_\gamma, E''_\gamma$ were all deterministically bounded, we will only get a probabilistic bound for E'''_γ. To obtain this bound we will need the coupling between the microscopic spin processes σ and $\tilde\sigma$.

Recall that for every z, after each jump the probability that $\tilde\sigma(\alpha^{-1}s, \varepsilon^{-1}z) \ne \sigma(\alpha^{-1}s, \varepsilon^{-1}z)$ is bounded by $C\gamma^{1-3\nu}$, where the constant C does not depend on z and the jump-time. We then get

$$\begin{aligned}\int_0^t A(\sigma(\alpha^{-1}s, \varepsilon^{-1}z))ds - \frac{2t}{\beta_c} &= \int_0^t A(\tilde\sigma(\alpha^{-1}s, \varepsilon^{-1}z))ds - \frac{2t}{\beta_c} \\ &\quad + \int_0^t A(\sigma(\alpha^{-1}s, \varepsilon^{-1}z))ds - A(\tilde\sigma(\alpha^{-1}s, \varepsilon^{-1}z))\, ds.\end{aligned}$$

For the term in the second line we get

$$\begin{aligned}&\mathbb{E}\Big|\int_0^t A(\sigma(\alpha^{-1}s, \varepsilon^{-1}z)) - A(\tilde\sigma(\alpha^{-1}s, \varepsilon^{-1}z))\, ds\Big| \\ &\quad\le \sup_{\bar\sigma\in\{\pm1,0\}} |A(\bar\sigma)| \int_0^t \mathbb{P}\Big(\sigma(\alpha^{-1}s, \varepsilon^{-1}z) \ne \tilde\sigma(\alpha^{-1}s, \varepsilon^{-1}z)\Big) ds \\ &\quad\le \sup_{\bar\sigma\in\{\pm1,0\}} |A(\bar\sigma)| \int_0^t \Big(\mathbb{P}(T_o > s) + C\gamma^{1-3\nu}\Big) ds \\ &\quad\le \sup_{\bar\sigma\in\{\pm1,0\}} |A(\bar\sigma)| \int_0^t \Big(e^{-\frac{s}{\alpha}} + C\gamma^{1-3\nu}\Big) ds \\ &\quad\le \sup_{\bar\sigma\in\{\pm1,0\}} |A(\bar\sigma)| (\alpha + Ct\gamma^{1-3\nu})\ .\end{aligned} \tag{3.18}$$

Here T_o is the holding time before the first jump.

For the other term, by Lemma 3.2, its second moment can be bounded as

$$\begin{aligned}&\mathbb{E}\Big(\int_0^t A(\tilde\sigma(\alpha^{-1}s, \varepsilon^{-1}z))ds - \frac{2t}{\beta_c}\Big)^2 \\ &\quad\le \int_0^t \int_0^t \mathbb{E}\bar A(\tilde\sigma(\alpha^{-1}s, \varepsilon^{-1}z))\bar A(\tilde\sigma(\alpha^{-1}s', \varepsilon^{-1}z))\, ds\, ds'\end{aligned}$$

$$\leq \int_0^t \int_0^t e^{-\frac{|s-s'|}{\alpha}}\, ds\, ds' \leq C\alpha\ .$$

So this term goes to zero as well. Therefore we have shown that the error term in (3.17) goes to zero and thus the theorem is proved. □

The following result will also be applied several times in the sequel.

Proposition 3.4 *For every $0 \leq s \leq t$ and $x \in \Lambda_\varepsilon$, one has*

$$\begin{aligned}\int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2\,(P^\gamma_{t-r} \star_\varepsilon K_\gamma)^2(z-x) \sum_{\bar\sigma\in\{\pm1,0\}} (\bar\sigma - \sigma(r,\varepsilon^{-1}z))^2 C_{\gamma,\mathfrak{m}}(r,z,\bar\sigma)\, dr \\ = \frac{2}{\beta_c}\int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2\,(P^\gamma_{t-r} \star_\varepsilon K_\gamma)^2(z-x)\, dr + \tilde E_t(s,x)\end{aligned} \tag{3.19}$$

where the process $\tilde E$ satisfies the bound

$$\mathbb{E}|\tilde E_t(s,x)|^p \leq C\gamma^{1-3\nu}\log(\gamma^{-1})^p \tag{3.20}$$

for every $p \geq 2$ and some constant $C = C(T,\nu,\mathfrak{m})$ depending linearly on $\mathfrak{m}$. Its extension $\mathsf{Ext}\tilde E_t(s,\cdot)$, which will still be denoted by $\tilde E_t(s,\cdot)$, satisfies

$$\mathbb{E}\|\mathsf{Ext}\tilde E_t(s,\cdot)\|^p_{L^p(\mathbb{T}^2)} \leq C\gamma^{1-4\nu}\log(\gamma^{-1})^{2p} \tag{3.21}$$

for every $p \geq 2$ and some constant $C = C(T,\nu,\mathfrak{m})$ depending linearly on $\mathfrak{m}$.

Proof. We first show that the sum over $\bar\sigma$ can be replaced by $A(\sigma(r,\varepsilon^{-1}z))$ (recall the definition of A in (3.11)) up to an error which is controlled deterministically. Turning to Fourier space, using (5.2) and Parseval's identity and the elementary bound $\int_0^s e^{-(s-r)a} dr \leq C(\frac{1}{s} + a)^{-1}$ for any $a > 0$, we obtain

$$\int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2\,(P^\gamma_{t-r} \star_\varepsilon K_\gamma)^2(z)\, dr \leq \sum_{\omega\in\{-N,\dots,N\}^2} \frac{|\hat K_\gamma(\omega)|^2}{t^{-1} + 2\gamma^{-b}(1-\hat K_\gamma(\omega))} \tag{3.22}$$

where $b = 2$ in the first regime and $b = 4$ in the second regime. We then use the estimates (5.3) and the first estimate in (5.6) to bound the sum over $|\omega| \leq C\gamma^{-1}$ (resp. $C\gamma^{-2}$) and the estimate (5.7) to bound the sum over $|\omega| \geq C\gamma^{-1}$ (resp. $C\gamma^{-2}$) in the first (resp. second) regime, which permits to conclude that the right hand side of (3.22) is bounded by $C\log(\gamma^{-1})$. Therefore, invoking (3.10), the left hand side of (3.19) is equal to

$$\int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2\,(P^\gamma_{t-r} \star_\varepsilon K_\gamma)^2(z-x)\, A(\sigma(\alpha^{-1}r,\varepsilon^{-1}z))\, dr$$

plus an error which is deterministically bounded by $C\gamma^{1-3\nu}\log\gamma^{-1}$.

We proceed as in the proof of Theorem 3.3, again making use of the process $\tilde\sigma$ constructed at the beginning of this section. Arguing as in (3.18) we can replace $A(\sigma(\alpha^{-1}r,\varepsilon^{-1}z))$ in the above integral by $A(\tilde\sigma(\alpha^{-1}r,\varepsilon^{-1}z))$ with an error satisfying the following *first* moment bound

$$\begin{aligned}\mathbb{E}\int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2\,(P^\gamma_{t-r} \star_\varepsilon K_\gamma)^2(z-x)\,\Big|A(\sigma(\alpha^{-1}r,\varepsilon^{-1}z)) - A(\tilde\sigma(\alpha^{-1}r,\varepsilon^{-1}z))\Big|\, dr \\ \leq \sup_{\bar\sigma\in\{\pm1,0\}} |A(\bar\sigma)| \int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2\,(P^\gamma_{t-r} \star_\varepsilon K_\gamma)^2(z-x)\,(e^{-\frac{r}{\alpha}} + C\gamma^{1-3\nu})\, dr\ .\end{aligned} \tag{3.23}$$

We claim that by a similar argument to the one leading to (3.22), the right hand side of (3.23) can be bounded by $C\gamma^{1-3\nu}\log\gamma^{-1}$. Indeed, for the term involving $C\gamma^{1-3\nu}$ this is immediately clear from the above $\log(\gamma^{-1})$ bound on (3.22). For the term with $e^{-\frac{r}{\alpha}}$ we divide the r-integral into an integral over $r \in [\gamma, s]$ and an integral over $r \in [0,\gamma]$. For the integral over $r \in [\gamma, s]$, we simply bound $e^{-\frac{r}{\alpha}} \le C\gamma$ (recall that $\alpha \approx \gamma^2$ in the first and $\alpha \approx \gamma^4$ in the second scaling regime), and the integration of the other factors is bounded by $C\log(\gamma^{-1})$ as above. For the integral over $r \in [0,\gamma]$, we bound $e^{-\frac{r}{\alpha}} \le 1$, and then since after applying Parseval's identity the only r-dependent factor inside the r-integral is $e^{-2(s-r)\gamma^{-b}(1-\hat{K}_\gamma(\omega))}$ and as this function is monotonically increasing in r, we have

$$\int_0^\gamma e^{-(s-r)\gamma^{-b}(1-\hat{K}_\gamma(\omega))}dr \le \frac{\gamma}{s}\int_0^s e^{-(s-r)\gamma^{-b}(1-\hat{K}_\gamma(\omega))}dr \ ;$$

applying the above $\log(\gamma^{-1})$ bound again we conclude that as claimed the right hand side of (3.23) is bounded by $C\gamma^{1-3\nu}\log\gamma^{-1}$.

Finally using the deterministic bound

$$\int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2 \,(P^\gamma_{t-r} \star_\varepsilon K_\gamma)^2(z-x)\,|A(\sigma(\alpha^{-1}r,\varepsilon^{-1}z)) - A(\tilde\sigma(\alpha^{-1}r,\varepsilon^{-1}z))|\,dr \\ \le C\log\gamma^{-1},$$

the above bound on the *first* moment can be upgraded to a bound on *all* stochastic moments. We get for any $p \ge 1$ that

$$\mathbb{E}\Big(\int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2 \,(P^\gamma_{t-r} \star_\varepsilon K_\gamma)^2(z-x)\,\Big|A(\sigma(\alpha^{-1}r,\varepsilon^{-1}z)) - A(\tilde\sigma(\alpha^{-1}r,\varepsilon^{-1}z))\Big|\,dr\Big)^p \\ \le C\gamma^{1-3\nu}(\log\gamma^{-1})^p. \tag{3.24}$$

To prove (3.19) it remains to control moments of the error term

$$\int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2 \,(P^\gamma_{t-r} \star_\varepsilon K_\gamma)^2(z-x)\Big(A(\tilde\sigma(\alpha^{-1}r,\varepsilon^{-1}z)) - \frac{2}{\beta_c}\Big)\,dr\ .$$

As before we use the centered random field $\bar{A}(\tilde\sigma(\alpha^{-1}r,\varepsilon^{-1}z)) = A(\tilde\sigma(\alpha^{-1}r,\varepsilon^{-1}z)) - \frac{2}{\beta_c}$ and write

$$\mathbb{E}\Big(\int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2 \,(P^\gamma_{t-r} \star_\varepsilon K_\gamma)^2(z-x)\bar{A}(\tilde\sigma(\alpha^{-1}r,\varepsilon^{-1}z))\,dr\Big)^2 \\ = \int_0^s\int_0^s \sum_{z\in\Lambda_\varepsilon}\sum_{z'\in\Lambda_\varepsilon} \varepsilon^4\,(P^\gamma_{t-r} \star_\varepsilon K_\gamma)^2(z-x)(P^\gamma_{t-r'} \star_\varepsilon K_\gamma)^2(z'-x) \\ \times \mathbb{E}\bar{A}(\tilde\sigma(\alpha^{-1}r,\varepsilon^{-1}z))\,\bar{A}(\tilde\sigma(\alpha^{-1}r',\varepsilon^{-1}z'))\,drdr'.$$

Applying Lemma 3.2, this turns into

$$
\begin{aligned}
&\mathbb{E}\Big(\int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2 \,(P^\gamma_{t-r}\star_\varepsilon K_\gamma)^2(z-x)\bar{A}(\tilde\sigma(\alpha^{-1}r,\varepsilon^{-1}z))\,dr\Big)^2\\
&\le C\varepsilon^2\int_0^s\int_0^s \sum_{z\in\Lambda_\varepsilon}\varepsilon^2\,(P^\gamma_{t-r}\star_\varepsilon K_\gamma)^2(z-x)(P^\gamma_{t-r'}\star_\varepsilon K_\gamma)^2(z-x)e^{\frac{|r-r'|}{\alpha}}\,drdr'\\
&\le C\varepsilon^2\sup_{r'\in[0,s]}\|P^\gamma_{t-r'}\star_\varepsilon K_\gamma\|_{L^\infty(\Lambda_\varepsilon)}\\
&\qquad\times\int_0^s\sum_{z\in\Lambda_\varepsilon}\varepsilon^2\,(P^\gamma_{t-r}\star_\varepsilon K_\gamma)^2(z-x)\Big(\int_0^s e^{\frac{|r-r'|}{\alpha}}dr'\Big)dr\\
&\le C\varepsilon^2\gamma^{-2b}\log(\gamma^{-1})\alpha\int_0^s\sum_{z\in\Lambda_\varepsilon}\varepsilon^2\,(P^\gamma_{t-r}\star_\varepsilon K_\gamma)^2(z-x)dr\\
&\le C\varepsilon^2\gamma^{-2b}(\log(\gamma^{-1}))^2\alpha,
\end{aligned}
$$

where in the third inequality we have use (5.9) and $b=1$ in the first regime and $b=2$ in the second regime. In the first regime (2.13) this expression is bounded by $\le C\gamma^4(\log(\gamma^{-1}))^2$ and in the second regime (2.14) it is bounded by $\le C\gamma^6(\log(\gamma^{-1}))^2$. As before we can upgrade this stochastic L^2 to a stochastic L^p bound by using a deterministic bound

$$
\int_0^s\sum_{z\in\Lambda_\varepsilon}\varepsilon^2\,(P^\gamma_{t-r}\star_\varepsilon K_\gamma)^2(z-x)\bar{A}(\tilde\sigma(\alpha^{-1}r,\varepsilon^{-1}z))\,dr\le C\log\gamma^{-1}.
$$

In both scaling regimes this yields a bound which is lower order with respect to (3.24) so that (3.20) follows.

To obtain the second bound (Eq. (3.21)) we sum (3.20) over $x\in\Lambda_\varepsilon$ to obtain

$$
\mathbb{E}\|\tilde{E}_t(s,\cdot)\|^p_{L^p(\Lambda_\varepsilon)}=\sum_{x\in\Lambda_\varepsilon}\varepsilon^2\mathbb{E}|\tilde{E}_t(s,x)|^p\le C\gamma^{1-3\nu}\log(\gamma^{-1})^p.
$$

To replace the L^p norm over Λ_ε by the L^p norm over the continuous torus and $\tilde{E}$ by its extension write using Jensen's inequality

$$
\begin{aligned}
&\int_{\mathbb{T}^2}|\mathsf{Ext}\tilde{E}_t(s,z)|^p dz\\
&=\int_{\mathbb{T}^2}\Big|\sum_{x\in\Lambda_\varepsilon}\varepsilon^2\tilde{E}_t(s,x)\mathsf{Ker}(x-z)\Big|^p dz\\
&\le\int_{\mathbb{T}^2}\Big(\sum_{x\in\Lambda_\varepsilon}\varepsilon^2|\tilde{E}_t(s,x)|^p|\mathsf{Ker}(x-z)|\Big)\Big(\sum_{x\in\Lambda_\varepsilon}\varepsilon^2|\mathsf{Ker}(x-z)|\Big)^{p-1}dz
\end{aligned}
\tag{3.25}
$$

where (as discussed in [MW16, Lemma A.6]) the extension kernel is given by

$$
\mathsf{Ker}(x-z)=\prod_{j=1}^2\frac{\sin\big(\frac{\pi}{2}(2N+1)(x_j-z_j)\big)}{\sin\big(\frac{\pi}{2}(x_j-z_j)\big)}
$$

so that we have that $\sum_{x\in\Lambda_\varepsilon}\varepsilon^2|\mathsf{Ker}(x-z)|\le C\log\gamma^{-1}$ uniformly in z. Plugging this estimate into (3.25) yields

$$\begin{aligned}&\int_{\mathbb{T}^2} |\mathsf{Ext}\tilde{E}_t(s,z)|^p dz\\&\leq C(\log\gamma^{-1})^{p-1}\Big(\sum_{x\in\Lambda_\varepsilon}\varepsilon^2|\tilde{E}_t(s,x)|^p\int_{\mathbb{T}^2}|\mathsf{Ker}(x-z)|\,dz\Big)\\&\leq C(\log\gamma^{-1})^p\|\tilde{E}_t(s,\cdot)\|^p_{L^p(\Lambda_\varepsilon)}\end{aligned}$$

so (3.21) follows as well. □

4 Wick powers and proof of the main theorem

The aim of this section is to prove Theorem 2.5. Since we will apply a discrete version of Da Prato-Debussche argument ([DPD03]) as in [MW16], an important step is to prove the convergence of the approximate Wick powers $Z_\gamma^{:n:}$ to the Wick powers. Fortunately, the work [MW16] treated the Wick powers with general n, though only $n \leq 3$ was needed therein; here we only need some minor modifications to their construction of Wick powers.

We start by recalling the definitions of the approximate Wick powers $Z_\gamma^{:n:}$. Recall that Z_γ is defined in (3.2). It will be convenient to work with the following family of approximations to $Z_\gamma(t,x)$. For $s \leq t$, we introduce

$$R_{\gamma,t}(s,x) \stackrel{\text{def}}{=} \int_{r=0}^{s} P^\gamma_{t-r}\, dM_\gamma(r,x)\;,$$

and extend $R_{\gamma,t}(s,\cdot)$ and $Z_\gamma(t,\cdot)$ to functions on all of $\mathbb{T}^2$ by trigonometric polynomials of degree $\leq N$ as (2.35). Note that for any t and any $x\in\mathbb{T}^2$, the process $R_{\gamma,t}(\cdot,x)$ is a martingale and $R_{\gamma,t}(t,\cdot) = Z_\gamma(t,\cdot)$.

The iterated integrals are then defined recursively as follows. For a fixed $t \geq 0$ and $x\in\mathbb{T}^2$, we set $R^{:1:}_{\gamma,t}(s,x) = R_{\gamma,t}(s,x)$. For $n\geq 2,\ t\geq 0$ and $x\in\Lambda_\varepsilon$, we set

$$R^{:n:}_{\gamma,t}(s,x) = n\int_{r=0}^{s} R^{:n-1:}_{\gamma,t}(r^-,x)\, dR_{\gamma,t}(r,x)\;. \tag{4.1}$$

We use the notation $R^{:n-1:}_{\gamma,t}(r^-,x)$ to denote the left limit of $R^{:n-1:}_{\gamma,t}(\cdot,x)$ at r. This definition ensures that $(R^{:n:}_{\gamma,t}(s,x))_{0\leq s\leq t}$ is a martingale. The extension of $R^{:n:}_{\gamma,t}(s,\cdot)$ to the entire $\mathbb{T}^2$ is also defined recursively, through its Fourier series

$$\hat{R}^{:n:}_{\gamma,t}(s,\omega) \stackrel{\text{def}}{=} n\int_{r=0}^{s}\frac{1}{4}\sum_{\tilde\omega\in\mathbb{Z}^2}\hat{R}^{:n-1:}_{\gamma,t}(r^-,\omega-\tilde\omega)\, d\hat{R}_{\gamma,t}(r,\tilde\omega)\;, \tag{4.2}$$

and set $R^{:n:}_{\gamma,t}(s,x) \stackrel{\text{def}}{=} \frac{1}{4}\sum_{\omega\in\mathbb{Z}^2}\hat{R}^{:n:}_{\gamma,t}(s,\omega)e^{i\pi\omega\cdot x}$. This definition coincides with (4.1) on Λ_ε, and for every $n\geq 2$ the function $R^{:n:}_{\gamma,t}(s,\cdot)\colon \mathbb{T}^2\to\mathbb{R}$ is a trigonometric polynomial of degree $\leq nN$. For any $n\geq 2$ and for $t\geq 0$, $x\in\mathbb{T}^2$ we define

$$Z^{:n:}_\gamma(t,x) \stackrel{\text{def}}{=} R^{:n:}_{\gamma,t}(t,x)\;. \tag{4.3}$$

Finally let $R^{:n:}_{\gamma,t,\mathfrak{m}}$ and $Z^{:n:}_{\gamma,\mathfrak{m}}$ be iterated stochastic integrals defined just as $R^{:n:}_{\gamma,t}$ and $Z^{:n:}_\gamma$ but with M_γ replaced by $M_{\gamma,\mathfrak{m}}$. Recall that $\mathfrak{m}$ is the parameter fixed in (3.5).

By the definition of $R_{\gamma,t}(s,x)$ and the quadratic variation of M_γ, one has

$$\begin{aligned}\langle R_{\gamma,t}(\cdot,x)\rangle_s = c_{\gamma,2}^2 \int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2 (P^\gamma_{t-r} \star_\varepsilon K_\gamma)^2(x-z) \\ \times \sum_{\bar\sigma\in\{\pm1,0\}} (\bar\sigma - \sigma(r,\varepsilon^{-1}z))^2 C_{\gamma,\mathfrak{m}}(r,z,\bar\sigma)\, dr\ .\end{aligned} \tag{4.4}$$

There exists a constant $\gamma_0 > 0$ (arising when we apply the kernel bounds in Section 5) such that the following results hold.

Proposition 4.1 *For every $n \in \mathbb{N}$, $p \geq 1$, $\nu > 0$, $T > 0$, $0 \leq \lambda \leq \frac{1}{2}$ and $0 < \kappa \leq 1$, there exists a constant $C = C(n,p,\nu,T,\lambda,\kappa)$ such that for every $0 \leq s \leq t \leq T$ and $0 < \gamma < \gamma_0$, one has*

$$\mathbb{E} \sup_{0\leq r\leq t} \|R^{:n:}_{\gamma,t}(r,\cdot)\|^p_{\mathcal{C}^{-\nu-2\lambda}} \leq C\, t^{\lambda p} + C\gamma^{p(1-\kappa)}\ , \tag{4.5}$$

$$\mathbb{E} \sup_{0\leq r\leq t} \|R^{:n:}_{\gamma,t}(r,\cdot) - R^{:n:}_{\gamma,s}(r\wedge s,\cdot)\|^p_{\mathcal{C}^{-\nu-2\lambda}} \leq C\,|t-s|^{\lambda p} + C\gamma^{p(1-\kappa)}\ , \tag{4.6}$$

$$\mathbb{E} \sup_{0\leq r\leq t} \|R^{:n:}_{\gamma,t}(r,\cdot) - R^{:n:}_{\gamma,t}(r\wedge s,\cdot)\|^p_{\mathcal{C}^{-\nu-2\lambda}} \leq C\ |t-s|^{\lambda p} + C\gamma^{p(1-\kappa)}\ . \tag{4.7}$$

The same bounds hold for $R^{:n:}_{\gamma,t,\mathfrak{m}}$.

Proposition 4.2 *For $x \in \Lambda_\varepsilon$, let*

$$Q_{\gamma,t}(s,x) = [R_{\gamma,t}(\cdot,x)]_s - \langle R_{\gamma,t}(\cdot,x)\rangle_s\ . \tag{4.8}$$

For any $t \geq 0$, $\kappa > 0$ and $1 \leq p < +\infty$, there exists $C = C(t,\kappa,p)$ such that for $0 < \gamma < \gamma_0$,

$$\mathbb{E} \sup_{x\in\Lambda_\varepsilon} \sup_{0\leq s\leq t} |Q_{\gamma,t}(s,x)|^p \leq C\gamma^{p(1-\kappa)}.$$

The same bound holds for $Q_{\gamma,t,\mathfrak{m}}$, that is, the same process as $Q_{\gamma,t}$ but defined via $M_{\gamma,\mathfrak{m}}$ instead of M_γ.

One important result is that these iterated integrals are almost Hermite polynomials with renormalization constant chosen as $[R_{\gamma,t}(\cdot,x)]_s$.

Proposition 4.3 *Define*

$$E^{:n:}_{\gamma,t}(s,x) \stackrel{\text{def}}{=} H_n(R_{\gamma,t}(s,x), [R_{\gamma,t}(\cdot,x)]_s) - R^{:n:}_{\gamma,t}(s,x)\ , \tag{4.9}$$

for any $x \in \mathbb{T}^2$. Here, we view $[R_{\gamma,t}(\cdot,x)]_s$ as defined on all of $\mathbb{T}^2$, by extending it as a trigonometric polynomial of degree $\leq N$. Then for any $n \in \mathbb{N}$, $\kappa > 0$, $t > 0$ and $1 \leq p < \infty$, there exists $C = C(n,p,t,\kappa) > 0$ such that for every sufficiently small $\gamma > 0$,

$$\mathbb{E} \sup_{x\in\mathbb{T}^2} \sup_{0\leq s\leq t} |E^{:n:}_{\gamma,t}(s,x)|^p \leq C\gamma^{p(1-\kappa)}.$$

The same bound holds for $E^{:n:}_{\gamma,t,\mathfrak{m}}$ - the same process as $E^{:n:}_{\gamma,t}$ but defined via $M_{\gamma,\mathfrak{m}}$ instead of M_γ.

Proof of Prop. 4.1 - 4.3. For the case of the Kac Ising model, these results are Prop 4.2, Lemma 5.1 and Prop 5.3 in [MW16]. Several modifications of these proofs are necessary for the case of our Blume-Capel model.

The first necessary modification is due to the difference in the scalings (2.13) and (2.14). This difference comes into play via the estimates on the kernels K_γ and P^γ_t used throughout the proofs. We list all these kernel estimates in Section 5. These estimates with modifications in the second regime lead to the desired bounds *mutatis mutandis*.

Another necessary modification of the proof for the case of our Blume-Capel model is due to the fact that the martingale we use to build $Z^{:n:}_\gamma$ is different. For Proposition 4.1, the only place where the martingale enters into play is [MW16, Lemma 4.1], which is a consequence of Burkholder-Davis-Gundy inequality. The proof of that lemma only used two facts that depend on the martingale. First, a jump of the spin at $\varepsilon^{-1}z$ causes a jump of size $2\delta^{-1}\varepsilon^2 K_\gamma(y-z)$ for $M_\gamma(y)$, and in our case this becomes an upper bound of the jump size since a spin could jump by 1 or 2. Second, in the quadratic variation of M_γ which was given by

$$\frac{d}{dt}\langle M_\gamma(\cdot,x), M_\gamma(\cdot,y)\rangle_t = 4c^2_{\gamma,2}\sum_{z\in\Lambda_\varepsilon}\varepsilon^2 K_\gamma(x-z)\,K_\gamma(y-z)\,C_\gamma(t,z)\ ,$$

and C_γ is a rate function therein which is bounded between 0 and 1. For our case, in the quadratic variation given in (3.8), one also has

$$0\le \sum_{\bar\sigma\in\{\pm1,0\}}(\bar\sigma-\sigma(s,\varepsilon^{-1}z))^2 C_\gamma(s,z,\bar\sigma)\le 5\ . \tag{4.10}$$

Since the desired bound in [MW16, Lemma 4.1] allows a proportionality constant, nothing else needs to be proved.

For Proposition 4.2, by Burkholder-Davis-Gundy inequality, one needs to bound the quadratic variation $\langle Q_{\gamma,t}(\cdot,x)\rangle_t$, which can be again explicitly expressed as in the case for $R_{\gamma,t}(\cdot,x)$ in (4.4); using the bound (4.10) one eventually obtains

$$\langle Q_{\gamma,t}(\cdot,x)\rangle_t \le \frac{C\varepsilon^6}{\alpha\delta^4}\int_0^t \sum_{z\in\Lambda_\varepsilon}\varepsilon^2 (P^\gamma_{t-s}\star_\varepsilon K_\gamma)^4(z)\,ds\ .$$

Using the bound $\|P^\gamma_{t-s}\star_\varepsilon K_\gamma\|_{L^\infty(\Lambda_\varepsilon)}\le C\frac{\gamma^2}{\varepsilon^2}$ and $(\varepsilon^2\gamma^4/\alpha\delta^4)\le 2\gamma^2$ which turn out to hold in *both* regimes, the proof of [MW16, Lemma 5.1] again goes through.

Proposition 4.3 is then a consequence of the first two propositions by the proof in [MW16], and therefore nothing needs to be re-proved. □

One then has the following tightness and convergence results.

Proposition 4.4 *For every* $\mathfrak{m}\in\mathbb{N}$ *and* $\nu>0$*, the family* $\{Z^{:n:}_{\gamma,\mathfrak{m}},\gamma\in(0,\frac13)\}$ *is tight on* $\mathcal{D}(\mathbb{R}_+,\mathcal{C}^{-\nu})$*. Any weak limit is supported on* $\mathcal{C}(\mathbb{R}_+,\mathcal{C}^{-\nu})$*. Furthermore, for any* $p\ge1$ *and* $T>0$*, we have*

$$\sup_{\gamma\in(0,\frac13)}\mathbb{E}\sup_{0\le t\le T}\|Z^{:n:}_{\gamma,\mathfrak{m}}(t,\cdot)\|^p_{\mathcal{C}^{-\nu}}<\infty\ . \tag{4.11}$$

Proof. Once Proposition 4.1 (in particular the bounds (4.5) and (4.6)) is shown, this tightness result follows in exactly the same way as [MW16, Proposition 5.4]. □

Recall that we have defined $Z^{:m:}$ below (2.25).

Proposition 4.5 *For every* $\mathfrak{m} \in \mathbb{N}$ *and* $n \in \mathbb{N}$, *the processes* $(Z^{:1:}_{\gamma,\mathfrak{m}}, \ldots, Z^{:n:}_{\gamma,\mathfrak{m}})$ *defined above converge (jointly) in law to* $(Z^{:1:}, \ldots, Z^{:n:})$ *with respect to the topology of* $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})^n$.

Proof. Since by Proposition 4.4 for every n, the family of vectors $(Z^{:1:}_{\gamma,\mathfrak{m}}, \ldots, Z^{:n:}_{\gamma,\mathfrak{m}})$, $\gamma \in (0, \frac{1}{3})$ is tight with respect to the topology of $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})^n$, we only need to show convergence of the finite dimensional distributions. We follow the diagonal argument as in [MW16, Theorem 6.2]. Define

$$R_t(s,x) \stackrel{\text{def}}{=} \sqrt{2/\beta_c} \int_{r=0}^{s} P_{t-r}\, dW(r,x) \;,$$

where β_c is a critical value of β as above. The process $s \mapsto R_t(s,x)$ for $s < t$ is a *continuous* martingale. For $n > 1$ define

$$R_t^{:n:}(s,x) \stackrel{\text{def}}{=} n \int_{r=0}^{s} R_t^{:n-1:}(r,x)\, dR_t(r,x) = H_n\left(R_t(s,x), \langle R_t(\cdot,x)\rangle_s\right) \;. \tag{4.12}$$

For $s < t$ $R_t^{:n:}(s,x)$ is a regular approximations of the limiting objects $Z^{:n:}(t,\cdot)$; indeed, as discussed in [MW16, (3.10)], for all $\nu > 0,\ 0 \le \lambda \le 1,\ p \ge 2$ and $T > 0$, there exists $C = C(\nu, \lambda, p, T)$ such that

$$\mathbb{E}\| Z^{:n:}(t,\cdot) - R_t^{:n:}(s,\cdot)\|^p_{\mathcal{C}^{-\nu-\lambda}} \le C|t-s|^{\frac{\lambda p}{2}} \tag{4.13}$$

for all $0 \le s \le t \le T$. Write

$$\begin{aligned} \mathbf{Z}_\gamma &= (Z^{:1:}_{\gamma,\mathfrak{m}}, \ldots, Z^{:n:}_{\gamma,\mathfrak{m}}) \;, \qquad & \mathbf{Z} &= (Z^{:1:}, \ldots, Z^{:n:}) \;, \\ \mathbf{R}_{\gamma,t} &= (R^{:1:}_{\gamma,t,\mathfrak{m}}, \ldots, R^{:n:}_{\gamma,t,\mathfrak{m}}) \;, \qquad & \mathbf{R}_t &= (R_t^{:1:}, \ldots, R_t^{:n:}) \;. \end{aligned}$$

Fix $K \in \mathbb{N}$ and $t_1 < t_2 < \ldots < t_K$. Let $F \colon (\mathcal{C}^{-\nu})^{n\times K} \to \mathbb{R}$ be bounded and uniformly continuous. For $s_1 < t_1, \ldots, s_K < t_K$,

$$\begin{aligned} &|\mathbb{E}\, F(\mathbf{Z}_\gamma(t_1), \ldots, \mathbf{Z}_\gamma(t_K)) - \mathbb{E}\, F(\mathbf{Z}(t_1), \ldots, \mathbf{Z}(t_K))| \\ &\quad \le \mathbb{E}\, |F(\mathbf{Z}_\gamma(t_1), \ldots, \mathbf{Z}_\gamma(t_K)) - F(\mathbf{R}_{\gamma,t_1}(s_1), \ldots, \mathbf{R}_{\gamma,t_K}(s_K))| \\ &\qquad + |\mathbb{E}\, F(\mathbf{R}_{\gamma,t_1}(s_1), \ldots, \mathbf{R}_{\gamma,t_K}(s_K)) - \mathbb{E}\, F(\mathbf{R}_{t_1}(s_1), \ldots, \mathbf{R}_{t_K}(s_K))| \\ &\qquad + \mathbb{E}\, |F(\mathbf{R}_{t_1}(s_1), \ldots, \mathbf{R}_{t_K}(s_K)) - F(\mathbf{Z}(t_1), \ldots, \mathbf{Z}(t_K))| \;. \end{aligned} \tag{4.14}$$

The estimates (4.13) and (4.7) yield moment bounds of arbitrary order of $\|\mathbf{Z}_\gamma(t_i) - \mathbf{R}_{\gamma,t_i}(s_i)\|_{(\mathcal{C}^{-\nu})^n}$ uniformly in γ. We can thus make the first and the third terms on the right-hand side of (4.14) small uniformly in γ by choosing $|t_i - s_i|$ small enough.

Some extra care has to be taken in the case of our model for the second term on the right-hand side of (4.14). By Proposition 4.3, it suffices to show that

$$H_\ell(R_{\gamma,t_i,\mathfrak{m}}(s_i,x), [R_{\gamma,t_i,\mathfrak{m}}(\cdot,x)]_{s_i}) \qquad \ell = 1, \ldots, n, \quad i = 1, \ldots, K$$

converges in law to $(\mathbf{R}_{t_1}(s_1), \ldots, \mathbf{R}_{t_K}(s_K))$ in $(\mathcal{C}^{-\nu})^K$. By (4.12) and Prop 4.2, it suffices to show the two convergences in law

$$(R_{\gamma,t_1,\mathfrak{m}}(s_1), \ldots, R_{\gamma,t_K,\mathfrak{m}}(s_K)) \xrightarrow[\gamma\to 0]{} (R_{t_1}(s_1), \ldots, R_{t_K}(s_K)) \;,$$

$$(\langle R_{\gamma,t_1,\mathfrak{m}}(\cdot,\cdot)\rangle_{s_1},\dots,\langle R_{\gamma,t_K,\mathfrak{m}}(\cdot,\cdot)\rangle_{s_K}) \underset{\gamma\to 0}{\longrightarrow} (\langle R_{t_1}(\cdot,\cdot)\rangle_{s_1},\dots,\langle R_{t_K}(\cdot,\cdot)\rangle_{s_K})\ ,$$

for a suitable topology, e.g. $(L^\infty)^K$ in the first convergence an $(L^p)^K$ for p large enough for the second convergence. For the first convergence, note that $R_{\gamma,t_i,\mathfrak{m}}(s_i) = P^\gamma_{t_i-s_i} Z_{\gamma,\mathfrak{m}}(s_i)$. [MW16, Corollary 8.7] then gives an error control if $P^\gamma_{t_i-s_i}$ is replaced by the continuous heat kernel $P_{t_i-s_i}$. So the first convergence follows from Theorem 3.3 (convergence of $Z_\gamma(t)$), continuity of the mapping $P_{t_i-s_i}$ and the continuous mapping theorem.

Regarding the second convergence, recall the explicit expression (4.4) for the quadratic variation $\langle R_{\gamma,t_i,\mathfrak{m}}(\cdot,x)\rangle_{s_i}$. The constant $c^2_{\gamma,2}$ is deterministically close to 1 by (2.15), and therefore Proposition 3.4 shows that the quadratic variation $\langle R_{\gamma,t_i,\mathfrak{m}}(\cdot,x)\rangle_{s_i}$ is given by

$$\frac{2}{\beta_c}\int_0^{s_i} \sum_{z\in\Lambda_\varepsilon} \varepsilon^2\,(P^\gamma_{t_i-r}\star_\varepsilon K_\gamma)^2(z-x)\,dr$$

up to an error $\tilde{E}_t(s)$ which satisfies $\mathbb{E}\|\tilde{E}_t(s)\|^p_{L^p(\mathbb{T}^2)} \to 0$. This expression in turn converges to the limiting object $\langle R_{t_i}(\cdot,\cdot)\rangle_{s_i}$ by the calculation as in [MW16, (6.14)]. □

We now summarize the results obtained above and prove our main result, Theorem 2.5. To show the convergence of discrete evolution (3.1) to the solution of

$$\begin{aligned} X(t,\cdot) = P_t X^0 + \int_0^t P_{t-s}\star\Big(\mathfrak{a}_1 X(s,\cdot) - \frac{a_c(4a_c-1)\beta_c^3}{3(2a_c+1)^2} X^{:3:}(s,\cdot)\Big)\,ds \\ + Z(t,\cdot) \qquad \text{on } \mathbb{T}^2 \end{aligned} \tag{4.15}$$

in the first regime and

$$\begin{aligned} X(t,\cdot) =& P_t X^0 + \int_0^t P_{t-s}\star\Big(-\frac{20}{9}X^{:5:}(s,\cdot) + \mathfrak{a}_3 X^{:3:}(s,\cdot) \\ &+\mathfrak{a}_1 X(s,\cdot)\Big)\,ds + Z(t,\cdot) \qquad \text{on } \mathbb{T}^2 \end{aligned} \tag{4.16}$$

in the second regime, we need to control the following error terms.

(1) The error E_γ in (3.1) arising from the Taylor expansion in Section 2.

(2) In the second regime the discrepancies caused by $C_{\beta,\theta} \neq -\frac{20}{9}$, the coefficient in front of $X^3_\gamma - 3\mathfrak{c}_\gamma X_\gamma$ in (2.18) is not exactly $\mathfrak{a}_3$, and the coefficient in front of X_γ in (2.18) is not exactly $\mathfrak{a}_1$; similarly in the first regime there are also such discrepancies of coefficients comparing with (2.16).

(3) The operator Ext which extends a function on Λ_ε to a function on $\mathbb{T}^2$ defined in (2.35) does not commute with powers. As in [MW16] this is dealt with by decomposing the field X_γ into a "high" and a "low" frequency part

$$X^{\mathrm{low}}_\gamma \stackrel{\mathrm{def}}{=} \sum_{2^k<\frac{N}{20}} \delta_k X_\gamma\ , \qquad X^{\mathrm{high}}_\gamma \stackrel{\mathrm{def}}{=} \sum_{2^k\geq\frac{N}{20}} \delta_k X_\gamma\ , \tag{4.17}$$

(we recall that $N \approx \gamma^{-2}$ in the first regime and $N \approx \gamma^{-3}$ in the second regime). For X^{low}_γ the operator Ext does commute with the powers appearing below and we need to control the error caused by the high frequencies.

(4) Recall that in the discussion on the limiting SPDE, the actual renormalization constant used to define the Wick powers $Z^{:n:}_\varepsilon$ in (2.25) is a time-dependent constant $\mathfrak{c}_\varepsilon(t)$, and the

time-dependent coefficients $\mathfrak{a}_k(t)$ is introduced in place of the time-independent ones $\mathfrak{a}_k$ in order to take care of the difference between $\mathfrak{c}_\varepsilon(t)$ and $\mathfrak{c}_\varepsilon$, i.e. to guarantee that (2.27) holds. For the discrete model, we have $\mathfrak{c}_\gamma \neq \mathfrak{c}_\varepsilon$, and we will introduce the approximate time-dependent renormalization constant

$$\mathfrak{c}_\gamma(s,x) \stackrel{\text{def}}{=} [R_{\gamma,s}(\cdot,x)]_s \tag{4.18}$$

(and extend this to all $x \in \mathbb{T}^2$ as a trigonometric polynomial). So we need to control the error caused by the fact that Eq. (2.27) does not exactly hold anymore if the subscript ε in (2.27) is replaced by γ.

(5) The error from $P_t^\gamma X_\gamma^0 \neq P_t X^0$.

(6) The processes $Z_{\gamma,\mathfrak{m}}^{:n:}$ are defined via iterated integrals, which are not exactly the same as Hermite polynomials with constant $\mathfrak{c}_\gamma(s,x)$ (see Prop. 4.3).

(7) $\Delta \neq \widetilde{\Delta}_\gamma$.

In the following Lemma we control the errors from (1)-(4). We will frequently use the fact that an $L^\infty(\Lambda_\varepsilon)$ bound on X_γ can be extended to an $L^\infty(\mathbb{T}^2)$ bound by loosing an arbitrarily small power of γ ([MW16, Lemma B.6]), and the fact that the L^∞ norm can be bounded by the $\mathcal{C}^{-\nu}$ norm of X_γ multiplied by a factor $\gamma^{-b\nu}$ ([MW16, Lemma B.3]) if $\hat{X}_\gamma$ has vanishing frequency larger than $\gamma^{-b\nu}$ ($b = 2, 3$ depending on the regime).

Before stating the lemma, we recall that the constant $\mathfrak{c}_\varepsilon$ is defined in (2.24), the constant $\mathfrak{c}_\varepsilon(t)$ is defined in (2.26), the constant $\mathfrak{c}_\gamma$ is defined in (2.36), the constant $\mathfrak{c}_\gamma(t,\cdot)$ is defined in (4.18), the constant $\mathfrak{a}_1$ (resp. $\mathfrak{a}_1$ and $\mathfrak{a}_3$) are introduced in (2.17) (resp. (2.21)) in the first (resp. second) regime. The constants $\mathfrak{a}_k^{(\varepsilon)}(t)$ are defined in (2.27), and here we will use the $\varepsilon \to 0$ limits of them: in the second regime, by (2.28) and (2.29) with $\mathfrak{a}_5$ substituted by $-\frac{9}{20}$ we define $\mathfrak{a}_1(s)$, $\mathfrak{a}_3(s)$ as $\varepsilon \to 0$ limits of $\mathfrak{a}_1^{(\varepsilon)}(s)$, $\mathfrak{a}_3^{(\varepsilon)}(s)$, namely

$$\mathfrak{a}_3(s) - \mathfrak{a}_3 = -\frac{9}{2}\bar{\mathfrak{c}}(s)\;, \qquad \mathfrak{a}_1(s) - \mathfrak{a}_1 = 3\mathfrak{a}_3\bar{\mathfrak{c}}(s) - \frac{27}{4}\bar{\mathfrak{c}}(s)^2\;, \tag{4.19}$$

where $\bar{\mathfrak{c}}(s) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0}(\mathfrak{c}_\varepsilon(s) - \mathfrak{c}_\varepsilon)$ (see (2.31) for existence of this limit). In the first regime we simply define $\mathfrak{a}_1(t) = 3\mathfrak{a}_3\bar{\mathfrak{c}}(s) + \mathfrak{a}_1 = -\frac{a_c(4a_c-1)\beta_c^3}{(2a_c+1)^2}\bar{\mathfrak{c}}(s) + \mathfrak{a}_1$.

Lemma 4.6 *For every $t \geq 0$, we have on $\mathbb{T}^2$ (we drop the space variables for readability)*

$$\begin{aligned} X_\gamma(t) = P_t^\gamma X_\gamma^0 + \int_0^t P_{t-s}^\gamma K_\gamma \star \Big(& -\frac{a_c(4a_c-1)\beta_c^3}{3(2a_c+1)^2}\Big(X_\gamma^3(s) - 3\mathfrak{c}_\gamma(s)X_\gamma(s)\Big) \\ & + \mathfrak{a}_1(s)X_\gamma(s) + \mathsf{Err}^{(1)}(s)\Big)\, ds + Z_\gamma(t)\;. \end{aligned} \tag{4.20}$$

in the first scaling regime and

$$\begin{aligned} X_\gamma(t) =& P_t^\gamma X_\gamma^0 + \int_0^t P_{t-s}^\gamma K_\gamma \star \Big(-\frac{9}{20}\Big(X_\gamma^5(s) - 10\mathfrak{c}_\gamma(s)X_\gamma^3(s) + 15\mathfrak{c}_\gamma(s)^2 X_\gamma(s)\Big) \\ & + \mathfrak{a}_3(s)\Big(X_\gamma^3(s) - 3\mathfrak{c}_\gamma(s)X_\gamma(s)\Big) + \mathfrak{a}_1(s)X_\gamma(s) + \mathsf{Err}^{(1)}(s)\Big)\, ds + Z_\gamma(t) \end{aligned} \tag{4.21}$$

in the second scaling regime, such that the following holds. For every $T > 0$ and $\kappa > 0$, there exists $C = C(T,\kappa,\nu)$ such that for all $0 \leq s \leq T$, $x \in \mathbb{T}^2$ and sufficiently small $\gamma > 0$

$$\begin{aligned} |\mathsf{Err}^{(1)}(s,x)| \leq\; & C\,\gamma^{-30\nu-\kappa}(\|X_\gamma(s,\cdot)\|_{\mathcal{C}^{-\nu}}^7 + 1) \\ & \times\Big(\gamma^{\frac{2}{3}}s^{-\frac{1}{3}} + \|X_\gamma^{\text{high}}(s,\cdot)\|_{L^\infty(\mathbb{T}^2)} + \|Q_{\gamma,s}(s,\cdot)\|_{L^\infty(\Lambda_\varepsilon)} + |\tilde{E}(s,x)|\Big)\;, \end{aligned} \tag{4.22}$$

where $\tilde{E}$ is defined in (3.19)*. Here* $\mathsf{Err}^{(1)}$ *is different in the two regimes but the bound holds for both regimes.*

Remark 4.7 Recall the stopping time $\tau_{\gamma,\mathfrak{m}}$ defined in (3.5). Denote by $X_{\gamma,\mathfrak{m}}$ the solution to (4.21) with Z_γ replaced by $Z_{\gamma,\mathfrak{m}}$ and $\mathsf{Err}^{(1)}$ replaced by $\mathsf{Err}^{(1)}_{\mathfrak{m}}$ which is equal to $\mathsf{Err}^{(1)}$ before the time $\tau_{\gamma,\mathfrak{m}}$ and is set to 0 after $\tau_{\gamma,\mathfrak{m}}$. Taking the $L^p(\mathbb{T}^2)$ norm on both sides of (4.22), one has the bound

$$\begin{aligned}\|\mathsf{Err}^{(1)}_{\mathfrak{m}}(s,\cdot)\|_{L^p(\mathbb{T}^2)} \le& C\gamma^{-(30\nu+\kappa)}\Big(\gamma^{\frac{2}{3}}s^{-\frac{1}{3}} + \|X^{\mathrm{high}}_\gamma(s,\cdot)\|_{L^\infty(\mathbb{T}^2)} \\ &+ \|Q_{\gamma,s}(s,\cdot)\|_{L^\infty(\Lambda_\varepsilon)} + \|\tilde{E}(s,\cdot)\|_{L^p(\mathbb{T}^2)}\Big)\;,\end{aligned} \tag{4.23}$$

where C depends on $T,\mathfrak{m},p,\kappa,\nu$.

Proof of Lemma 4.6. We first consider the second regime. With the choice of parameters as in (2.21), or equivalently (2.19) and (2.20), the discrete evolution (3.1) can be written as

$$\begin{aligned}X_\gamma(t,\cdot) =& P^\gamma_t X^0_\gamma + \int_0^t P^\gamma_{t-s} K_\gamma \star \Big(C_{\beta,\theta} X^5_\gamma(s,\cdot) + \Big(\frac{9}{2}\mathfrak{c}_\gamma + \mathfrak{a}_3\Big) X^3_\gamma(s,\cdot) \\ &+ \Big(-3\mathfrak{c}_\gamma \mathfrak{a}_3 - \frac{27}{4}\mathfrak{c}^2_\gamma + \mathfrak{a}_1\Big) X_\gamma(s,\cdot) + E_\gamma(s,\cdot)\Big)\, ds + Z_\gamma(s,\cdot) \qquad \text{on } \Lambda_\varepsilon\end{aligned}$$

We apply Ext on both sides, and compare it with the continuous equation (4.21). We then have

$$\mathsf{Err}^{(1)} = \mathsf{err}^{(1)} + \mathsf{err}^{(2)} + \mathsf{err}^{(3)}\;, \tag{4.24}$$

where the error terms are given by

$$\begin{aligned}\mathsf{err}^{(1)}(s) &= E_\gamma(s) + \Big(C_{\beta,\theta} + \frac{9}{20}\Big)\mathsf{Ext}(X^5_\gamma(s))\;, \\ \mathsf{err}^{(2)}(s) &= -\frac{9}{20}\Big(\mathsf{Ext}\;(X^5_\gamma(s)) - (\mathsf{Ext}\;X_\gamma(s))^5\Big) \\ &\quad + \Big(\frac{9}{2}\mathfrak{c}_\gamma + \mathfrak{a}_3\Big)\Big(\mathsf{Ext}\;(X^3_\gamma(s)) - (\mathsf{Ext}\;X_\gamma(s))^3\Big)\;, \\ \mathsf{err}^{(3)}(s) &= \Big(\frac{9}{2}\mathfrak{c}_\gamma + \mathfrak{a}_3 - \frac{9}{2}\mathfrak{c}_\gamma(s) - \mathfrak{a}_3(s)\Big)\; X^3_\gamma(s) \\ &\quad - \Big(\frac{27}{4}\mathfrak{c}^2_\gamma + 3\mathfrak{a}_3\mathfrak{c}_\gamma - \mathfrak{a}_1 - \frac{27}{4}\mathfrak{c}_\gamma(s)^2 - 3\mathfrak{a}_3(s)\mathfrak{c}_\gamma(s) + \mathfrak{a}_1(s)\Big)\; X_\gamma(s)\;,\end{aligned}$$

where in the expression of $\mathsf{err}^{(3)}$ and also below we simply denote $X_\gamma = \mathsf{Ext}X_\gamma$. The analysis for $\mathsf{err}^{(1)}$ and $\mathsf{err}^{(2)}$ follow essentially the same way as in [MW16, Proof of Lemma 7.1], so we will only write down the bounds we eventually obtain for these errors.

For the first term $\mathsf{err}^{(1)}$, using the assumption (2.21) on (β,θ), and the definition of $C_{\beta,\theta}$, one has $|C_{\beta,\theta} + \frac{9}{20}| \le C\gamma^2\mathfrak{c}_\gamma$. Then by the definition of E_γ in (2.12), and that $\mathfrak{c}_\gamma$ has only logarithmic divergence, we can finally get that for any arbitrary small $\kappa > 0$

$$\|\mathsf{err}^{(1)}(s,\cdot)\|_{L^\infty(\mathbb{T}^2)} \le C(\kappa,\nu)\gamma^{2-\kappa-30\nu}(\|X_\gamma(s,\cdot)\|^7_{\mathcal{C}^{-\nu}} + 1)\;.$$

For the second term $\mathsf{err}^{(2)}$, by decomposing X_γ into low and high modes as in (4.17), we can obtain the bound

$$\|\mathsf{err}^{(2)}(s,\cdot)\|_{L^\infty(\mathbb{T}^2)} \le C(\kappa)\gamma^{-\kappa-15\nu}\|X^{\mathrm{high}}_\gamma(s,\cdot)\|_{L^\infty(\mathbb{T}^2)}\|X_\gamma(s,\cdot)\|^4_{\mathcal{C}^{-\nu}}\;. \tag{4.25}$$

In order to control the term $\mathsf{err}^{(3)}$, we first consider the quantity

$$\mathfrak{c}_\gamma - \mathfrak{c}_\gamma(s,x) + \lim_{\varepsilon\to 0}(\mathfrak{c}_\varepsilon(s) - \mathfrak{c}_\varepsilon)\ , \tag{4.26}$$

which is called $\mathfrak{c}_\gamma - \mathfrak{c}_\gamma(s,x) + A - A(s)$ in [MW16, Proof of Lemma 7.1] (see the definition of $A_\varepsilon(s)$ below [MW16, (3.11)]); note that the $\varepsilon \to 0$ limit is well-defined as discussed around (2.31) in the proof of Lemma 2.3. By the definition of $\mathfrak{c}_\gamma$ in (2.36), the definition of $\mathfrak{c}_\gamma(s,x)$ in (4.18), and (2.31), we have that for $x \in \mathbb{T}^2$, (4.26) is equal to

$$\sum_{\substack{\omega\in\{-N,\dots,N\}^2\\ \omega\neq 0}} \frac{|\hat{K}_\gamma(\omega)|^2}{4\beta_c\gamma^{-b}(1-\hat{K}_\gamma(\omega))} - [R_{\gamma,s}(\cdot,x)]_s + \frac{s}{2\beta_c} - \sum_{\substack{\omega\in\mathbb{Z}^2\\ \omega\neq 0}} \frac{\exp(-2s\pi^2|\omega|^2)}{4\beta_c\pi^2|\omega|^2}\ .$$

Here $b=4$ and $\beta_c = 3$ since we are considering the second regime. Recall from (4.8) that for $x\in\Lambda_\varepsilon$, $[R_{\gamma,r}(\cdot,x)]_r = \langle R_{\gamma,r}(\cdot,x)\rangle_r + Q_{\gamma,r}(s,x)$. According to (4.4) we get for $x \in \Lambda_\varepsilon$

$$\begin{aligned}
&\langle R_{\gamma,s}(\cdot,x)\rangle_s\\
&= c_{\gamma,2}^2 \int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2 (P^\gamma_{s-r}\star_\varepsilon K_\gamma)^2(x-z) \sum_{\bar\sigma\in\{\pm1,0\}} (\bar\sigma - \sigma(r,\varepsilon^{-1}z))^2 C_{\gamma,\mathfrak{m}}(r,z,\bar\sigma)\,dr\\
&= \frac{2}{\beta_c}\int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2 (P^\gamma_{s-r}\star_\varepsilon K_\gamma)^2(x-z)\ dr + \mathsf{err}^{(4)}(s,x) + \tilde{E}_s(s,x)\\
&= \frac{1}{2\beta_c}\int_0^s \sum_{\omega\in\{-N,\dots,N\}^2} \exp\Big(-\frac{2r}{\gamma^b}(1-\hat{K}_\gamma(\omega))\Big)\,|\hat{K}_\gamma(\omega)|^2\,dr + \mathsf{err}^{(4)}(s,x) + \tilde{E}_s(s,x)\\
&= \frac{s}{2\beta_c} + \sum_{\substack{\omega\in\{-N,\dots,N\}^2\\ \omega\neq 0}} \frac{|\hat{K}_\gamma(\omega)|^2}{4\beta_c\gamma^{-b}(1-\hat{K}_\gamma(\omega))}\Big(1-e^{-\frac{2s}{\gamma^b}\left(1-\hat{K}_\gamma(\omega)\right)}\Big) + \mathsf{err}^{(4)}(s,x) + \tilde{E}_s(s,x)
\end{aligned}$$

where $\mathsf{err}^{(4)}$ is the second line above with $c_{\gamma,2}^2$ replaced by 1, and $\tilde{E}$ is defined in (3.19). By $|c_{\gamma,2}^2 - 1| \le \gamma^2$ and $\int_0^s \sum_{z\in\Lambda_\varepsilon} \varepsilon^2 \left(P^\gamma_{t-r}\star_\varepsilon K_\gamma\right)^2(z-x)\,dr \le C\log\gamma^{-1} \le C(\kappa)\gamma^{-\kappa}$ one has $|\mathsf{err}^{(4)}(s,x)| \le C\gamma^{2-\kappa}$. Proposition 3.4 gives the stochastic bound on $\tilde{E}_s(s,x)$.

Therefore up to the terms $Q_{\gamma,s}(s,x)$, $\mathsf{err}^{(4)}(s,x)$ and $\tilde{E}_s(s,x)$, the quantity (4.26) is equal to

$$\sum_{\substack{\omega\in\{-N,\dots,N\}^2\\ \omega\neq 0}} \frac{|\hat{K}_\gamma(\omega)|^2}{4\beta_c\gamma^{-b}(1-\hat{K}_\gamma(\omega))} e^{-\frac{2s}{\gamma^b}\left(1-\hat{K}_\gamma(\omega)\right)} - \sum_{\substack{\omega\in\mathbb{Z}^2\\ \omega\neq 0}} \frac{\exp(-2s\pi^2|\omega|^2)}{4\beta_c\pi^2|\omega|^2}\ . \tag{4.27}$$

By discussing the cases of $|\omega| < \gamma^{-2}$ and $|\omega| \ge \gamma^{-2}$ applying Lemma 5.1 - 5.3, we find that (4.27) is bounded by $C\gamma^{\frac{2}{3}}s^{-\frac{1}{3}}$.

Now to really bound the coefficients appearing in $\mathsf{err}^{(3)}(s,x)$, note that the coefficient of $X^3_\gamma(s)$ in $\mathsf{err}^{(3)}(s,x)$ is then equal to

$$\frac{9}{2}\mathfrak{c}_\gamma + \mathfrak{a}_3 - \frac{9}{2}\mathfrak{c}_\gamma(s) - \mathfrak{a}_3(s) = \frac{9}{2}(\mathfrak{c}_\gamma - \mathfrak{c}_\gamma(s) + \bar{\mathfrak{c}}(s))$$

which is exactly the quantity (4.26) we have bounded times $\frac{9}{2}$. Furthermore, the absolute value of the coefficient of $X_\gamma(s)$ in $\mathsf{err}^{(3)}(s,x)$ is

$$\Big|\frac{27}{4}\mathfrak{c}_\gamma^2 + 3\mathfrak{a}_3\mathfrak{c}_\gamma - \mathfrak{a}_1 - \frac{27}{4}\mathfrak{c}_\gamma(s)^2 - 3\mathfrak{a}_3(s)\mathfrak{c}_\gamma(s) + \mathfrak{a}_1(s)\Big|$$

$$
\begin{aligned}
&= \Big|\frac{27}{4}\Big(\mathfrak{c}_\gamma^2 - \mathfrak{c}_\gamma(s)^2\Big) + 3\Big(\mathfrak{a}_3\mathfrak{c}_\gamma + \frac{9}{2}\bar{\mathfrak{c}}(s)\mathfrak{c}_\gamma(s) - \mathfrak{a}_3\mathfrak{c}_\gamma(s)\Big) + 3\mathfrak{a}_3\bar{\mathfrak{c}}(s) - \frac{27}{4}\bar{\mathfrak{c}}(s)^2\Big| \\
&= \frac{27}{4}\Big|\mathfrak{c}_\gamma(s) + \mathfrak{c}_\gamma - \bar{\mathfrak{c}}(s) + \frac{4}{9}\mathfrak{a}_3\Big| \cdot \Big|\mathfrak{c}_\gamma - \mathfrak{c}_\gamma(s) + \bar{\mathfrak{c}}(s)\Big| \\
&\le C(\kappa)\gamma^{-\kappa}|\mathfrak{c}_\gamma - \mathfrak{c}_\gamma(s) + \bar{\mathfrak{c}}(s)|
\end{aligned}
$$

where in the second line we applied (4.19), the third line is obtained by elementary factorization, and in the last line we applied the logarithmic bound for the renormalization constants. So the bound of this coefficient again boils down to the bound on (4.26).

The $\tilde{E}$ dependent terms in $\mathsf{err}^{(3)}$ are

$$
-\frac{9}{2}\tilde{E}_s(s,x)X_\gamma^3(s,x) + \frac{27}{4}\Big(\mathfrak{c}_\gamma(s) + \mathfrak{c}_\gamma - \bar{\mathfrak{c}}(s) + \frac{4}{9}\mathfrak{a}_3\Big)\tilde{E}_s(s,x)X_\gamma(s,x) \tag{4.28}
$$

whose absolute value is bounded by

$$
C(\nu,\kappa)\gamma^{-10\nu-\kappa}\Big(\|X_\gamma(s,\cdot)\|_{\mathcal{C}^{-\nu}}^3 + 1\Big)|\tilde{E}_s(s,x)| \ .
$$

Summarizing all the above bounds we obtain (4.22).

The proof for the first regime is analogous is thus omitted; in particular we can obtain bounds with slightly larger (but still negative) powers of γ and lower powers of $\|X_\gamma(s,\cdot)\|_{\mathcal{C}^{-\nu}}$ than that in (4.22) but the latter is sufficient for our purpose. □

The error (5) is bounded by [MW16, Lemma 7.3] as

$$
\sup_{0\le t\le T}\|P_t^\gamma X_\gamma^0 - P_t X^0\|_{\mathcal{C}^{-\nu}} \le C\|X^0 - X_\gamma^0\|_{\mathcal{C}^{-\nu}} + \bar{C}\gamma^{\frac{\kappa}{2}} \to 0 \tag{4.29}
$$

for every $T > 0$, where $\bar{C}$ depends on ν, κ, T and $\|X_\gamma^0\|_{\mathcal{C}^{-\nu+\kappa}}$.

In the sequel, we let $\bar{n} = 3$ in the first regime and $\bar{n} = 5$ in the second regime.

At this stage, note that if we define

$$
\overline{X}_{\gamma,\mathfrak{m}}(t,\cdot) \stackrel{\text{def}}{=} P_t X^0 + Z_{\gamma,\mathfrak{m}}(t,\cdot) + \mathcal{S}_T(Z_{\gamma,\mathfrak{m}}, Z_{\gamma,\mathfrak{m}}^{:2:}, \cdots, Z_{\gamma,\mathfrak{m}}^{:\bar{n}:})(t,\cdot) \ , \tag{4.30}
$$

where $\mathcal{S}_T$ is the solution map defined in Theorem 2.4, then by the convergence in law of $(Z_{\gamma,\mathfrak{m}}, Z_{\gamma,\mathfrak{m}}^{:2:}, \cdots, Z_{\gamma,\mathfrak{m}}^{:\bar{n}:})$ with respect to the topology of $L^\infty([0,T],\mathcal{C}^{-\nu})^{\bar{n}}$ to $(Z, Z^{:2:}, \cdots, Z^{:\bar{n}:})$, and by the continuity of the map $\mathcal{S}_T$ as stated in Theorem 2.4, one has that $\overline{X}_{\gamma,\mathfrak{m}}$ converges in law to X.

Therefore, it remains to compare $\overline{X}_{\gamma,\mathfrak{m}}$ and $X_{\gamma,\mathfrak{m}}$. The idea is to follow a discrete version of Da Prato-Debussche argument [DPD03], namely, setting

$$
\begin{aligned}
v_{\gamma,\mathfrak{m}}(t,x) &\stackrel{\text{def}}{=} X_{\gamma,\mathfrak{m}}(t,x) - Z_{\gamma,\mathfrak{m}}(t,x) - P_t^\gamma X_\gamma^0(t,x) \qquad x \in \mathbb{T}^2 \ , \\
\overline{v}_{\gamma,\mathfrak{m}}(t,x) &\stackrel{\text{def}}{=} \overline{X}_{\gamma,\mathfrak{m}}(t,x) - Z_{\gamma,\mathfrak{m}}(t,x) - P_t X^0(t,x) \qquad x \in \mathbb{T}^2 \ ,
\end{aligned} \tag{4.31}
$$

and we compare $v_{\gamma,\mathfrak{m}}$ and $\overline{v}_{\gamma,\mathfrak{m}}$. Define

$$
\widetilde{Z}_{\gamma,\mathfrak{m}}^{:k:} \stackrel{\text{def}}{=} \sum_{\ell=0}^{k}(P_t^\gamma X_\gamma^0)^{k-\ell} Z_{\gamma,\mathfrak{m}}^{:\ell:} \ , \qquad \overline{Z}_{\gamma,\mathfrak{m}}^{:k:} \stackrel{\text{def}}{=} \sum_{\ell=0}^{k}(P_t X^0)^{k-\ell} Z_{\gamma,\mathfrak{m}}^{:\ell:} \ . \tag{4.32}
$$

Note that if the above Wick powers were defined via Hermite polynomials rather than iterated integrals then the above identities would follow from basic properties of Hermite polynomials $H_k(x+y) = \sum_{\ell=0}^k x^\ell H_{k-\ell}(y)$.

Now it is straightforward to check that $\overline{v}_{\gamma,\mathfrak{m}}$ satisfies

$$\overline{v}_{\gamma,\mathfrak{m}}(t) = -\int_0^t P_{t-s}\overline{\Psi}_{\gamma,\mathfrak{m}}(s)\,ds \;, \tag{4.33}$$

where we have set

$$\overline{\Psi}_{\gamma,\mathfrak{m}}(s) \stackrel{\text{def}}{=} \frac{a_c(4a_c-1)\beta_c^3}{3(2a_c+1)^2}\sum_{k=0}^{3}\binom{3}{k}\overline{Z}^{:k:}_{\gamma,\mathfrak{m}}(s)\,\overline{v}^{3-k}_{\gamma,\mathfrak{m}}(s) - \mathfrak{a}_1(s)\left(\overline{v}_{\gamma,\mathfrak{m}}(s)+\overline{Z}_{\gamma,\mathfrak{m}}(s)\right) \tag{4.34}$$

in the first regime and

$$\begin{aligned}\overline{\Psi}_{\gamma,\mathfrak{m}}(s) \stackrel{\text{def}}{=} \frac{9}{20}\sum_{k=0}^{5}\binom{5}{k}\overline{Z}^{:k:}_{\gamma,\mathfrak{m}}(s)\,\overline{v}^{5-k}_{\gamma,\mathfrak{m}}(s) - \mathfrak{a}_3(s)\sum_{k=0}^{3}\binom{3}{k}\overline{Z}^{:k:}_{\gamma,\mathfrak{m}}(s)\,\overline{v}^{3-k}_{\gamma,\mathfrak{m}}(s)\\ - \mathfrak{a}_1(s)\left(\overline{v}_{\gamma,\mathfrak{m}}(s)+\overline{Z}_{\gamma,\mathfrak{m}}(s)\right)\;.\end{aligned} \tag{4.35}$$

in the second regime. On the other hand, by Lemma 4.6 and (4.31), $v_{\gamma,\mathfrak{m}}$ satisfies (on $\mathbb{T}^2$)

$$v_{\gamma,\mathfrak{m}}(t) = -\int_0^t P^{\gamma}_{t-s}K_\gamma \star \left(\Psi_{\gamma,\mathfrak{m}}(s) + \mathsf{Err}^{(1)}_{\mathfrak{m}} + \mathsf{Err}^{(2)}_{\mathfrak{m}}(s,\cdot)\right)ds \;, \tag{4.36}$$

where $\Psi_{\gamma,\mathfrak{m}}(s)$ is defined in the same way as (4.34) or (4.35) with $\overline{Z}^{:k:}_{\gamma,\mathfrak{m}}, \overline{Z}_{\gamma,\mathfrak{m}}$ replaced by $\widetilde{Z}^{:k:}_{\gamma,\mathfrak{m}}, \widetilde{Z}_{\gamma,\mathfrak{m}}$ and $\overline{v}_{\gamma,\mathfrak{m}}$ replaced by $v_{\gamma,\mathfrak{m}}$. Here the term $\mathsf{Err}^{(1)}_{\mathfrak{m}}$ was estimated in Lemma 4.6, and $\mathsf{Err}^{(2)}_{\mathfrak{m}}$ is to control the error (6) mentioned before Lemma 4.6, namely the discrepancy between Hermite polynomials and iterated integrals.

As in [MW16, Lemma 7.4] it is straightforward to check using (4.32) that in both regimes one has for $0 \le s \le T$

$$\|\mathsf{Err}^{(2)}_{\mathfrak{m}}(s,\cdot)\|_{L^\infty(\mathbb{T}^2)} \le C(T,\nu,\kappa)\Big(1 + s^{-3\nu-\kappa} + \|v_{\gamma,\mathfrak{m}}\|^4_{\mathcal{C}^{\frac12}}\Big)\sum_{k=2}^{5}\|E^{:k:}_{\gamma,s,\mathfrak{m}}(s,\cdot)\|_{L^\infty(\mathbb{T}^2)} \tag{4.37}$$

where $E^{:n:}_{\gamma,t}(s,x)$ was introduced in Proposition 4.3. The following estimate holds in both regimes.

Lemma 4.8 *For every $0 \le t \le T$ and sufficiently small $\gamma > 0$, we have*

$$\begin{aligned}\|\overline{v}_{\gamma,\mathfrak{m}}(t,\cdot) - v_{\gamma,\mathfrak{m}}(t,\cdot)\|_{\mathcal{C}^{\frac12}} \le{}& \overline{C}_1\int_0^t (t-s)^{-\frac13}s^{-\frac16}\|\overline{v}_{\gamma,\mathfrak{m}}(s,\cdot) - v_{\gamma,\mathfrak{m}}(s,\cdot)\|_{\mathcal{C}^{\frac12}}\,ds\\ &+ \overline{C}_1(\gamma^{\frac{\kappa}{2}} + \|X^0_\gamma - X^0\|_{\mathcal{C}^{-\nu}}) + \mathsf{Err}^{(3)}(t)\;,\end{aligned} \tag{4.38}$$

where the constant $\overline{C}_1$ depends on ν,κ,T, $\|X^0\|_{\mathcal{C}^{-\nu+\kappa}}$, $\|X^0_\gamma\|_{\mathcal{C}^{-\nu+\kappa}}$ as well as the random quantities $\sup_{0\le s\le T}\|\overline{v}_{\gamma,\mathfrak{m}}(s,\cdot)\|_{\mathcal{C}^{\frac12}}$, $\sup_{0\le s\le T}\|v_{\gamma,\mathfrak{m}}(s,\cdot)\|_{\mathcal{C}^{\frac12}}$, and

$$\sup_{0\le s\le T}\|Z^{:k:}_{\gamma,\mathfrak{m}}(s,\cdot)\|_{\mathcal{C}^{-\nu}} \quad \text{for } k = 1,\ldots,\bar{n}\;.$$

There exists some $p \ge 2$, such that the error term $\mathsf{Err}^{(3)}$ satisfies that for every $T \ge 0$ and $0 < \lambda \le \frac12$

$$\mathbb{E}\sup_{0\le t\le T}|\mathsf{Err}^{(3)}(t)|^p \le \overline{C}_2\gamma^\lambda\;, \tag{4.39}$$

for a constant $\overline{C}_2 = \overline{C}_2(p,T,\lambda)$.

Proof. Using (4.33) - (4.36), we get that for any $t \geq 0$ and $\gamma > 0$,

$$\begin{aligned} \overline{v}_{\gamma,\mathfrak{m}}(t,\cdot) - v_{\gamma,\mathfrak{m}}(t,\cdot) = &- \int_0^t \left(P_{t-s} - P^\gamma_{t-s} \star K_\gamma\right) \overline{\Psi}_{\gamma,\mathfrak{m}}(s)\, ds \\ &- \int_0^t P^\gamma_{t-s} \star K_\gamma \star \left(\overline{\Psi}_{\gamma,\mathfrak{m}}(s) - \Psi_{\gamma,\mathfrak{m}}(s)\right) ds \\ &+ \int_0^t P^\gamma_{t-s} \star K_\gamma \star \left(\mathsf{Err}^{(1)}_{\mathfrak{m}}(s,\cdot) + \mathsf{Err}^{(2)}_{\mathfrak{m}}(s,\cdot)\right) ds\ , \end{aligned} \tag{4.40}$$

where $\overline{\Psi}_{\gamma,\mathfrak{m}}(s)$ was defined in (4.35) and $\Psi_{\gamma,\mathfrak{m}}(s)$ was defined below (4.36). The rest of the proof relies on the crucial multiplicative inequality [MW16, Lemma A.5] which is the linchpin around which the Da Prato-Debussche argument revolves (see [DPD03, Proposition 2.1] for a similar result); it states that if $\beta < 0 < \nu$ with $\nu + \beta > 0$, then there exists a constant C depending only on ν and β such that

$$\|Z_1\, Z_2\|_{\mathcal{C}^\beta} \leq C \|Z_1\|_{\mathcal{C}^\nu}\, \|Z_2\|_{\mathcal{C}^\beta}\ . \tag{4.41}$$

Proceeding as in the proof of [MW16, Lemma 7.5], which uses the above multiplicative inequality, together with the (discrete) heat kernel estimates in Sec. 8 of that reference, we can bound $\|\overline{\Psi}_{\gamma,\mathfrak{m}}(s)\|_{\mathcal{C}^{-\nu}}$ in (4.40) in terms of $\|\overline{v}_{\gamma,\mathfrak{m}}(s,\cdot)\|_{\mathcal{C}^{\frac12}}$ and $\|\overline{Z}^{:k:}_{\gamma,\mathfrak{m}}(s,\cdot)\|_{\mathcal{C}^{-\nu}}$ where $\nu < \frac12$, and the latter quantity is by (4.32) further bounded in terms of $\|Z^{:k:}_{\gamma,\mathfrak{m}}(s,\cdot)\|_{\mathcal{C}^{-\nu}}$ and $\|X^0\|_{\mathcal{C}^{-\nu}}$. Therefore the $\mathcal{C}^{\frac12}$ norm of the first term on the RHS of (4.40) can be eventually bounded by $C\gamma^{\frac12}$ where C may depend on all the quantities stated in the lemma, and the small factor $\gamma^{\frac12}$ arises from a bound on $\|(P_t - P^\gamma_t \star K_\gamma)\|_{\mathcal{C}^{-\nu}\to\mathcal{C}^{\frac12}}$.

The $\mathcal{C}^{\frac12}$ norm of the second term on the RHS of (4.40) can be bounded in the same way using the multiplicative inequality (4.41) and heat kernel estimates, by

$$C\int_0^t (t-s)^{-\frac13} s^{-\frac16} \|\overline{v}_{\gamma,\mathfrak{m}}(s) - v_{\gamma,\mathfrak{m}}(s)\|_{\mathcal{C}^{\frac12}}\, ds + C\|X^0_\gamma - X^0\|_{\mathcal{C}^{-\nu}} + C\gamma^{\frac{\kappa}{2}}\ ,$$

where again C may depend on all the quantities stated in the lemma.

Now we consider the $\mathcal{C}^{\frac12}$ norm of the last term on the RHS of (4.40). We use [MW15, Rem. 3.6 and Prop. 3.7] which state that the space L^p is continuously embedded in $\mathcal{B}^0_{p,\infty}$ and the latter is further continuously embedded in $\mathcal{B}^\alpha_{\infty,\infty}$ (i.e. the space $\mathcal{C}^\alpha$) provided that $\alpha + 2/p = 0$. Thus applying (4.23), we have that for any $\bar\kappa > 0$ there exists $C = C(p,\bar\kappa)$ such that

$$\begin{aligned} \Big\| \int_0^t P^\gamma_{t-s} \star K_\gamma \star \mathsf{Err}^{(1)}_{\mathfrak{m}}(s,\cdot)\, ds \Big\|_{\mathcal{C}^{\frac12}} &\leq C \int_0^t (t-s)^{-\frac14 - \frac1p - \bar\kappa} \|\mathsf{Err}^{(1)}_{\mathfrak{m}}(s,\cdot)\|_{L^p(\mathbb{T}^2)}\, ds \\ &\leq C\gamma^{-(30\nu+\kappa)} \int_0^t (t-s)^{-\frac14-\frac1p-\bar\kappa} \Big(\gamma^{\frac23} s^{-\frac13} + \|X^{\mathrm{high}}_\gamma(s,\cdot)\|_{L^\infty(\mathbb{T}^2)} \\ &\qquad + \|Q_{\gamma,s}(s,\cdot)\|_{L^\infty(\Lambda_\varepsilon)} + \|\tilde E(s,\cdot)\|_{L^p(\mathbb{T}^2)} \Big)\, ds \\ &\leq C\gamma^{\frac23-(30\nu+\kappa)} \int_0^t (t-s)^{-\frac14-\frac1p-\bar\kappa} s^{-\frac13} ds \\ &\qquad + C\gamma^{-(30\nu+\kappa)} \Big(\int_0^t (t-s)^{-(\frac14+\frac1p+\bar\kappa)p_\star}\, ds \Big)^{\frac{1}{p_\star}} \\ &\qquad \times \Big(\|X^{\mathrm{high}}_\gamma\|_{L^\infty(\mathbb{T}^2\times[0,T])} + \|Q_{\gamma,s}\|_{L^\infty(\Lambda_\varepsilon\times[0,T])} + \|\tilde E\|_{L^p(\mathbb{T}^2\times[0,T])} \Big) \end{aligned} \tag{4.42}$$

where $p_\star$ is such that $\frac{1}{p_\star}+\frac{1}{p}=1$. Choosing (and fixing from now on) p sufficiently large (depending only on $\bar\kappa$) the above expression can be bounded by

$$C(T,p,\bar\kappa)\gamma^{-(30\nu+\kappa)}\Big(\gamma^{\frac{2}{3}}+\|X_\gamma^{\text{high}}\|_{L^\infty(\mathbb{T}^2\times[0,T])}+\|Q_{\gamma,s}\|_{L^\infty(\Lambda_\varepsilon\times[0,T])}+\|\tilde E\|_{L^p(\mathbb{T}^2\times[0,T])}\Big)\;.$$

We have Proposition 3.4 to bound $\tilde E$, Proposition 4.2 to bound Q_γ. Regarding the term X_γ^{high}, which is equal to $Z_\gamma^{\text{high}}+v_{\gamma,\mathfrak{m}}^{\text{high}}+(P_s^\gamma X_\gamma^0)^{\text{high}}$, we can bound $\|\cdot\|_{L^\infty(\mathbb{T}^2)}$ of the last two quantities by $C\gamma^1\|\cdot\|_{\mathcal{C}^{\frac12}}$. Finally for Z_γ^{high}, by [MW16, Lemmas 4.6] with minor changes in the proof due to the scaling-regime-dependent definition (4.17) and kernel estimates in Section 5, one has $\mathbb{E}\|X_\gamma^{\text{high}}(s,\cdot)\|_{L^\infty}^p\le C\gamma^{p(1-\kappa)}$. Therefore by choosing ν,κ small enough depending on the previously fixed p one has that

$$\mathbb{E}\sup_{0\le t\le T}\Big\|\int_0^t P_{t-s}^\gamma\star K_\gamma\star \mathsf{Err}_{\mathfrak{m}}^{(1)}(s,\cdot)\,ds\Big\|_{\mathcal{C}^{\frac12}}^p\le C(p,T)\gamma^{\frac12}\;.$$

Similarly for $\mathsf{Err}_{\mathfrak{m}}^{(2)}$, invoking Proposition 4.3 to bound $E_{\gamma,s,\mathfrak{m}}^{:k:}$, one has

$$\mathbb{E}\sup_{0\le t\le T}\Big\|\int_0^t P_{t-s}^\gamma\star K_\gamma\star \mathsf{Err}_{\mathfrak{m}}^{(2)}(s,\cdot)\,ds\Big\|_{\mathcal{C}^{\frac12}}^p\le C(p,T)\gamma^{\frac{p}{2}}\;.$$

Therefore (4.39) is obtained. □

Now we prove our main theorem of the article.

Proof of Theorem 2.5. The proof is essentially the same as [MW16]; we give the proof for completeness. Our arguments hold for both scaling regimes. For r and $\mathfrak{m}\ge 1$, we define the events $\mathcal{A}_{\mathsf{r}}^Z=\mathcal{A}_{\mathsf{r}}^Z(\gamma,\mathfrak{m})$, and $\mathcal{A}^{\mathsf{E}}=\mathcal{A}^{\mathsf{E}}(\gamma,\mathfrak{m})$ by

$$\begin{aligned}\mathcal{A}_{\mathsf{r}}^Z&\stackrel{\text{def}}{=}\{\|Z_{\gamma,\mathfrak{m}}^{:k:}\|_{\mathcal{C}^{-\nu}}\le \mathsf{r}\text{ on }[0,T],\quad k=1,\dots,5\}\;,\\ \mathcal{A}^{\mathsf{E}}&\stackrel{\text{def}}{=}\{\sup_{0\le t\le T}|\mathsf{Err}^{(3)}(t)|\le\gamma^{\frac{1}{2p}}\}\;,\end{aligned}$$

where p is the constant in (4.39). For every $\mathfrak{m},\mathsf{r}\ge 1$ and every bounded uniformly continuous mapping $F:\mathcal{D}([0,T],\mathcal{C}^{-\nu})\to\mathbb{R}$, we have

$$\begin{aligned}|\mathbb{E}(F(X_{\gamma,\mathfrak{m}}))-\mathbb{E}(F(X))|&\le|\mathbb{E}(F(\overline{X}_{\gamma,\mathfrak{m}}))-\mathbb{E}(F(X))|\\&+\mathbb{E}\Big(\big|F\big(\overline{X}_{\gamma,\mathfrak{m}}\big)-F\left(X_{\gamma,\mathfrak{m}}\right)\big|\mathbf{1}_{\mathcal{A}_{\mathsf{r}}^Z\cap\mathcal{A}^{\mathsf{E}}}\Big)+\|F\|_{L^\infty}\mathbb{P}\Big(\overline{\mathcal{A}}_{\mathsf{r}}^Z\cup\overline{\mathcal{A}}^{\mathsf{E}}\Big)\;.\end{aligned}\tag{4.43}$$

Recall that $\overline{X}_{\gamma,\mathfrak{m}}$ converges in law to X, see (4.30) and the discussion below it.

To bound the second term on RHS of (4.43), note that on the event $\mathcal{A}_{\mathsf{r}}^Z$ and by continuity of $\mathcal{S}_T$ (Theorem 2.4), we have $\sup_{0\le t\le T}\|\overline{v}_{\gamma,\mathfrak{m}}(t)\|_{\mathcal{C}^{\frac12}}\le C(T,\mathsf{r})$ for some finite constant $C(T,\mathsf{r})$. Apply Gronwall's inequality to the bound obtained in Lemma 4.8, one has that on the event $\mathcal{A}_{\mathsf{r}}^Z\cap\mathcal{A}^{\mathsf{E}}$

$$\|v_{\gamma,\mathfrak{m}}(t,\cdot)-\overline{v}_{\gamma,\mathfrak{m}}(t,\cdot)\|_{\mathcal{C}^{\frac12}}\le C\left(\gamma^{\frac{\kappa}{2}}+\|X_\gamma^0-X^0\|_{\mathcal{C}^{-\nu}}\right)\tag{4.44}$$

for all $t\ge0$ such that $\|v_{\gamma,\mathfrak{m}}(t)\|_{\mathcal{C}^{\frac12}}\le C(T,\mathsf{r})+2$. In particular for γ small enough, the right hand side of (4.44) is bounded by 1. By continuity of v_γ and $\bar v_\gamma$ (which follow by definition

(4.31) - the jumps in the evolution of X_γ are all contained in the part $Z_{\gamma,\mathfrak{m}}$), the bound (4.44) must actually hold for all $t \in [0,T]$.

This together with (4.29), (4.31) implies that the second term on RHS of (4.43) vanishes.

Regarding the last term in (4.43), it follows from (4.39) i.e. the bound for $\mathsf{Err}^{(3)}(t)$ and Chebyshev's inequality that $\lim_{\gamma\to 0}\mathbb{P}(\mathcal{A}^{\mathsf{E}}) = 1$. For the event $\mathcal{A}^Z_{\mathsf{r}}$, we know that the limiting quantities $\sup_{0\le t\le T}\|Z^{:k:}(t)\|_{\mathcal{C}^{-\nu}}$ are finite a.s.; on the other hand it is easy to argue that the stopping time that $\|Z^{:k:}_{\gamma,\mathfrak{m}}(t)\|_{\mathcal{C}^{-\nu}}$ first exceeds the value r will converge to [2] the stopping time that $\|Z^{:k:}(t)\|_{\mathcal{C}^{-\nu}}$ first exceeds the same value r. Thus we can choose r large enough, so that $\liminf_{\gamma\to 0}\mathbb{P}(\mathcal{A}^Z_{\mathsf{r}})$ is arbitrarily close to 1.

This proves that $X_{\gamma,\mathfrak{m}}$ converges in law to X as γ tends to 0, for any fixed value of $\mathfrak{m}$. We can remove $\mathfrak{m}$ by the same reasoning as above. The stopping time $\tau_{\gamma,\mathfrak{m}}$ defined in (3.5) converges in law to [3] the stopping time $\tau_{\mathfrak{m}}$ defined in the same way for X, for every $\mathfrak{m}$. Moreover, we know from Theorem 2.4 that $\sup_{0\le t\le T+1}\|X(t)\|_{\mathcal{C}^{-\nu}}$ is a.s. finite. Hence by choosing $\mathfrak{m} = \mathfrak{m}(T,\varepsilon)$ sufficiently large, $\liminf_{\gamma\to 0}\mathbb{P}(X_{\gamma,\mathfrak{m}} = X_\gamma)$ can be made arbitrarily close to 1. Therefore we have proved that X_γ also converges in law to X.

This concludes the proof of Theorem 2.5. Note that item (2) of the theorem is clearly just the degenerate case of the item (1) that the cubic term equals zero and therefore one obtains a linear limit. □

5 Appendix: Kernel estimates

We need some estimates about K_γ and P^γ. In the case of the first scaling regime (2.13), these estimates are proved in [MW16, Section 8]. For the second scaling regime (2.14), we list all these results, without proving them since the proofs follow exactly the same way except that one simply applies the new scaling relations.

We begin with the Fourier transforms of these kernels. For $\omega \in \{-N,\dots,N\}^2$,

$$\hat{K}_\gamma(\omega) = \sum_{x\in\Lambda_\varepsilon} \varepsilon^2\, K_\gamma(x)\, e^{-i\pi\omega\cdot x} = c_{\gamma,1}\sum_{x\in\gamma\mathbb{Z}^2_\star}\gamma^2\,\mathfrak{K}(x)\, e^{-i\pi(\varepsilon/\gamma)\omega\cdot x}\;, \tag{5.1}$$

where $\mathfrak{K}$ is the smooth function introduced in (2.2), $\gamma\mathbb{Z}^2_\star \stackrel{\text{def}}{=} \gamma\mathbb{Z}^2\setminus\{0\}$, and note that $\varepsilon/\gamma \approx \gamma$ in the first regime and $\varepsilon/\gamma\approx\gamma^2$ in the second regime. Also,

$$\hat{P}^\gamma_t(\omega) = \exp\Big(t\gamma^{-b}(\hat{K}_\gamma(\omega)-1)\Big)\;, \tag{5.2}$$

where $b=2$ in the first regime and $b=4$ in the second regime.

We now list some estimates which state that some properties of $\widehat{\mathfrak{K}}(\gamma\omega)$ (resp. $\widehat{\mathfrak{K}}(\gamma^2\omega)$) also hold for $\hat{K}_\gamma$ in the first (resp. second) regime, uniformly in γ.

Lemma 5.1 *The following statement holds with $b=1$ in the first regime and $b=2$ in the second regime. There exists $C>0$ such that for all $0<\gamma<\frac{1}{3}$ and for $|\omega|\le\gamma^{-b}$ we have for $j=1,2$*

$$|\gamma^{-2b}(1-\hat{K}_\gamma(\omega)) - \pi^2|\omega|^2| \le C\gamma^b|\omega|^3\;, \tag{5.3}$$

$$|-\gamma^{-2b}\partial_j\hat{K}_\gamma(\omega) - 2\pi^2\omega_j| \le C\gamma^b|\omega|^2\;, \tag{5.4}$$

$$|-\gamma^{-2b}\partial_j^2\hat{K}_\gamma(\omega) - 2\pi^2| \le C\gamma^b|\omega|\;. \tag{5.5}$$

[2] outside a countable set of r that $\|Z^{:k:}_{\gamma,\mathfrak{m}}(t)\|_{\mathcal{C}^{-\nu}}$ attains r as a local maximum with positive probability

[3] outside a countable set for the same reason

Lemma 5.2 *The following statements hold with $b = 1$ in the first regime and $b = 2$ in the second regime. There exists $C > 0$ such that for all $0 < \gamma < \frac{1}{3}$, $\omega \in [-N - \frac{1}{2}, N + \frac{1}{2}]^2$ and $j = 1, 2$,*

(1) (Estimates most useful for $|\omega| \leq \gamma^{-b}$)

$$|\hat{K}_\gamma(\omega)| \leq 1 \,, \quad |\partial_j \hat{K}_\gamma(\omega)| \leq C\gamma^b(|\gamma^b\omega| \wedge 1) \,, \quad |\partial_j^2 \hat{K}_\gamma(\omega)| \leq C\gamma^{2b} \,. \tag{5.6}$$

(2) (Estimates most useful for $|\omega| \geq \gamma^{-b}$)

$$|\gamma^b\omega|^2 \, |\hat{K}_\gamma(\omega)| \leq C, \quad |\gamma^b\omega|^2 \, |\partial_j \hat{K}_\gamma(\omega)| \leq C\gamma^b, \quad |\gamma^b\omega|^2 \, |\partial_j^2 \hat{K}_\gamma(\omega)| \leq C\gamma^{2b}. \tag{5.7}$$

Furthermore, there exist constants $C_1 > 0$ and $\gamma_0 > 0$ such that for all $0 < \gamma < \gamma_0$ and $\omega \in [-N - \frac{1}{2}, N + \frac{1}{2}]^2$,

$$1 - \hat{K}_\gamma(\omega) \geq \frac{1}{C_1}(|\gamma^b\omega|^2 \wedge 1) \,. \tag{5.8}$$

Lemma 5.3 *Let $\gamma_0 > 0$ be the constant introduced in Lemma 5.2. For every $T > 0$, there exists a constant $C = C(T)$ such that for all $0 < \gamma < \gamma_0$, $0 \leq t \leq T$ and $x \in \mathbb{T}^2$, we have*

$$|P_t^\gamma \star K_\gamma(x)| \leq C\big(t^{-1}\big(\log(\gamma^{-1})\big)^2 \wedge \gamma^{-2b}\log(\gamma^{-1})\big) \,, \tag{5.9}$$

where $b = 1$ in the first regime and $b = 2$ in the second regime.

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