

SHARP INTERFACE LIMIT FOR INVARIANT MEASURES OF A STOCHASTIC ALLEN-CAHN EQUATION

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ABSTRACT. The invariant measure of a one-dimensional Allen-Cahn equation with an additive space-time white noise is studied. This measure is absolutely continuous with respect to a Brownian bridge with a density which can be interpreted as a potential energy term. We consider the sharp interface limit in this setup. In the right scaling this corresponds to a Gibbs type measure on a growing interval with decreasing temperature. Our main result is that in the limit we still see exponential convergence towards a curve of minimizers of the energy if the interval does not grow too fast. In the original scaling the measures concentrate on configurations with precisely one jump.

1. INTRODUCTION

Reaction-diffusion equations can be used to model phase separation and boundary evolutions in various physical contexts. Typically behavior of boundaries or geometric evolution laws are studied with the help of such equations. Often in such models one includes an extra noise term. This may happen for various reasons – the noise may be a simplified model for the effect of additional degrees of freedom that are not reflected in the reaction-diffusion equation. From a numerical point of view noise may improve stability in the simulations. In some systems there is even a justification for an extra noise term from a scaling limit of microscopic particle systems.

1.1. Setup and first main result. The system considered here is the case of a symmetric bistable potential with two wells of equal depth. To be more precise, for a small parameter $\varepsilon > 0$ we are interested in the equation

$$(1.1) \quad \begin{aligned} \partial_t u(x, t) &= \Delta u(x, t) - \varepsilon^{-2\gamma} F'(u(x, t)) + \varepsilon^{(1-\gamma)/2} \sqrt{2} \partial_x \partial_t W(x, t) \\ u(-1, t) &= -1 \quad u(1, t) = 1, \end{aligned}$$

for $(x, t) \in (-1, 1) \times \mathbb{R}_+$. Here F is supposed to be a smooth (at least C^3) symmetric double-well potential i.e. we assume that F satisfies the following properties:

$$(1.2) \quad \begin{cases} (a) & F(u) \geq 0 \quad \text{and} \quad F(u) = 0 \quad \text{iff} \quad u = \pm 1, \\ (b) & F' \text{ admits exactly three zeros } \{\pm 1, 0\} \text{ and } F''(0) < 0, F''(\pm 1) > 0, \\ (c) & F \text{ is symmetric, } \forall u \geq 0 \quad F(u) = F(-u). \end{cases}$$

A typical example is $F(u) = \frac{1}{2}(u^2 - 1)^2$. The expression $\partial_x \partial_t W(x, t)$ is a formal expression denoting space-time white noise. Such equation can be given rigorous sense in various ways, for example in the sense of mild solutions ([Iwa87, DPZ92]) or using Dirichlet forms [AR91]. We are interested in the behavior of the system in

the sharp interface limit $\varepsilon \downarrow 0$. The parameter $\gamma > 0$ is a scaling factor. Our result will be valid for $\gamma < \frac{2}{3}$.

We study the behavior of the invariant measure of (1.1). This measure can be described quite explicitly as follows ([DPZ96, RVE05]): Let $\tilde{\nu}^\varepsilon$ be the law of a rescaled Brownian bridge on $[-1, 1]$ with boundary points ± 1 . More precisely $\tilde{\nu}^\varepsilon$ is the law of a Gaussian process $(\tilde{u}(s), s \in [-1, 1])$ with expectations $\mathbb{E}[\tilde{u}(s)] = s \quad \forall s \in [-1, 1]$ and covariance $\text{Cov}(\tilde{u}(s), \tilde{u}(s')) = \varepsilon^{1-\gamma}(s \wedge s' + 1 - \frac{(s+1)(s'+1)}{2})$. Another equivalent way to characterize $\tilde{\nu}^\varepsilon$ is to say that it is a Gaussian measure on $L^2[-1, 1]$ with expectation function $s \mapsto s$ and covariance operator $\varepsilon^{1-\gamma}(-\Delta)^{-1}$ where Δ denotes the one-dimensional Dirichlet Laplacian. Even another equivalent way is to say that $\tilde{u}(s)$ is the solution to the stochastic differential equation (SDE)

$$d\tilde{u}(s) = \varepsilon^{\frac{1-\gamma}{2}} dB(s) \quad \tilde{u}(-1) = -1$$

with some Brownian motion $B(s)$ conditioned on $\tilde{u}(1) = 1$. Then the invariant measure $\tilde{\mu}^\varepsilon$ of (1.1) is absolutely continuous with respect to $\tilde{\nu}^\varepsilon$ and is given as

$$(1.3) \quad \tilde{\mu}^\varepsilon(d\tilde{u}) = \frac{1}{Z^\varepsilon} \exp\left(-\frac{1}{\varepsilon^{1+\gamma}} \int_{-1}^1 F(\tilde{u}(s)) ds\right) \tilde{\nu}^\varepsilon(d\tilde{u}).$$

Here $Z^\varepsilon = \int \exp\left(-\frac{1}{\varepsilon^{1+\gamma}} \int_{-1}^1 F(\tilde{u}(s)) ds\right) \tilde{\nu}^\varepsilon(d\tilde{u})$ is the appropriate normalization constant.

Often important intuition on a measure on path space can be gained from considering Feynman's heuristic interpretation. In our context this heuristic interpretation states that $\tilde{\nu}^\varepsilon(d\tilde{u})$ is proportional to a measure

$$\exp\left(-\frac{1}{\varepsilon^{1-\gamma}} \int_{-1}^1 \frac{|\tilde{u}'(s)|^2}{2} ds\right) d\tilde{u}$$

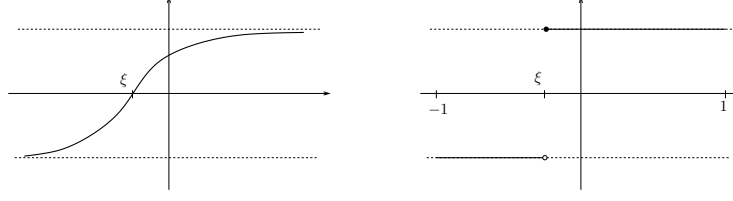
where $d\tilde{u}$ is a flat reference measure on path space. Of course this picture is non-rigorous: Such a measure $d\tilde{u}$ does not exist and the quantity $\int_{-1}^1 \frac{|\tilde{u}'(s)|^2}{2} ds$ is almost surely not finite under $\tilde{\nu}^\varepsilon(d\tilde{u})$. Nonetheless, it is rigorous on the level of finite-dimensional distributions and various classical statements about Brownian motion such as Schilder's theorem or Girsanov's theorem have an interpretation in terms of this heuristic picture. The measure $\tilde{\mu}^\varepsilon(d\tilde{u})$ can then be interpreted as proportional to

$$\exp\left(-\frac{1}{\varepsilon^{1+\gamma}} \int_{-1}^1 F(\tilde{u}(s)) ds - \frac{1}{\varepsilon^{1-\gamma}} \int_{-1}^1 \frac{|\tilde{u}'(s)|^2}{2} ds\right) d\tilde{u}.$$

As one wants to observe an effect which results from the interaction of the *potential term* $\frac{1}{\varepsilon^{1+\gamma}} \int F(\tilde{u}(s)) ds$ and the *kinetic energy term* $\frac{1}{\varepsilon^{1-\gamma}} \int \frac{|\tilde{u}'(s)|^2}{2} ds$ it seems reasonable to transform the system in a way that guarantees that these terms scale with the same power of ε . This transformation is given by stretching the random functions onto a growing interval $[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$. More precisely consider the operators

$$T^\varepsilon : L^2[-1, 1] \rightarrow L^2[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}] \quad T^\varepsilon \tilde{u}(s) = \tilde{u}(\varepsilon^\gamma s).$$

Then consider the pushforward measures $\mu^\varepsilon = T^\varepsilon_\# \tilde{\mu}^\varepsilon$. These measures are again absolutely continuous with respect to Gaussian measures: ν^ε is the Gaussian measure on $L^2[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$ with expectation function $s \mapsto \varepsilon^\gamma s$ and covariance operator $\varepsilon(-\Delta)^{-1}$. The other equivalent characterizations for $\tilde{\nu}^\varepsilon$ can be adapted with the

FIGURE 1. The instanton shape m_ξ and the function $-\mathbf{1}_{[-1, \xi]} + \mathbf{1}_{[\xi, 1]}$.

right powers of ε . The measure μ^ε is then given as

$$\mu^\varepsilon(du) = \frac{1}{Z^\varepsilon} \exp\left(-\varepsilon^{-1} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^\varepsilon(du).$$

Note that the normalization constant Z^ε is the same as above. In the Feynman picture this suggests that $\mu^\varepsilon(du)$ is proportional to

$$\exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \left[\frac{|u'(s)|^2}{2} + F(u(s))\right] ds\right) du.$$

This motivates to study the energy functional appearing in the exponent: For functions $u: \mathbb{R} \rightarrow \mathbb{R}$ defined on the whole line with boundary conditions $u(\pm\infty) = \pm 1$ consider the energy functional

$$\mathcal{E}(u) = \int_{-\infty}^{\infty} \left[\frac{|u'(s)|^2}{2} + F(u(s))\right] ds - C_*.$$

Here C_* is a constant chosen in a way to guarantee that the minimizers of \mathcal{E} with the right boundary conditions verify $\mathcal{E}(u) = 0$. This is the one-dimensional version of the well known real Ginzburg-Landau energy functional. There is a unique minimizer m of \mathcal{E} subject to the condition $m(0) = 0$ and all the other minimizers are obtained via translation of m . More details on the energy functional and the minimizers can be found in Section 2. Denote by M the set of all these minimizers and by $m + L^2(\mathbb{R}) := \{u: \mathbb{R} \rightarrow \mathbb{R}, u - m \in L^2(\mathbb{R})\}$ and $m + H^1(\mathbb{R}) := \{u: \mathbb{R} \rightarrow \mathbb{R}, u - m \in H^1(\mathbb{R})\}$ the spaces of functions with the right boundary values. Note that every random function distributed according to $\mu^\varepsilon(du)$ can be considered as function in $m + L^2(\mathbb{R})$ by trivial extension with ± 1 outside of $[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$. In this way $\mu^\varepsilon(du)$ can be interpreted as measure on $m + L^2(\mathbb{R})$. We can now state the main result of this work:

Theorem 1.1. *Assume $0 < \gamma < \frac{2}{3}$. Assume $p = 2$ or $p = \infty$. Then there exist positive constants c_0 and δ_0 such that for every $0 < \delta \leq \delta_0$ one has*

$$(1.4) \quad \limsup_{\varepsilon \downarrow 0} \varepsilon \log \mu^\varepsilon \left\{ \text{dist}_{L^p}(u, M) \geq \delta \right\} \leq -c_0 \delta^2.$$

In particular the measures μ^ε concentrate around the set of minimizers exponentially fast.

The crucial step in the proof is to find a bound on the exponential decay of the normalization constant Z^ε . This lower bound can be found in Section 4. The asymptotic behavior of Z^ε is given in Corollary 4.15.

On the fixed interval $[-1, 1]$ this bound implies the following:

Corollary 1.2. *Assume $0 < \gamma < \frac{2}{3}$. Then the measures $\tilde{\mu}^\varepsilon(du)$ are tight for $\varepsilon \downarrow 0$ as measures on $L^2[-1, 1]$. Every limiting measure $\tilde{\mu}$ is concentrated on random functions of the type*

$$\tilde{u}(s) = -\mathbf{1}_{[-1, \xi]} + \mathbf{1}_{[\xi, 1]},$$

where $\xi \in [-1, 1]$.

Remark 1.3. It is expected that the measures $\tilde{\mu}^\varepsilon(du)$ converge toward a measure $\tilde{\mu}$. Comparison with the dynamical results (e.g. [Fun95]) suggests that the phase separation point ξ should be distributed uniformly on $[-1, 1]$ in the limit.

Remark 1.4. Note that by Schilder's theorem together with an exponential tilting argument (such as [dH00] Theorem III.17 on page 34), in the case where $\gamma = 0$ the measures $\tilde{\mu}^\varepsilon$ concentrate exponentially fast around the unique minimizer of

$$u \mapsto \int_{-1}^1 \left[\frac{|u'(s)|^2}{2} + F(u(s)) \right] ds,$$

under the appropriate boundary conditions. In particular the weak limit is a Dirac measure on this minimizer, which is not a step function.

Remark 1.5. One can remark that by an application of Girsanov's theorem the measure $\tilde{\mu}^\varepsilon$ can be considered as the distribution of the solution of an SDE which is conditioned on the right boundary values (see [RY99] Chapter VIII §3 and also [HSV07, RVE05]). It could be possible to obtain concentration results such as Theorem 1.2 by studying this SDE with the help of large deviation theory (see for example [Sug95]). We do not follow such an approach but conclude from Theorem 1.1 which is obtained by a discretization argument.

Remark 1.6. The reader might consider it unusual to work with $\tilde{\mu}^\varepsilon$ as measure on $L^2[-1, 1]$ instead of $C[-1, 1]$ or the space of càdlàg functions $D[-1, 1]$. The class of *continuous* processes is closed under weak convergence of measures on $D[-1, 1]$ such that tightness on this space cannot hold. But in fact the tightness holds in every topology τ in which the rescaled profiles $m_\xi(\varepsilon^\gamma x)$ converge to step functions and in which convergence in $L^\infty[-1, 1]$ implies convergence in τ .

1.2. Motivation and related works. The Allen-Cahn equation without noise was introduced in [AC79] to model the dynamics of interfaces between different domains of different lattice structure in crystals and has been studied since in various contexts. In the one-dimensional case the dynamics of the deterministic equation are well understood [Che04, CP89, OR07] and can be described as follows: If one starts with arbitrary initial data, solutions will quickly tend to configurations which are locally constant close to ± 1 possibly with many transition layers that roughly look like the instanton shapes m introduced above. Then these interfaces move extremely slowly until eventually some two transition layers meet and annihilate. After that the dynamics continue very slowly with less interfaces.

In the two or more dimensions no such metastable behavior occurs. Solutions tend very quickly towards configurations which are locally constant with interfaces of width ε . Then on a slower scale these interfaces evolve according to motion by mean curvature (see [Ilm93] and the references therein).

Stochastic systems which are very similar to (1.1) have been studied in the classic paper by Farris and Jona-Lasinio [FJL82]. In the nineties Funaki [Fun95]

and Brascocco, de Masi, Presutti [BDMP95] studied the one-dimensional equation in the case where the initial data is close to the instanton shape and showed that in an appropriate scaling the solution will stay close to such a shape. Then due to the random perturbation a dynamic along the one-parameter family of such shapes can be observed on a much faster time scale than in the deterministic case. Our result Theorem 1.2 corresponds to this on the level of invariant measures.

If the process does not start in a configuration with a single interface, it is believed that these different interfaces also follow a random induced dynamic which is much quicker than in the deterministic case. Different interfaces should annihilate when they meet [FVE02]. More recently there were also investigations of the same system on a much bigger space interval where due to entropic effects noise induced nucleation should occur. This phenomenon has been studied on the level of invariant measures [RVE05]. The limiting process should be related to the Brownian web which has recently been investigated e.g. in [FINR06].

From the point of view of statistical physics Theorem 1.1 can be interpreted as quite natural. In fact the Feynman picture suggests to view μ^ε as a Gibbs measure with energy \mathcal{E} and decreasing temperature ε . On a fixed interval the result of Theorem 1.1 would therefore simply state that with decreasing temperature the Gibbs measure concentrates around the energy minimizers exponentially fast. On a rigorous level such results follow from standard large deviation theory (see e.g. [dH00, DS89]). Our result states that the entropic effects which originate from considering growing intervals do not change this picture. In fact also this is not very surprising - analysis of similar spin systems suggests that even on intervals that grow exponentially in ε^{-1} one should not observe more than one jump. But it is not clear if one can say anything about the shape of the interface in this settings.

1.3. Structure of the paper. In Section 2 results about the energy landscape of the Ginzburg-Landau energy functional are summarized. In particular we discuss in some detail the minimizers of \mathcal{E} and introduce tubular coordinates close to the curve of minimizers. The energy landscape is studied in terms of these tubular coordinates. In Section 3 some Gaussian concentration inequalities are discussed. In particular the discretization of the measure ν^ε is given and some error bounds are proven. The proof of Theorem 1.1 can then be found in Section 4. We will follow the convention that C denotes a generic constant which may change from line to line. Constants that appear several times will be numbered c_1, c_2, \dots . Auxiliary scaling factors γ_1 to γ_3 that are supposed to satisfy a number of conditions will be introduced. These conditions are satisfied if γ_1 and γ_3 are very small and $\gamma_2 < 1$ is very close to 1.

2. THE ENERGY FUNCTIONAL

In this section we discuss properties of the Ginzburg-Landau energy functional. We introduce the one parameter family of minimizers which we think of as a one-dimensional submanifold of the infinite-dimensional space of possible configurations. Then we discuss tubular coordinates of a neighborhood of this curve as well as a Taylor expansion of the energy landscape in these tubular coordinates. These ideas are mostly classical and go back to [CP89, Fun95, OR07]. Finally we give a discretized version of the minimizers and prove some error bounds.

For a function u defined on the whole real line consider the following energy functional:

$$\mathcal{E}(u) = \int_{\mathbb{R}} \left[\frac{1}{2} |u'(s)|^2 + F(u(s)) \right] ds - C_*,$$

where the constant C_* is chosen in a way to guarantee that the minimum of \mathcal{E} on the set of functions with the right boundary conditions is 0. In fact let m be the standing wave solution of the Allen-Cahn equation:

$$(2.1) \quad m''(s) - F'(m(s)) = 0 \quad \forall s \in \mathbb{R}, \quad m(\pm s) \rightarrow \pm 1 \quad \text{for } s \rightarrow \infty.$$

As (2.1) is invariant under translations one can assume $m(0) = 0$. Then the solution can be found by solving the system

$$(2.2) \quad m'(s) - \sqrt{2F(m(s))} = 0 \quad \forall s \in \mathbb{R}, \quad m(0) = 0 \quad m(\pm\infty) = \pm 1.$$

Note that the assumptions (1.2) on F imply that \sqrt{F} is C^1 such that the solution to (2.2) is unique. The translations of m will be denoted by $m_\xi(s) = m(s - \xi)$. Note that the m_ξ are not the only solutions to (2.1) but that all the other solutions are either periodic or diverge such that the m_ξ are the only nonconstant critical points of \mathcal{E} with finite energy. In fact the m_ξ are global minimizers of \mathcal{E} subject to its boundary conditions. Completing the squares yields:

$$(2.3) \quad \begin{aligned} & \int_{\mathbb{R}} \left[\frac{1}{2} |u'(s)|^2 + F(u(s)) \right] ds \\ &= \int_{-\infty}^{\infty} \frac{1}{2} \left(u'(s) - \sqrt{2F(u(s))} \right)^2 + \sqrt{2F(u(s))} u'(s) ds \\ &\geq \int_{u(-\infty)}^{u(\infty)} \sqrt{2F(u)} du. \end{aligned}$$

The term in the bracket is nonnegative and it vanishes if and only if u solves (2.2). In the sequel we will write

$$M = \{m_\xi, \xi \in \mathbb{R}\} \quad \text{and} \quad C_* = \int_{\mathbb{R}} \frac{1}{2} [|m'(s)|^2 + F(m(s))] ds.$$

For notational convenience we introduce the function $G(u) = \int_0^u \sqrt{2F(\tilde{u})} d\tilde{u}$. Then equation (2.3) states that $\int_{\mathbb{R}} \frac{1}{2} |u'(s)|^2 + F(u(s)) ds \geq G(u(\infty)) - G(u(-\infty))$. Note that the assumptions (1.2) on F imply that G is a strictly increasing C^4 function with $G(0) = 0$. In the case of the standard double-well potential $F(u) = \frac{1}{2}(u^2 - 1)^2$ a calculation yields

$$m(s) = \tanh(s) \quad \text{and} \quad C_* = \frac{4}{3}.$$

Equation (2.2) shows that in general m can be given implicitly as

$$(2.4) \quad s = \int_0^m \frac{1}{\sqrt{2F(\tilde{m})}} d\tilde{m}.$$

By expanding F around 1 one obtains exponential convergence to ± 1 for $s \rightarrow \pm\infty$. To be more precise there exist positive constants c_1 and c_2 such that

$$(2.5) \quad \begin{cases} |1 \mp m(\pm s)| \leq c_1 \exp(-c_2 s) & s \geq 0 \\ |m'(\pm s)| \leq c_1 c_2 \exp(-c_2 s) & s \geq 0 \\ |m''(\pm s)| \leq c_1 c_2^2 \exp(-c_2 s) & s \geq 0. \end{cases}$$

Recall that $m + L^2(\mathbb{R}) = \{u : u - m \in L^2(\mathbb{R})\}$. Due to (2.5) for all ξ one has $m - m_\xi \in L^2(\mathbb{R})$ such that the definition of the space $m + L^2$ is independent of the choice of minimizer.

We now introduce the concept of Fermi coordinates which was first used in this context in [CP89, Fun95]: Recall that for a function $u \in m + L^2(\mathbb{R})$ we write $\text{dist}_{L^2}(u, M) := \inf_{\xi \in \mathbb{R}} \|u - m_\xi\|_{L^2(\mathbb{R})}$. If $\text{dist}_{L^2}(u, M)$ is small enough there exists a unique $\xi \in \mathbb{R}$ such that $\text{dist}(u, M) = \|u - m_\xi\|_{L^2(\mathbb{R})}$ and one has

$$(2.6) \quad \langle u - m_\xi, m'_\xi \rangle_{L^2(\mathbb{R})} = 0.$$

In fact the last equality (2.6) can easily be seen by differentiating $\xi \mapsto \|u - m_\xi\|_{L^2(\mathbb{R})}^2$. This has a simple geometric interpretation. The function m'_ξ can be seen as tangent vector to the curve M in m_ξ and the relation (2.6) can be interpreted as $v := u - m_\xi$ being normal to the tangent space in m_ξ . We will denote the space

$$N_\xi := \{v \in L^2(\mathbb{R}) : \langle v, m'_\xi \rangle_{L^2(\mathbb{R})} = 0\}$$

and interpret it as the normal space to M in m_ξ . For $u = m_\xi + v$ with $v \in N_\xi$ we will call the pair (ξ, v) Fermi or tubular coordinates of u .

One obtains information about the behavior of the energy functional close to M by considering the linearized Schrödinger type operators

$$\mathcal{A}_\xi = -\Delta + F''(m_\xi)$$

with domain of definition $H^2(\mathbb{R}) \subset L^2(\mathbb{R})$. The operator \mathcal{A}_ξ is selfadjoint and nonnegative (see e.g. [Fun95]) and the eigenspace corresponding to the eigenvalue 0 is spanned by the function m'_ξ . This can be understood quite easily: The fact that the operator is nonnegative corresponds to the functional \mathcal{E} attaining its minimum at m_ξ and the fact that m'_ξ is an eigenfunction to the eigenvalue 0 corresponds to the translational invariance of \mathcal{E} . The following more detailed description of the spectral behavior of \mathcal{A}_ξ is taken from [OR07] Proposition 3.2 on page 391:

Lemma 2.1. *There exists a constant $c_3 > 0$ such that if $u \in H^1(\mathbb{R})$ satisfies*

$$(i) \quad u(\xi) = 0 \quad \text{or} \quad (ii) \quad \int_{\mathbb{R}} u(s) m'_\xi(s) \, ds = 0,$$

then

$$(2.7) \quad c_3 \|u\|_{L^2(\mathbb{R})}^2 \leq \int_{\mathbb{R}} [u'(s)^2 + F''(m_\xi(s)) u(s)^2] \, ds.$$

This can be used to obtain the following description of the energy landscape. Similar results were already obtained in [Fun95] and [OR07]:

Proposition 2.2. (i) *There exist positive constants c_0, c_4, δ_1 such that for u with Fermi coordinates $u = m_\xi + v$ and $\|v\|_{H^1(\mathbb{R})} \leq \delta_1$ one has:*

$$(2.8) \quad c_0 \|v\|_{H^1(\mathbb{R})}^2 \leq \mathcal{E}(u) \leq c_4 \|v\|_{H^1(\mathbb{R})}^2.$$

(ii) *There exists a $\delta_0 > 0$ such that for $\delta \leq \delta_0$ the relation $\text{dist}_{H^1}(u, M) \geq \delta$ implies*

$$(2.9) \quad \mathcal{E}(u) \geq c_0 \delta^2.$$

Here $\text{dist}_{H^1}(u, M) = \inf_{\xi \in \mathbb{R}} \|u - m_\xi\|_{H^1(\mathbb{R})}$. Statement (i) will be used as a local description of the energy landscape close to the curve of minimizers whereas the statement (ii) will be useful as a rough lower bound for the energy away from the curve. For the proof of Proposition 2.2 one needs the following lemma:

Lemma 2.3. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $u \in m + L^2$ with $\mathcal{E}(u) \leq \delta$ then there exists $\xi \in \mathbb{R}$ such that*

$$\|u - m_\xi\|_{L^\infty(\mathbb{R})} \leq \varepsilon.$$

Furthermore, ξ can be chosen in a such a way that $u(\xi) = 0$.

Proof. For a small $\delta > 0$ assume $\mathcal{E}(u) \leq \delta$. We want to find a $\xi \in \mathbb{R}$ such that by choosing δ sufficiently small we can deduce that $\|u - m_\xi\|_{L^\infty(\mathbb{R})}$ becomes arbitrarily small. As $\mathcal{E}(u) < \infty$ we have $u \in m + H^1$ and in particular $u \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Note that a similar calculation as (2.3) implies that $\mathcal{E}(u) \geq (G(\sup_{s \in \mathbb{R}} u(s)) - G(\inf_{s \in \mathbb{R}} u(s))) - (G(1) - G(-1))$. Therefore by the properties of G by choosing δ sufficiently small, one can assume that $\|u\|_{L^\infty(\mathbb{R})} \leq 2$. By the assumptions (1.2) on F there exists a C such that for $u \in [-2, 2]$ one has

$$F(u) \geq C \min(|u - 1|, |u + 1|)^2,$$

and in particular we know that for every interval I the H^1 -norm of $\min(|u - 1|, |u + 1|)$ can be controlled by the energy. As u is continuous and converges to ± 1 as s goes to $\pm\infty$, there exists a ξ with $u(\xi) = 0$. Without loss of generality one can assume that $\xi = 0$. We will show that in this case $\|u - m\|_{L^\infty(\mathbb{R})}$ can be made arbitrarily small.

According to (2.5) for every $\varepsilon > 0$ there exists T such that for $s \geq T$ one has $|m(s) - 1| \leq \varepsilon$ and for $s \leq -T$ it holds that $|m(s) + 1| \leq \varepsilon$. We will first give a bound on $u - m$ in $[-T, T]$. We consider only the case $s \geq 0$ the other one being similar. Note that as according to (2.3)

$$\mathcal{E}(u) = \int_{\mathbb{R}} \frac{1}{2} \left(u'(s) - \sqrt{2F(u)} \right)^2 ds,$$

one can write

$$(2.10) \quad \begin{aligned} u'(s) &= \sqrt{F(u(s))} + r(s) \\ u(0) &= 0 \end{aligned}$$

where $\int_0^T r(s)^2 ds \leq 2\delta$ and using Cauchy-Schwarz inequality

$$\int_0^T |r(s)| ds \leq \sqrt{2T\delta}.$$

Thus using (2.2) one obtains for $v = |u - m|$

$$(2.11) \quad v(t) \leq \int_0^t \left| \sqrt{F(u(s))} - \sqrt{F(m(s))} \right| + |r(s)| ds \leq C \int_0^t v(s) ds + \int_0^t |r(s)| ds,$$

where the constant C is given by $C = \sup_{u \in [-2, 2]} \frac{d}{du} \left(\sqrt{F(u)} \right)$. Thus Gronwall's Lemma implies

$$|v(s)| \leq \int_0^s |r(t)| e^{C(s-t)} dt,$$

and so $\sup_{s \in [0, T]} |v(s)| \leq \sqrt{2T\delta} e^{CT}$. Thus by choosing δ small enough one can assure that $\sup_{s \in [0, T]} |v(s)| \leq \frac{\varepsilon}{2}$.

Now let us focus on the case $s \in [-T, T]^c$. We will again only focus on $s \geq T$. Note that by the above calculations and the choice of T one has $u(-T) \leq -(1 - \varepsilon)$ and $u(T) \geq 1 - \varepsilon$. Therefore, using

$$\begin{aligned} \int_{-\infty}^{-T} \frac{u'(s)^2}{2} + F(u(s)) ds + \int_{-T}^T \frac{u'(s)^2}{2} + F(u(s)) ds + \int_T^{\infty} \frac{u'(s)^2}{2} + F(u(s)) ds \\ \leq G(1) - G(-1) + \delta, \end{aligned}$$

as well as

$$\int_{-T}^T \frac{u'(s)^2}{2} + F(u(s)) ds \geq G(u(T)) - G(u(-T)),$$

we get

$$\int_T^{\infty} \frac{u'(s)^2}{2} + F(u(s)) ds \leq (G(1) - G(u(T))) - (G(-1) - G(u(-T))) + \delta \leq C\varepsilon + \delta,$$

where $C = 2 \sup_{u \in [-2, 2]} F(u)$. Therefore, by using the fact that $\int_T^{\infty} \frac{u'(s)^2}{2} + F(u(s))$ controls the H^1 -norm and thus also the L^∞ -norm of $\min(|u - 1|, |u + 1|)$ on $[T, \infty)$, one can conclude that possibly by choosing a smaller δ one obtains $\sup_{s \in [T, \infty)} v(s) \leq C\varepsilon$. Thus by redefining ε one obtains the desired result. \square

Proof. (Of Proposition 2.2): (i) First of all remark that for $v \in N_\xi$ one has

$$(2.12) \quad \tilde{c}_0 \|v\|_{H^1(\mathbb{R})}^2 \leq \langle v, \mathcal{A}_\xi v \rangle_{L^2(\mathbb{R})} \leq \tilde{c}_4 \|v\|_{H^1(\mathbb{R})}^2.$$

In fact Lemma 2.1 (ii) implies that

$$(2.13) \quad c_3 \|v\|_{L^2(\mathbb{R})}^2 \leq \langle v, \mathcal{A}_\xi v \rangle_{L^2(\mathbb{R})}.$$

To get the lower bound in (2.12) write

$$(2.14) \quad \begin{aligned} \langle \mathcal{A}v, v \rangle_{L^2(\mathbb{R})} &= \|\nabla v\|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}} F''(m(s))v^2(s) ds \\ &\geq \|v\|_{H^1(\mathbb{R})}^2 - (c_5 + 1) \|v\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

where $c_5 = \max_{|v| \leq 1} F''(v)$. Then (2.12) follows with $\tilde{c}_0 = \frac{c_3}{c_3 + c_5 + 1}$. In fact if $\|v\|_{L^2} \leq \frac{1}{c_3 + c_0 + 1} \|v\|_{H^1}$ one can use (2.14) and one can use (2.13) else. The upper bound in (2.12) is immediate noting that $\sup_{u \in [-1, +1]} |F''(u)| < \infty$.

In order to obtain (2.8) one writes:

$$(2.15) \quad \mathcal{E}(u) = \frac{1}{2} \langle \mathcal{A}_\xi v, v \rangle + \int_{\mathbb{R}} U(s, \xi, v) ds,$$

where

$$U(s, \xi, v) = F(m_\xi(s) + v(s)) - F(m_\xi(s)) - F'(m_\xi(s))v(s) - \frac{1}{2}F''(m_\xi(s))v(s)^2.$$

Here equation (2.1) is used. Using the Sobolev embedding $\|v\|_{L^\infty(\mathbb{R})} \leq C\|v\|_{H^1(\mathbb{R})}$ one obtains by Taylor formula

$$(2.16) \quad \left| \int_{\mathbb{R}} U \right| \leq \frac{1}{6} \sup_{|v| \leq C\delta_1 + 1} |F'''(v)| \|v\|_{L^3(\mathbb{R})}^3 \leq C\|v\|_{L^\infty(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}^2 \leq C\|v\|_{H^1(\mathbb{R})}^3.$$

This implies the inequality (2.8).

(ii) To show the second statement, first note that there exists a $\tilde{\delta}_0 > 0$ such that if $\mathcal{E}(u) \leq \tilde{\delta}_0$ there exists a ξ such that

$$(2.17) \quad c_0 \|u - m_\xi\|_{H^1(\mathbb{R})}^2 \leq \mathcal{E}(u).$$

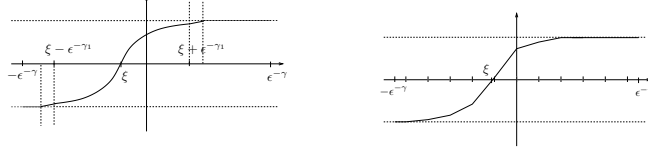


FIGURE 2. The approximated wavelike shapes m_ξ^ε and $m_\xi^{N,\varepsilon}$.

In fact choosing ξ as in Lemma 2.3 and noting that if one uses the case (i) of Lemma 2.1 instead (ii) one sees that inequalities (2.12) and (2.16) remain valid for $v = u - m_\xi$. Then by using the L^∞ bound on v from Lemma 2.3 instead of Sobolev embedding in the last step of (2.16) one obtains the above statement. In order to obtain (2.9) choose $\delta_0 = \sqrt{\frac{\tilde{\delta}_0}{c_0}}$ and assume $\text{dist}_{H^1}(u, M) \geq \delta$ for a $\delta \leq \delta_0$. If $\mathcal{E}(u) \geq \tilde{\delta}_0$ the bound (2.9) holds automatically. Otherwise (2.17) holds and gives the desired estimate. \square

We now pass to some bounds on approximated wave shapes. To this end fix $\gamma_1 < \gamma$. This parameter will be fixed throughout the paper. Denote by m^ε the profile m cut off outside of $[-\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_1}]$. More precisely assume that m^ε is a smooth monotone function that coincides with m on $[-\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_1}]$ and that verifies $m^\varepsilon(s) = \pm 1$ for $\pm s \geq \varepsilon^{-\gamma_1} + 1$. Furthermore, assume that on the intervals $[\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_1} + 1]$ (respectively $[-\varepsilon^{-\gamma_1} - 1, -\varepsilon^{-\gamma_1}]$) one has $m(s) \leq m^\varepsilon(s) \leq 1$ (resp. $m(s) \geq m^\varepsilon(s) \geq -1$). Due to (2.5) one can also assume that $|(m^\varepsilon)'(s)| \leq 2c_1c_2e^{-c_2\varepsilon^{-\gamma_1}}$ on both of these intermediate intervals. Then define $m_\xi^\varepsilon(s) = m^\varepsilon(s - \xi)$.

Furthermore, for $N \in \mathbb{N}$ and $k \in \{-N, -(N-1), \dots, (N-1), N\}$ set $s_k^{N,\varepsilon} = \frac{k\varepsilon^{-\gamma}}{N}$ and define

$$(2.18) \quad m_\xi^{N,\varepsilon}(s) = \begin{cases} m_\xi^\varepsilon(s) & \text{if } s = s_k^{N,\varepsilon} \text{ for } k = -(N-1), \dots, (N-1) \\ \text{the linear interpolation between these points.} & \end{cases}$$

One then gets the following bound:

Lemma 2.4. *For ε small enough and $\xi \in [-\varepsilon^{-\gamma} + \varepsilon^{-\gamma_1} + 1, \varepsilon^{-\gamma} - \varepsilon^{-\gamma_1} - 1]$ one has*

- (i) $\|m_\xi - m_\xi^\varepsilon\|_{L^2(\mathbb{R})} \leq C \exp(-c_2\varepsilon^{-\gamma_1})$
 $\|(m_\xi)' - (m_\xi^\varepsilon)'\|_{L^2(\mathbb{R})} \leq C \exp(-c_2\varepsilon^{-\gamma_1}).$
- (ii) $\|m_\xi - m_\xi^{N,\varepsilon}\|_{L^2(\mathbb{R})} \leq C\varepsilon^{-\gamma_1/2} \frac{\varepsilon^{-2\gamma}}{N^2}$
 $\|(m_\xi)' - (m_\xi^{N,\varepsilon})'\|_{L^2(\mathbb{R})} \leq C\varepsilon^{-\gamma_1/2} \frac{\varepsilon^{-\gamma}}{N}.$

Proof. To see (i) write

$$\begin{aligned} \|m_\xi - m_\xi^\varepsilon\|_{L^2(\mathbb{R})}^2 &\leq \int_{\varepsilon^{-\gamma_1}}^{\infty} (m(s) - m^\varepsilon(s))^2 ds + \int_{-\infty}^{-\varepsilon^{-\gamma_1}} (m(s) - m^\varepsilon(s))^2 ds \\ &\leq 2 \int_{\varepsilon^{-\gamma_1}}^{\infty} c_1^2 \exp(-2c_2s) ds \leq C \exp(-2c_2\varepsilon^{-\gamma_1}) \end{aligned}$$

and

$$\begin{aligned} \|m'_\xi - (m_\xi^\varepsilon)'\|_{L^2(\mathbb{R})}^2 &\leq \int_{\varepsilon^{-\gamma_1}}^{\infty} \left(m'(s) - (m^\varepsilon)'(s)\right)^2 ds + \int_{-\infty}^{\varepsilon^{-\gamma_1}} \left(m'(s) - (m^\varepsilon)'(s)\right)^2 ds \\ &\leq C \exp(-2c_2\varepsilon^{-\gamma_1}). \end{aligned}$$

Here one uses the inequalities (2.5) as well as the properties of m^ε .

To see (ii) write

$$(2.19) \quad \|m'_\xi - (m_\xi^{N,\varepsilon})'\|_{L^2(\mathbb{R})} \leq \|m'_\xi - (m_\xi^\varepsilon)'\|_{L^2(\mathbb{R})} + \|(m_\xi^\varepsilon)' - (m_\xi^{N,\varepsilon})'\|_{L^2(\mathbb{R})}.$$

To bound the second term assume without loss of generality that $\xi = 0$ and write

$$(2.20) \quad \begin{aligned} \|(m^\varepsilon)' - (m^{N,\varepsilon})'\|_{L^2(\mathbb{R})}^2 &= \sum_{k=-N}^{N-1} \int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} \left((m^\varepsilon)'(s) - (m^{N,\varepsilon})'(s)\right)^2 ds \\ &= \sum_{k=-N^\varepsilon}^{N^\varepsilon-1} \int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} \left((m^\varepsilon)'(s) - (m^{N,\varepsilon})'(s)\right)^2 ds. \end{aligned}$$

In the second equality $N^\varepsilon = \lceil \varepsilon^{-\gamma_1} \frac{N}{\varepsilon^{-\gamma}} \rceil$. Here we use the fact that u^ε is constant outside of $[-\varepsilon^{-\gamma_1}, \varepsilon^{-\gamma_1}]$ and therefore coincides with its piecewise linearization. The integrals can be bounded using the Poincaré inequality:

$$(2.21) \quad \int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} \left((m^\varepsilon)'(s) - (m_\xi^{N,\varepsilon})'(s)\right)^2 ds \leq \frac{\varepsilon^{-2\gamma}}{N^2\pi^2} \int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} (m^\varepsilon)''(s) ds \leq \frac{\varepsilon^{-3\gamma}}{N^3\pi^2} \sup_{s \in \mathbb{R}} |(m^\varepsilon)''(s)|^2.$$

Plugging this into (2.19) one gets:

$$\|(m^\varepsilon)' - (m^{N,\varepsilon})'\|_{L^2}^2 \leq \varepsilon^{-\gamma_1} \frac{\varepsilon^{-2\gamma}}{N^2\pi^2} \sup_{s \in \mathbb{R}} |(m^\varepsilon)''(s)|^2.$$

Due to (i) the term involving $|m'_\xi - (m_\xi^\varepsilon)'|$ can be absorbed in the constant for ε small enough. This yields the second estimate in (ii). For the bound on $\|m'_\xi - (m_\xi^\varepsilon)'\|_{L^2(\mathbb{R})}$ one proceeds in the same manner with another use of Poincaré inequality. The details are left to the reader. \square

3. GAUSSIAN ESTIMATES

In this section concentration properties of some discretized Gaussian measures are discussed and the bounds which are needed in Section 4 are provided. To this end we recall a classical Gaussian concentration inequality. Then we introduce the discretized version of the Gaussian reference measure ν^ε and give an error bound. We also study another discretized measure which can be viewed as a discretized massive Gaussian free field.

Let E be a separable Banach space equipped with its Borel- σ -field \mathcal{F} and norm $\|\cdot\|$. Recall that a probability measure μ on (E, \mathcal{F}) is called Gaussian if for every η in the dual space X^* the pushforward measure $\eta_\# \mu$ is Gaussian. For the moment all Gaussian measures are assumed to be centered i.e. for all $\eta \in X^*$ it holds $\int \eta(x) \mu(dx) = 0$. Denote by

$$\sigma = \sup_{\eta \in X^*, \|\eta\|_{X^*} \leq 1} \left(\int \eta(x)^2 \mu(dx) \right)^{1/2}.$$

Note that σ is finite [Led96]. Then one has the following classical concentration inequality (see [Led96] page 203):

$$\mu\left(y; \|y\| \geq \int \|x\| \mu(dx) + r\right) \leq e^{-r^2/2\sigma^2}.$$

There are several ways to prove this inequality. It can, for example, be derived as a consequence of the Gaussian isoperimetric inequality.

The difficulty in applying this inequality to specific examples is to evaluate the quantities σ and $\int \|x\| \mu(dx)$. This is easier in the case where E is a Hilbert space. Then a centered Gaussian measure μ is uniquely characterized by the covariance operator Σ which satisfies

$$\int \langle \eta_1, x \rangle \langle \eta_2, x \rangle \mu(dx) = \langle \eta_1, \Sigma \eta_2 \rangle \quad \forall \eta_1, \eta_2 \in E.$$

It is known [DPZ92] that Σ must be a nonnegative symmetric trace class operator. Then σ^2 is the spectral radius of Σ and using Jensen's inequality one obtains

$$\int \|x\| \mu(dx) \leq \left(\int \|x\|^2 \mu(dx) \right)^{1/2} = (\text{Tr } \Sigma)^{1/2}.$$

Therefore, one can write

Lemma 3.1. *Let μ be a centered Gaussian measure on a Hilbert space E with covariance operator Σ . Then one has*

$$(3.1) \quad \mu(x; \|x\| \geq (\text{Tr } \Sigma)^{1/2} + r) \leq e^{-r^2/2\sigma^2}.$$

We now want to use this inequality to study the behavior of the measure ν^ε under discretization. To this end fix an integer N and consider piecewise affine functions $u \in L^2[-\varepsilon^{-\gamma}, -\varepsilon^{-\gamma}]$ of the following type

$$(3.2) \quad u(x) = \begin{cases} \pm 1 & \text{for } x = \pm \varepsilon^{-\gamma} \\ \text{arbitrary} & \text{for } x = s_k^{N,\varepsilon} \quad k = -(N-1), \dots, (N-1) \\ \text{the linear interpolation between those points,} & \end{cases}$$

and denote by $H^{N,\varepsilon}$ the affine space of all such functions. Recall that $s_k^{N,\varepsilon} = \frac{k\varepsilon^{-\gamma}}{N}$. The space $H^{N,\varepsilon}$ can canonically be identified with \mathbb{R}^{2N-1} . In particular typical finite-dimensional objects such as Lebesgue- and codimension one Hausdorff measures make sense on $H^{N,\varepsilon}$. Denote these measures by $\mathcal{L}^{N,\varepsilon}$ and $\mathcal{H}^{N,\varepsilon}$. There are several bilinear forms on (the tangential space of) $H^{N,\varepsilon}$ which will be important in the sequel: The H^1 - and L^2 -scalar product correspond to the fact that $H^{N,\varepsilon}$ is a subset of $m + H^1$. But there is also the Euclidean scalar product $\langle u, v \rangle = \sum_{k=-(N-1)}^{(N-1)} u(s_k^{N,\varepsilon})v(s_k^{N,\varepsilon})$ which determines the behavior of the measures $\mathcal{L}^{N,\varepsilon}$ and $\mathcal{H}^{N,\varepsilon}$.

Recall that ν^ε is the distribution of a Gaussian process $(u(s), s \in [-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}])$ with $\mathbb{E}[u(s)] = \varepsilon^\gamma s$ and $\text{Cov}(u(s), u(s')) = \varepsilon \left(s \wedge s' + \varepsilon^{-\gamma} - \frac{(s+\varepsilon^{-\gamma})(s'+\varepsilon^{-\gamma})}{2\varepsilon^{-\gamma}} \right)$. According to the Kolmogorov-Chentsov Theorem we can assume that u has continuous paths. Consider now the piecewise linearization of u^N of u :

$$u^N(s) = \begin{cases} \pm 1 & \text{for } s = \pm \varepsilon^{-\gamma} \\ u(s) & \text{for } x = s_k^{N,\varepsilon} \quad k = -(N-1), \dots, (N-1) \\ \text{the linear interpolation between those points.} & \end{cases}$$

Lemma 3.2. (i) *The distribution of u^N is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^{N,\varepsilon}$ on $H^{N,\varepsilon}$. The density is given by*

$$(3.3) \quad \frac{1}{\sqrt{(\varepsilon 2\pi)^{2N-1}}} \left(\frac{N}{\varepsilon^{-\gamma}}\right)^N (2\varepsilon^{-\gamma})^{1/2} \exp(-\varepsilon^{\gamma-1}) \exp\left(-\frac{1}{2\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u'(s)|^2 ds\right).$$

(ii) *The random function $u - u^N$ consists of $2N$ independent rescaled Brownian bridges. To be more precise for each $k \in \{-N, \dots, (N-1)\}$ the process $(u(s) - u^N(s) : s \in [s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}])$ is a centered Gaussian process with covariance*

$$(3.4) \quad \text{Cov}(u(s) - u^N(s), u(s') - u^N(s')) = \varepsilon \left(s \wedge s' - s_k^{N,\varepsilon} - \frac{N}{\varepsilon^{-\gamma}} (s - s_k^{N,\varepsilon})(s' - s_k^{N,\varepsilon}) \right).$$

These processes are mutually independent and independent of u^N .

Proof. (i) The measure ν^ε can be considered as the distribution of a rescaled Brownian motion u on $[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$ starting at $u(-\varepsilon^{-\gamma}) = -1$ and conditioned on $u(\varepsilon^{-\gamma}) = 1$. Therefore, the finite-dimensional distributions can be obtained by finite-dimensional conditioning:

$$\begin{aligned} & \nu^\varepsilon \left(u(s_{-(N-1)}^{N,\varepsilon}) \in dx_{-(N-1)}, \dots, u(s_{N-1}^{N,\varepsilon}) \in dx_{(N-1)} \right) \\ &= \left(\prod_{i=-N}^{(N-1)} \frac{1}{\sqrt{(\varepsilon 2\pi)\delta}} \exp\left(-\frac{(x_{i+1} - x_i)^2}{2\varepsilon\delta}\right) \right) \left(\frac{1}{\sqrt{(\varepsilon 2\pi)2\varepsilon^{-\gamma}}} \exp\left(-\frac{(1 - (-1))^2}{4\varepsilon^{-\gamma}\varepsilon}\right) \right)^{-1} \\ &= \frac{1}{\sqrt{(\varepsilon 2\pi)^{2N-1}}} \delta^{-N} \sqrt{2\varepsilon^{-\gamma}} \exp(\varepsilon^{\gamma-1}) \exp\left(-\frac{1}{2\varepsilon} \sum_{i=-N}^{N-1} \delta \frac{(x_{i+1} - x_i)^2}{\delta^2}\right) \end{aligned}$$

Here $\delta = \frac{\varepsilon^{-\gamma}}{N}$ and $x_{\pm N} = \pm 1$. By noting that the Riemann sum appearing in the last line is equal to the integral of the squared derivative of the piecewise linearization one obtains the result.

(ii) Denote for $k = -N, \dots, (N-1)$ and $s \in [0, \delta]$ by $\tilde{u}_k(s) = u(s_k^{N,\varepsilon} + s) - u^N(s_k^{N,\varepsilon} + s) = u(s_k^{N,\varepsilon} + s) - \left(1 - \frac{s}{\delta}\right)u(s_k^{N,\varepsilon}) - \frac{s}{\delta}u(s_{k+1}^{N,\varepsilon})$. We want to show that the processes $(\tilde{u}_k(s), s \in [0, \delta])$ possess the right covariances and are mutually independent and independent of u^N . To this end calculate for $s, s' \in [0, \delta]$ and $i = -N, \dots, (N-1)$:

$$\begin{aligned} \text{Cov}(\tilde{u}_k(s), \tilde{u}_k(s')) &= \text{Cov} \left[u(s_k^{N,\varepsilon} + s) - \left(1 - \frac{s}{\delta}\right)u(s_k^{N,\varepsilon}) - \frac{s}{\delta}u(s_{k+1}^{N,\varepsilon}), \right. \\ & \quad \left. u(s_k^{N,\varepsilon} + s') - \left(1 - \frac{s'}{\delta}\right)u(s_k^{N,\varepsilon}) - \frac{s'}{\delta}u(s_{k+1}^{N,\varepsilon}) \right]. \end{aligned}$$

By plugging in the explicit expression for the covariances of the $u(s)$ and some tedious but elementary calculations one obtains the desired expression. In a similar way one can see that for $i \neq j$ one has

$$\text{Cov}(\tilde{u}_j(s), \tilde{u}_i(s')) = 0 \quad \text{and} \quad \text{Cov}(\tilde{u}_j(s), u^N(t)) = 0$$

for all $s, s' \in [0, \delta]$ and $t \in [-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$. \square

Denote the Gaussian normalization constant by

$$Z_1^{N,\varepsilon} := \left(\frac{1}{\sqrt{(2\pi\varepsilon)^{2N-1}}} \left(\frac{N}{\varepsilon^{-\gamma}}\right)^N \sqrt{2\varepsilon^{-\gamma}} \exp(-\varepsilon^{\gamma-1}) \right)^{-1}.$$

We define the discrete Dirichlet-Laplace operator $\Delta_{N,\varepsilon}$ as

$$(3.5) \quad \Delta_{N,\varepsilon}^{k,j} = \frac{N}{\varepsilon^{-\gamma}} \begin{cases} -2 & \text{for } k = j \\ 1 & \text{for } |k - j| = 1 \\ 0 & \text{else.} \end{cases}$$

Then a direct computation shows

$$(3.6) \quad Z_1^{N,\varepsilon} = \sqrt{(2\pi\varepsilon)^{2N-1}} \exp(\varepsilon^{\gamma-1}) (\det(-\Delta_{N,\varepsilon}))^{-1/2}.$$

We now want to apply the Gaussian concentration inequality to obtain a bound on the probability of large $u - u^N$:

Lemma 3.3. *The following bounds hold:*

(1) *L^2 -bound on the whole line:*

$$(3.7) \quad \nu^\varepsilon \left(\|u - u^N\|_{L^2[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]} \geq \sqrt{\varepsilon \frac{\varepsilon^{-2\gamma}}{3N}} + r \right) \leq \exp \left(-\frac{r^2 \pi^2 N^2}{\varepsilon^{1-2\gamma}} \right)$$

(2) *L^2 -bound on the short intervals:*

$$(3.8) \quad \nu^\varepsilon \left(\|u - u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]} \geq \sqrt{\varepsilon \frac{\varepsilon^{-2\gamma}}{6N^2}} + r \right) \leq \exp \left(-\frac{r^2 \pi^2 N^2}{\varepsilon^{1-2\gamma}} \right).$$

(3) *L^∞ -bound on the whole line:*

$$(3.9) \quad \nu^\varepsilon \left(\|u - u^N\|_{L^\infty[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]} \geq r \right) \leq 4N \exp \left(-\frac{r^2 N}{8\varepsilon^{1-\gamma}} \right).$$

Proof. Let us consider (3.7) first. Note that $u - u^N$ is a centered Gaussian process such that Lemma 3.1 can be applied. The expected L^2 -norm can be calculated as follows:

$$\begin{aligned} \nu^\varepsilon \left[\|u - u^N\|_{L^2[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]}^2 \right] &= \sum_{k=-N}^{N-1} \nu^\varepsilon \|\tilde{u}_k\|_{L^2}^2 = \sum_{k=-N}^{N-1} \int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} \nu^\varepsilon (\tilde{u}(s)^2) ds \\ &= \sum_{k=-N}^{N-1} \int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} \varepsilon \left(s - s_k^{N,\varepsilon} - \frac{(s - s_k^{N,\varepsilon})^2}{\frac{\varepsilon^{-\gamma}}{N}} \right) ds = 2N\varepsilon \frac{1}{6} \left(\frac{\varepsilon^{-\gamma}}{N} \right)^2. \end{aligned}$$

Here for the third equality equation (3.4) is used.

To get information about the spectral radius of the covariance operator Σ calculate for $f, g \in L^2[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$:

$$\begin{aligned} \langle f, \Sigma g \rangle &= \nu^\varepsilon [\langle f, u - u^N \rangle \langle g, u - u^N \rangle] \\ &= \sum_{k=-N}^{N-1} \int_{s_k^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} \varepsilon \left(s \wedge s' - \frac{(s - s_k^{N,\varepsilon})(s' - s_k^{N,\varepsilon})}{\frac{\varepsilon^{-\gamma}}{N}} \right) f(s)g(s') ds. \end{aligned}$$

Here in the last step the independence of the different bridges is used as well as formula (3.3). Note that the integral kernel in the last line is the Green function of the negative Dirichlet-Laplace operator on the interval $[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]$. Denoting this operator by $\varepsilon(-\Delta_{T_k})^{-1}$ one can write

$$\langle f, \Sigma g \rangle = \sum_{k=-N}^{N-1} \langle f, \varepsilon(-\Delta_{T_k})^{-1} g \rangle_{L^2(T_k)}.$$

The spectral decomposition of the inverse Dirichlet-Laplace operator on intervals of length T is well known. In fact on $L^2[0, T]$ the smallest eigenvalue λ_0 and the corresponding eigenfunction $e_0(x)$ are given as:

$$e_0(s) = \sin\left(\frac{\pi s}{T}\right) \quad \text{and} \quad \lambda_0 = \frac{\varepsilon T^2}{\pi^2}.$$

The spectral radius of $\varepsilon(-\Delta_{T_k})^{-1}$ is thus given as

$$\sigma_k^2 = \varepsilon \frac{\varepsilon^{-2\gamma}}{(N\pi)^2}.$$

Therefore, one can write

$$\begin{aligned} \sigma^2 &= \sup_{f, \|f\|=1} \langle f, \Sigma f \rangle = \sup_{f, \|f\|=1} \sum_{k=-N}^{N-1} \langle f, \varepsilon(\Delta_{T_k})^{-1} g \rangle_{L^2(T_k)} \\ &\leq \sup_{f, \|f\|=1} \sum_{k=-N}^{N-1} \sigma_k^2 \langle f, f \rangle_{L^2(T_k)} = \varepsilon \left(\frac{\varepsilon^{-\gamma}}{\pi N}\right)^2 \sup_{f, \|f\|=1} \langle f, f \rangle. \end{aligned}$$

On the other hand by taking f as a linear combination of the eigenfunctions on the shorter intervals one obtains

$$\sigma^2 = \varepsilon \left(\frac{\varepsilon^{-\gamma}}{\pi N}\right)^2.$$

Thus equation (3.1) gives the desired result. The proof of (3.8) proceeds in the same manner.

To prove the third statement (3.9) note that by Lemma 3.2, the deviations of a the random function u from the piecewise linearizations u^N between the points $s_k^{N, \varepsilon}$ are independent Brownian bridges. Therefore, such a process $\left(u(s_k^{N, \varepsilon} + s) - u^N(s_k^{N, \varepsilon} + s), 0 \leq s \leq \frac{\varepsilon^{-\gamma}}{N}\right)$ has the same distribution as $\varepsilon^{\frac{1}{2}} \left(B_s - \frac{sN}{\varepsilon^{-\gamma}} B_{\frac{\varepsilon^{-\gamma}}{N}}\right)$ for a Brownian motion B defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. So one can write

$$\begin{aligned} &\nu^\varepsilon \left(\|u(s) - u^N(s)\|_{L^\infty[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]} \geq r \right) \\ &\leq \sum_{k=-N}^{N-1} \nu^\varepsilon \left(\max_{s_k^{N, \varepsilon} \leq s \leq s_{k+1}^{N, \varepsilon}} |u(s) - u^N(s)| \geq r \right) \\ &\leq 2N \mathbb{P} \left(\max_{0 \leq s \leq \frac{\varepsilon^{-\gamma}}{N}} \left| \varepsilon^{1/2} \left(B_s - \frac{sN}{\varepsilon^{-\gamma}} B_{\frac{\varepsilon^{-\gamma}}{N}} \right) \right| \geq r \right) \\ &\leq 2N \mathbb{P} \left(\max_{0 \leq s \leq \frac{\varepsilon^{-\gamma}}{N}} |B_s| \geq \frac{r}{2\varepsilon^{1/2}} \right). \end{aligned}$$

Using the exponential version of the maximal inequality for martingales (see Proposition 1.8 in Chapter II in [RY99]) one can see that

$$\nu^\varepsilon \left(\|u(s) - u^N(s)\|_{L^\infty[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]} \geq r \right) \leq 4N \exp \left(-\frac{r^2 N}{8\varepsilon^{1-\gamma}} \right).$$

□

We now want to study the properties of another discrete Gaussian measure. In fact denote by $H_0^{N,\varepsilon}$ the space of affine functions defined as in 3.2 with the only change that they are assumed to possess zero boundary conditions. The Lebesgue measure on this space is defined in the same manner. For a fixed constant κ consider the centered probability measure ϱ whose density with respect to $\mathcal{L}^{N,\varepsilon}$ is proportional to

$$\exp\left(-\kappa \frac{\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s)|^2 + |\nabla u(s)|^2 ds}{2\varepsilon}\right).$$

In fact this measure is a version of what is known in the literature as discrete massive free field, discrete Ornstein-Uhlenbeck bridge or pinned $\nabla\phi$ surface model [She07, HSVW05]. The H^1 -norm in the exponent can be rewritten in terms of the finite-dimensional coordinates. In fact for $u \in H_0^{N,\varepsilon}$ with $u(s_k^{N,\varepsilon}) = u_k$ for $k = -N, \dots, N$ one has

$$\begin{aligned} \|u\|_{H^1(\mathbb{R})}^2 &= \|u\|_{L^2(\mathbb{R})}^2 + \|u'\|_{L^2(\mathbb{R})}^2 \\ (3.10) \quad &= \frac{1}{3} \frac{\varepsilon^{-\gamma}}{N} \sum_{k=-N}^N u_k^2 + u_{k+1}^2 + u_k u_{k+1} + \frac{N}{\varepsilon^{-\gamma}} \sum_{k=-N}^N u_k^2 + u_{k+1}^2 - 2u_k u_{k+1} \\ &= \sum_{k,j=-(N-1)}^{N-1} u_k \left(I_{N,\varepsilon}^{k,j} - \Delta_{N,\varepsilon}^{k,j} \right) u_j, \end{aligned}$$

where the $(2N-1) \times (2N-1)$ matrix $(I_{N,\varepsilon}^{k,j})$ is given as

$$(3.11) \quad I_{N,\varepsilon}^{k,j} = \frac{1}{3} \frac{\varepsilon^{-\gamma}}{N} \begin{cases} 2 & \text{for } k = j \\ \frac{1}{2} & \text{for } |k - j| = 1 \\ 0 & \text{else,} \end{cases}$$

and the discrete Laplace operator $\Delta_{N,\varepsilon}$ is defined as in (3.5). Denote the normalization constant

$$Z_2^{N,\varepsilon} = \int \exp\left(-\kappa \frac{\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s)|^2 + |\nabla u(s)|^2 ds}{2\varepsilon}\right) \mathcal{L}^{N,\varepsilon}(du).$$

Lemma 3.4. (i) $Z_2^{N,\varepsilon}$ is given as

$$(3.12) \quad \sqrt{\left(\frac{2\varepsilon\pi}{\kappa}\right)^{2N-1}} \det(-\Delta_{N,\varepsilon} + I_{N,\varepsilon})^{-\frac{1}{2}}.$$

(ii) In the sense of symmetric matrices we have the following Poincaré inequality

$$(3.13) \quad I_{N,\varepsilon} \leq C\varepsilon^{-\gamma}(-\Delta_{N,\varepsilon}).$$

(iii) We have the following bound: For $r \geq 0$

$$(3.14) \quad \varrho^{N,\varepsilon} \left\{ u: \|u\|_{H^1} \geq \sqrt{\frac{(2N-1)\varepsilon}{\kappa}} + r \right\} \leq \exp(-\kappa r^2/2\varepsilon).$$

Proof. (i) To see this one only has to note that $\kappa\varepsilon^{-1}(-\Delta_{N,\varepsilon} + I_{N,\varepsilon})$ is the inverse covariance matrix of this finite-dimensional Gaussian measure.

(ii) Using the usual Poincaré inequality one can write for $u \in H_0^{N,\varepsilon}$

$$\begin{aligned} - \sum_{k,j=-(N-1)}^{N-1} u(s_k^{N,\varepsilon}) \Delta_{N,\varepsilon}^{k,j} u(s_j^{N,\varepsilon}) &= \|u'\|_{L^2[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]}^2 \\ &\geq C\varepsilon^\gamma \|u\|_{L^2[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]}^2 = C\varepsilon^\gamma \sum_{k,j=-(N-1)}^{N-1} u(s_k^{N,\varepsilon}) I_{N,\varepsilon}^{k,j} u(s_j^{N,\varepsilon}). \end{aligned}$$

(iii) To see (3.14) write with a finite-dimensional change of variables:

$$\begin{aligned} \varrho^{N,\varepsilon} \{u: \|u\|_{H^1} \geq r\} &= \frac{1}{Z_2^{N,\varepsilon}} \int_{\{u: \|u\|_{H^1} \geq r\}} \exp\left(-\kappa \frac{\|u\|_{H^1}^2}{2\varepsilon}\right) \mathcal{L}^{N,\varepsilon}(du) \\ &= \sqrt{\left(\frac{\kappa}{2\varepsilon\pi}\right)^{2N-1}} \int_{\{\sum_{k=-(N-1)}^{N-1} x_k^2 \geq r\}} \exp\left(-\kappa \frac{\sum_{k=-(N-1)}^{N-1} x_k^2}{2\varepsilon}\right) dx_{-(N-1)} \dots dx_{N-1}. \end{aligned}$$

In fact here one uses the linear transformation that transforms a Gaussian random variable on a finite-dimensional space to a Gaussian random variable with Id covariance matrix. Therefore, the problem reduces to considering a vector of $2N - 1$ independent centered Gaussian random variables X_k with variance $\frac{\varepsilon}{\kappa}$. The expectation

$$\mathbb{E} \left[\sum_{k=-(N-1)}^{N-1} X_k^2 \right] = \frac{(2N-1)\varepsilon}{\kappa}$$

and the spectral radius

$$\sigma^2 = \frac{\varepsilon}{\kappa}$$

are calculated easily such that (3.1) gives the desired result. \square

4. CONCENTRATION AROUND A CURVE IN INFINITE-DIMENSIONAL SPACE

In this section we give the proof of Theorem 1.1. To this end we consider the finite-dimensional measure

$$\mu^{N,\varepsilon}(du) = \frac{1}{Z^{N,\varepsilon}} \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^{N,\varepsilon}(du),$$

with the normalization constant $Z^{N,\varepsilon} = \int \exp\left(-\frac{1}{\varepsilon} \int F(u(s)) ds\right) \nu^{N,\varepsilon}(du)$. Note that although $\nu^{N,\varepsilon}$ is given by the finite-dimensional marginals of ν^ε , the measure $\mu^{N,\varepsilon}$ does not coincide with the finite-dimensional distribution of μ^ε . The strategy is now as follows: In Proposition 4.4 a lower bound on the discrete normalization constant $Z^{N,\varepsilon}$ is given. This is achieved by calculating the integral in a tubular neighborhood of the set of minimizers M . Then in Proposition 4.8 the rough energy bound given in Proposition 2.2 is used to conclude concentration of the discretized measure $\mu^{N,\varepsilon}$ around the curve of minimizers. Finally Lemma 4.13 gives a bound on the discretization error which allows to finish the proof of concentration around the curve of minimizers in the continuous case.

Recall the following version of the coarea formula:

Lemma 4.1. *Let f be a Lipschitz function $f : A \subseteq E \rightarrow I \subseteq \mathbb{R}$, where E is a n -dimensional Euclidean space and A is an open subset and I some interval. Denote by $\mathcal{L}^n, \mathcal{L}^1$ and \mathcal{H}^{n-1} the Lebesgue measure on E , on \mathbb{R} and the $(n-1)$ -dimensional Hausdorff measure on E respectively. Suppose that the gradient (which exists \mathcal{L}^n -a.e.) Df does not vanish \mathcal{L}^n a.e. in A . Then for every nonnegative measurable test function $\varphi : A \rightarrow \mathbb{R}$ one has the following formula:*

$$(4.1) \quad \int_A \varphi(x) \mathcal{L}^n(dx) = \int_I \mathcal{L}^1(d\xi) \int_{f^{-1}(\xi)} \mathcal{H}^{n-1}(dx) \frac{1}{|Df(x)|_E} \varphi(x).$$

In order to apply formula (4.1) to $\mu^{N,\varepsilon}$ one needs the following:

Lemma 4.2. *Consider the function $f : A \rightarrow \mathbb{R}$, where $A := \{x \in m + L^2 : \text{dist}_{L^2}(x, M) < \beta\}$ is the open set in which the Fermi coordinates are defined, given by*

$$f(x) = f(m_\xi + s) = \xi,$$

where $x = m_\xi + s$ are the Fermi coordinates of x . Then f is Fréchet differentiable and one has

$$(4.2) \quad Df(x)[h] = Df(m_\xi + s)[h] = \frac{\langle m'_\xi, h \rangle}{|m'_\xi|^2 - \langle s, m''_\xi \rangle}.$$

Proof. The differentiability follows from the implicit function theorem. To calculate the derivative at $x = m_\xi + s$ in direction h consider the function

$$\Phi(v, w) = \langle m_\xi - m_w + s + vh, m'_w \rangle,$$

defined in an environment of $(0, \xi) \in \mathbb{R}^2$. Noting that one has $\Phi(v, f(m_\xi + s + vh)) = 0$ one can write

$$0 = \partial_v \Phi(v, f(m_\xi + s + vh))|_{v=0} + \partial_w \Phi(v, f(m_\xi + s + vh))|_{v=0} Df(m_\xi + s)[h].$$

Observing that

$$\partial_v \Phi(v, f(m_\xi + s + vh))|_{v=0} = \langle h, m'_\xi \rangle$$

and

$$\partial_w \Phi(v, f(m_\xi + s + vh))|_{v=0} = -\langle m'_\xi, m'_\xi \rangle + \langle s, m''_\xi \rangle$$

concludes the proof. \square

We want to apply the coarea formula to the function f just defined, restricted to $H^{N,\varepsilon}$. There is a slight inconvenience which originates from the fact that the norm of the gradient which appears in 4.1 is the norm in the finite-dimensional space E whereas the gradient of the function f is a function in $L^2(\mathbb{R})$. To resolve this is the content of the next lemma:

Lemma 4.3. *Let $g : m + L^2(\mathbb{R}) \rightarrow \mathbb{R}$ be a Fréchet differentiable function and denote by $\nabla g(x)$ its L^2 -gradient at point x . Consider then the function \tilde{g} defined on \mathbb{R}^{2N-1} obtained by composition of the embedding $\mathbb{R}^{2N-1} \rightarrow H^{N,\varepsilon}$ and g . Denote by $\tilde{\nabla} \tilde{g}$ its gradient. Then one has the following inequality:*

$$\|\tilde{\nabla} \tilde{g}\|_{\mathbb{R}^{2N-1}} \leq 2 \sqrt{\frac{\varepsilon^{-\gamma}}{N}} \|\nabla g\|_{L^2}.$$

Proof. We calculate the derivative of \tilde{g} in direction $\tilde{e}_k = (0, \dots, 0, 1, 0, \dots, 0)$ with the 1 on k -th position. Embedding \tilde{e}_k into $H^{N,\varepsilon}$ gives the hat function

$$(4.3) \quad e_k(s) = \begin{cases} 0 & \text{for } s \notin [s_{k-1}^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}] \\ \frac{N}{\varepsilon^{-\gamma}} \left(s - s_{k-1}^{N,\varepsilon} \right) & \text{for } s \in]s_{k-1}^{N,\varepsilon}, s_k^{N,\varepsilon}[\\ \frac{N}{\varepsilon^{-\gamma}} \left(s_{k+1}^{N,\varepsilon} - s \right) & \text{for } s \in]s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]. \end{cases}$$

Therefore, one obtains

$$(\tilde{\nabla} \tilde{g})_k = \int_{\mathbb{R}} e_k(s) \nabla g(s) ds = \int_{s_{k-1}^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} e_k(s) \nabla g(s) ds.$$

Applying Cauchy-Schwarz inequality and using $\|e_k\|_\infty \leq 1$ one gets:

$$(4.4) \quad \begin{aligned} \|\tilde{\nabla} \tilde{g}\|_{\mathbb{R}^{2N-1}}^2 &= \sum_{k=-(N-1)}^{N-1} \left(\int_{s_{k-1}^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} e_k(s) \nabla g(s) ds \right)^2 \leq 2 \frac{\varepsilon^{-\gamma}}{N} \sum_{k=-(N-1)}^{N-1} \int_{s_{k-1}^{N,\varepsilon}}^{s_{k+1}^{N,\varepsilon}} (\nabla g(s))^2 ds \\ &\leq 2 \frac{\varepsilon^{-\gamma}}{N} 2 \|\nabla g\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

□

Now we are ready to derive a lower bound on the normalization constant $Z^{N,\varepsilon}$ of the finite-dimensional approximation of μ^ε . Recall that $\mu^{N,\varepsilon}(du) = \frac{1}{Z^{N,\varepsilon}} \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^{N,\varepsilon}(du)$ where $\nu^{N,\varepsilon}$ is a discretized Brownian bridge. One gets the following bound:

Proposition 4.4. *Assume $N = N(\varepsilon)$ grows like $\varepsilon^{-\gamma_2}$ for ε decreasing to 0. Assume that*

$$(4.5) \quad 0 < \gamma_1 < \gamma < \gamma_2 < 1.$$

Then the following bound holds for ε small enough:

$$(4.6) \quad \begin{aligned} Z^{N,\varepsilon} &\geq \exp\left(-\frac{C_*}{\varepsilon}\right) \exp\left(-2C \left(\frac{\varepsilon^{-2\gamma-\gamma_1}}{\varepsilon N^2}\right)\right) \sqrt{\frac{N}{\varepsilon^{-\gamma}}} (\varepsilon^{-\gamma} - \varepsilon^{-\gamma_1} - 1) \\ &\quad \sqrt{\varepsilon} \sqrt{\frac{\varepsilon^{-1-\gamma}}{N}} \exp\left(-C \frac{\varepsilon^{-4\gamma-1-\gamma_1}}{N^4}\right) \exp(-\varepsilon^{\gamma-1}) c_4^{-\frac{2N-1}{2}} (1 + C\varepsilon^{-\gamma})^{-\frac{2N-1}{2}}. \end{aligned}$$

In particular, if one assumes that

$$(4.7) \quad -2\gamma - \gamma_1 + 2\gamma_2 > 0$$

one obtains

$$(4.8) \quad \liminf_{\varepsilon \downarrow 0} \varepsilon \log Z^{N,\varepsilon} \geq -C_*.$$

Proof. Using the definition of $\nu^{N,\varepsilon}$ one can write

$$\begin{aligned}
Z^{N,\varepsilon} &= \int_{H^{N,\varepsilon}} \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^{N,\varepsilon}(du) \\
&= \frac{1}{Z_1^{N,\varepsilon}} \exp\left(-\frac{C_*}{\varepsilon}\right) \int_{H^{N,\varepsilon}} \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right. \\
&\quad \left.- \frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \frac{1}{2} |u'(s)|^2 ds + \frac{C_*}{\varepsilon}\right) \mathcal{L}^{N,\varepsilon}(du) \\
&= \frac{1}{Z_1^{N,\varepsilon}} \exp\left(-\frac{C_*}{\varepsilon}\right) \int_{H^{N,\varepsilon}} \exp\left(-\frac{1}{\varepsilon} \mathcal{E}(u)\right) \mathcal{L}^{N,\varepsilon}(du).
\end{aligned} \tag{4.9}$$

Recall that $Z_1^{N,\varepsilon} = \int \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \frac{1}{2} |u'(s)|^2 ds\right) \mathcal{L}^{N,\varepsilon}(du)$ is the normalization constant of the discretized Brownian bridge and $\mathcal{L}^{N,\varepsilon}$ is the Lebesgue measure on the finite-dimensional space $H^{N,\varepsilon}$. In order to find a lower bound on $Z^{N,\varepsilon}$ we can restrict the integration to a tubular neighborhood of M . More precisely set $I_\varepsilon := [-\varepsilon^{-\gamma} + \varepsilon^{-\gamma_1}, \varepsilon^{-\gamma} - \varepsilon^{-\gamma_1}]$ and

$$A := \left\{ u \in H^{N,\varepsilon} : u = m_\xi + v : \langle v, m'_\xi \rangle_{L^2(\mathbb{R})} = 0 \text{ for some } \xi \in I_\varepsilon \text{ and } \|v\|_{H^1(\mathbb{R})} \leq \delta \right\},$$

for some δ to be determined later. For the moment we will only assume δ to be small enough to be able to apply Funaki's estimate (2.8) on the energy landscape. Furthermore, denote by

$$A_\xi := \left\{ u \in H^{N,\varepsilon} : u = m_\xi + v : \langle v, m'_\xi \rangle = 0 \text{ and } \|v\|_{H^1(\mathbb{R})} \leq \delta \right\}.$$

Using Funaki's estimate (2.8) for $u = m_\xi + v \in A$ one can write

$$\exp\left(-\frac{1}{\varepsilon} \mathcal{E}(u)\right) \geq \exp\left(-\frac{c_4}{\varepsilon} \|v\|_{H^1(\mathbb{R})}^2\right).$$

Note the v is not an element of the discretized space $H^{N,\varepsilon}$ but a general function in $L^2(\mathbb{R})$ that needs not vanish outside of $[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]$. But v can be well approximated by a function $v^{N,\varepsilon} = u - m_\xi^{N,\varepsilon} \in H_0^{N,\varepsilon}$. In fact using Lemma 2.4 one gets

$$\|v^{N,\varepsilon} - v\|_{H^1(\mathbb{R})} = \|m_\xi^{N,\varepsilon} - m_\xi\|_{H^1(\mathbb{R})} \leq C \frac{\varepsilon^{-\gamma}}{N} \varepsilon^{-\frac{\gamma_1}{2}}.$$

Putting this together one gets:

$$\begin{aligned}
Z^{N,\varepsilon} Z_1^{N,\varepsilon} \exp\left(\frac{C_*}{\varepsilon}\right) &\geq \int_A \exp\left(-\frac{c_4}{\varepsilon} \|v\|_{H^1(\mathbb{R})}^2\right) \mathcal{L}^{N,\varepsilon}(du) \\
&\geq \exp\left(-2C \left(\frac{\varepsilon^{-2\gamma-\gamma_1}}{\varepsilon N^2}\right)\right) \int_A \exp\left(-\frac{2c_4}{\varepsilon} \|v^{N,\varepsilon}\|_{H^1(\mathbb{R})}^2\right) \mathcal{L}^{N,\varepsilon}(du).
\end{aligned} \tag{4.10}$$

Let us concentrate on the integral term in equation (4.10). Using the coarea formula (4.1) one gets:

$$\int_A \exp\left(-\frac{2c_4}{\varepsilon} \|v^{N,\varepsilon}\|_{H^1}^2\right) \mathcal{L}^{N,\varepsilon}(du) = \int_{I_\varepsilon} d\xi \int_{A_\xi} \frac{1}{|\tilde{\nabla} \tilde{f}|} \exp\left(-\frac{2c_4}{\varepsilon} \|v^{N,\varepsilon}\|_{H^1}^2\right) \mathcal{H}^{N,\varepsilon}(du). \tag{4.11}$$

where $\mathcal{H}^{N,\varepsilon}$ is the codimension one Hausdorff measure on $H^{N,\varepsilon}$. Using Lemma 4.2 and the observation from Lemma 4.3 one knows:

$$\frac{1}{|\tilde{\nabla} \tilde{f}|} \geq \frac{1}{2} \sqrt{\frac{N}{\varepsilon^{-\gamma}}} \frac{\|m'_\xi\|_{L^2(\mathbb{R})}^2 - \langle v, m''_\xi \rangle_{L^2(\mathbb{R})}}{\|m'_\xi\|_{L^2(\mathbb{R})}}.$$

By choosing a smaller δ if necessary this can be bounded uniformly from below on A by $C\sqrt{\frac{N}{\varepsilon^{-\gamma}}}$ such that one gets:

$$(4.12) \quad \begin{aligned} & \int_A \exp\left(-\frac{2c_4}{\varepsilon} \|v^{N,\varepsilon}\|_{H^1(\mathbb{R})}^2\right) \mathcal{L}^{N,\varepsilon}(du) \\ & \geq C\sqrt{\frac{N}{\varepsilon^{-\gamma}}} \int_{I_\varepsilon} d\xi \int_{A_\xi} \exp\left(-\frac{2c_4}{\varepsilon} \|v^{N,\varepsilon}\|_{H^1(\mathbb{R})}^2\right) \mathcal{H}^{N,\varepsilon}(du). \end{aligned}$$

Let us focus on the last integral. By a linear change of coordinates one can write

$$(4.13) \quad \int_{A_\xi} \exp\left(-\frac{2c_4}{\varepsilon} \|v^{N,\varepsilon}\|_{H^1(\mathbb{R})}^2\right) \mathcal{H}^{N,\varepsilon}(du) = \int_{B_\xi} \exp\left(-\frac{2c_4}{\varepsilon} \|v\|_{H^1(\mathbb{R})}^2\right) \mathcal{H}^{N,\varepsilon}(dv),$$

where $B_\xi = \left\{v \in H_0^{N,\varepsilon} : \langle v, m'_\xi \rangle_{L^2(\mathbb{R})} = \langle m_\xi - m_\xi^{N,\varepsilon}, m'_\xi \rangle_{L^2(\mathbb{R})} \text{ and } \|v\|_{H^1(\mathbb{R})} \leq \delta\right\}$. In order to conclude, we need the following lemma:

Lemma 4.5. *Let E be a finite-dimensional Euclidean space with Lebesgue measure \mathcal{L} and codimension 1 Hausdorff measure \mathcal{H} . Let $a^* = \langle a, \cdot \rangle \in E^*$ be a linear form and $x \mapsto \langle x, \Sigma x \rangle$ be a symmetric, positive bilinear form. Furthermore, write for $b \in \mathbb{R}$ and $\delta > 0$*

$$\tilde{B}^{b,\delta^2} = \{x \in E : a^* x = b \text{ and } \langle x, \Sigma x \rangle \leq \delta^2\}.$$

Furthermore, set $d^2 = \inf_{x \in \tilde{B}^{b,\infty}} \langle x, \Sigma x \rangle$ and let n be a Σ -unit normal vector on $\tilde{B}^{0,\infty}$, i.e. $\langle n, \Sigma n \rangle = 1$ for all $x \in \tilde{B}^{0,\infty}$ and $\langle n, \Sigma x \rangle = 0$. Then one has for every b

$$(4.14) \quad \int_{\langle x, \Sigma x \rangle \leq \delta^2 - d^2} \exp(-\langle x, \Sigma x \rangle) \mathcal{L}(dx) \leq 2\delta \sqrt{\frac{1}{\langle \Sigma n, \Sigma n \rangle}} \exp(d^2) \int_{\tilde{B}^{b,\delta^2}} \exp(-\langle x, \Sigma x \rangle) \mathcal{H}(dx).$$

Furthermore, one has the following expressions for d^2 :

$$(4.15) \quad d^2 = \frac{b^2}{\langle a, \Sigma^{-1} a \rangle}, \quad \text{and} \quad \langle a, \Sigma^{-1} a \rangle = \left(\sup_{\eta: \langle \eta, \Sigma \eta \rangle = 1} a^*(\eta) \right)^2.$$

The vector n satisfies $n = \pm \frac{\Sigma^{-1} a}{\sqrt{\langle a, \Sigma^{-1} a \rangle}}$ such that

$$(4.16) \quad \langle \Sigma n, \Sigma n \rangle = \frac{1}{\langle a, \Sigma^{-1} a \rangle} \left(\sup_{\eta: \langle \eta, \Sigma \eta \rangle = 1} a^*(\eta) \right)^2.$$

Proof. (Of Lemma 4.5): Using the coarea formula one can write:

$$\begin{aligned}
(4.17) \quad & \int_{\langle x, \Sigma x \rangle \leq \delta^2 - d^2} \exp(-\langle x, \Sigma x \rangle) \mathcal{L}(dx) \\
& \leq \int_{-\delta}^{\delta} \int_{\tilde{B}^{0, \delta^2 - d^2}} \exp(-\langle y + \lambda n, \Sigma(y + \lambda n) \rangle) \sqrt{\frac{1}{\langle \Sigma n, \Sigma n \rangle}} \mathcal{H}(dy) d\lambda \\
& \leq \sqrt{\frac{1}{\langle \Sigma n, \Sigma n \rangle}} \int_{-\delta}^{\delta} \int_{\tilde{B}^{0, \delta^2 - d^2}} \exp(-\langle y, \Sigma y \rangle) \mathcal{H}(dy) d\lambda \\
& = 2\delta \sqrt{\frac{1}{\langle \Sigma n, \Sigma n \rangle}} \int_{\tilde{B}^{0, \delta^2 - d^2}} \exp(-\langle y, \Sigma y \rangle) \mathcal{H}(dy) \\
& = 2\delta \sqrt{\frac{1}{\langle \Sigma n, \Sigma n \rangle}} \exp(d^2) \int_{\tilde{B}^{0, \delta^2 - d^2}} \exp(-\langle y + dn, \Sigma(y + dn) \rangle) \mathcal{H}(dy) \\
& = 2\delta \sqrt{\frac{1}{\langle \Sigma n, \Sigma n \rangle}} \exp(d^2) \int_{\tilde{B}^{b, \delta^2}} \exp(-\langle y, \Sigma y \rangle) \mathcal{H}(dy).
\end{aligned}$$

The other assertions are elementary. \square

In order to apply this lemma to the case $E = H_0^{N, \varepsilon}$, $a^*(v) = \langle v, m'_\xi \rangle_{L^2(\mathbb{R})}$ $b = \langle m_\xi - m_\xi^{N, \varepsilon}, m'_\xi \rangle_{L^2(\mathbb{R})}$ and $\langle v, \Sigma v \rangle = \frac{2c_4}{\varepsilon} \|v\|_{H^1(\mathbb{R})}^2$ one needs to evaluate the constants d and $\langle \Sigma n, \Sigma n \rangle$ in this context. This is the subject of the next lemma. Note that the scalar product $\langle \cdot, \cdot \rangle$ from Lemma 4.5 corresponds to the Euclidean scalar product on $H_0^{N, \varepsilon}$.

Lemma 4.6. *One has for ε small enough:*

- (i) $\langle m_\xi - m_\xi^{N, \varepsilon}, m'_\xi \rangle_{L^2(\mathbb{R})} \leq C\varepsilon^{-\gamma_1/2} \frac{\varepsilon^{-2\gamma}}{N^2}$,
- (ii) $d^2 \leq C \frac{\varepsilon^{-4\gamma-1-\gamma_1}}{N^4}$,
- (iii) $\langle \Sigma n, \Sigma n \rangle \geq C \frac{\varepsilon^{-1-\gamma}}{N}$.

Proof. (Of Lemma 4.6) (i) Applying Cauchy-Schwarz inequality one gets

$$(4.18) \quad \langle m_\xi - m_\xi^{N, \varepsilon}, m'_\xi \rangle_{L^2(\mathbb{R})} \leq \|m_\xi - m_\xi^{N, \varepsilon}\|_{L^2(\mathbb{R})} \|m'_\xi\|_{L^2(\mathbb{R})} \leq C\varepsilon^{-\gamma_1/2} \frac{\varepsilon^{-2\gamma}}{N^2}.$$

Here Lemma 2.4 was used.

(ii) In order to get a lower bound on $\langle a, \Sigma^{-1}a \rangle$ we use the variational principle given in (4.15). Without loss of generality we may assume that $\xi = 0$. In the present context (4.15) reads:

$$(4.19) \quad d^2 = \frac{b^2}{\langle a, \Sigma^{-1}a \rangle}, \quad \text{and} \quad \langle a, \Sigma^{-1}a \rangle = \left(\sup_{\eta \in H_0^{N, \varepsilon}: \frac{2c_4}{\varepsilon} \|\eta\|_{H^1} = 1} \langle m', \eta \rangle_{L^2(\mathbb{R})} \right)^2.$$

Thus in order to find a lower bound on $\langle a, \Sigma^{-1}a \rangle$ one needs to choose an appropriate test function $\eta \in H_0^{N, \varepsilon}$. To this end set $\bar{k} = \inf \{k: s_k^{N, \varepsilon} > 1\}$. Recall that the $s_k^{N, \varepsilon} = \frac{k\varepsilon^{-\gamma}}{N}$ define the discretization of functions in $H_0^{N, \varepsilon}$. For ε small enough one

has $1 \leq s_{\bar{k}}^{N,\varepsilon} \leq 2$. Set

$$(4.20) \quad \bar{\eta}(s) = \begin{cases} 0 & \text{for } s \leq -s_{\bar{k}}^{N,\varepsilon} \\ \frac{s+s_{\bar{k}}^{N,\varepsilon}}{s_{\bar{k}}^{N,\varepsilon}} & \text{for } -s_{\bar{k}}^{N,\varepsilon} < s \leq 0 \\ \frac{s_{\bar{k}}^{N,\varepsilon}-s}{s_{\bar{k}}^{N,\varepsilon}} & \text{for } 0 < s \leq s_{\bar{k}}^{N,\varepsilon} \\ 0 & \text{for } s_{\bar{k}}^{N,\varepsilon} < s. \end{cases}$$

Then $\bar{\eta} \in H_0^{N,\varepsilon}$ and

$$(4.21) \quad \frac{2c_4}{\varepsilon} \|\bar{\eta}\|_{H^1}^2 = \frac{2c_4}{\varepsilon} \left(\frac{2}{s_{\bar{k}}^{N,\varepsilon}} + \frac{2}{3}s_{\bar{k}}^{N,\varepsilon} \right) \leq \frac{20c_4}{3\varepsilon}.$$

On the other hand m' is strictly positive and we can write

$$(4.22) \quad \langle \bar{\eta}, m' \rangle_{L^2(\mathbb{R})} = \int_{-2}^2 \bar{\eta}(s)m'(s)ds \geq \inf_{s \in [-2,2]} m'(s) \int_{-2}^2 \bar{\eta}(s)ds \geq \inf_{s \in [-2,2]} m'(s).$$

Thus we get $\langle a, \Sigma^{-1}a \rangle \geq C\varepsilon$. Using (4.18) one obtains:

$$d^2 \leq C \frac{\varepsilon^{-4\gamma-1-\gamma_1}}{N^4}$$

(iii) To derive a lower bound on $\langle \Sigma n, \Sigma n \rangle$ first note that

$$(4.23) \quad \langle a, \Sigma^{-1}a \rangle = \left(\sup_{\langle \eta, \Sigma \eta \rangle = 1} \langle a, \eta \rangle \right)^2 = \left(\sup_{\eta \in H_0^{N,\varepsilon} : \frac{2c_4}{\varepsilon} \|\eta\|_{H^1}^2 = 1} \langle \eta, m' \rangle_{L^2(\mathbb{R})} \right)^2 \\ \leq \|m'\|_{L^2(\mathbb{R})}^2 \frac{\varepsilon}{2c_4}.$$

To bound the second factor choose the test function $\bar{\eta}$ as above in (4.20). Then one gets noting $\|\bar{\eta}\|_{\infty} \leq 1$

$$\langle \bar{\eta}, \bar{\eta} \rangle = \sum_{k=-\bar{k}}^{\bar{k}} \left(\bar{\eta}(s_{\bar{k}}^{N,\varepsilon}) \right)^2 \leq C \frac{N}{\varepsilon^{-\gamma}}.$$

We obtain the desired estimate from (4.16) together with (4.23) and (4.22). \square

End of proof of Proposition 4.4: Applying Lemma 4.5 and 4.6 to equation (4.13) one gets:

$$(4.24) \quad \int_{B_{\xi}} \exp\left(-\frac{2c_4}{\varepsilon} \|v\|_{H^1(\mathbb{R})}^2\right) \mathcal{H}^{N,\varepsilon}(dv) \\ \geq C \frac{\sqrt{\varepsilon}}{\delta} \sqrt{\frac{\varepsilon^{-1-\gamma}}{N}} \exp\left(-C \frac{\varepsilon^{-4\gamma-1-\gamma_1}}{N^4}\right) \int_B \exp\left(-\frac{2c_4}{\varepsilon} \|v\|_{H^1(\mathbb{R})}^2\right) \mathcal{L}^{N,\varepsilon}(dv),$$

where $B = \left\{ v \in H_0^{N,\varepsilon} : \frac{2c_4}{\varepsilon} \|v\|_{H^1}^2 \leq \frac{2c_4}{\varepsilon} \delta^2 - d^2 \right\}$. Due to Lemma 4.6 (ii) and (4.5) $\frac{\varepsilon d^2}{2c_4} \downarrow 0$ as $\varepsilon \downarrow 0$ such that the last integral in (4.24) can be bounded from below by

$$(4.25) \quad \int_{\|v\|_{H^1}^2 \leq \delta^2} \exp\left(-\frac{2c_4}{\varepsilon} \|v\|_{H^1(\mathbb{R})}^2\right) \mathcal{L}^{N,\varepsilon}(dv) = Z_2^{N,\varepsilon} \varrho^{N,\varepsilon} \left(\|v\|_{H^1}^2 \leq \frac{\delta^2}{2} \right).$$

Here ϱ is the massive Gaussian free field discussed in Lemma 3.4 for $\kappa = 2c_4$. Lemma 3.4 together with (4.5) yields that $\varrho^{N,\varepsilon} \left(\|v\|_{H^1}^2 \leq \frac{\delta^2}{2} \right) \rightarrow 1$ for $\varepsilon \downarrow 0$. This probability can thus be bounded from below by $\frac{1}{2}$ for ε small enough. Thus the following lemma finishes the proof. \square

Lemma 4.7. *The Gaussian normalization constants $Z_1^{N,\varepsilon}$ and $Z_2^{N,\varepsilon}$ satisfy the following:*

$$(4.26) \quad \exp(-\varepsilon^{\gamma-1}) c_4^{-\frac{2N-1}{2}} (1 + C\varepsilon^{-\gamma})^{-\frac{2N-1}{2}} \leq \frac{Z_2^{N,\varepsilon}}{Z_1^{N,\varepsilon}} \leq \exp(-\varepsilon^{\gamma-1}) c_4^{-\frac{2N-1}{2}}.$$

Proof. By Lemma 3.4 and equation (3.6)

$$\frac{Z_2^{N,\varepsilon}}{Z_1^{N,\varepsilon}} = (2c_4)^{-\frac{2N-1}{2}} \exp(-\varepsilon^{\gamma-1}) \left(\frac{\det(-\Delta_{N,\varepsilon})}{\det(I_{N,\varepsilon} - \Delta_{N,\varepsilon})} \right)^{\frac{1}{2}}.$$

By the Poincaré inequality (3.13) one has

$$-\Delta_{N,\varepsilon} \leq (I_{N,\varepsilon} - \Delta_{N,\varepsilon}) \leq (1 + C\varepsilon^{-\gamma}) (-\Delta_{N,\varepsilon})$$

in the sense of symmetric matrices. This implies

$$\det(-\Delta_{N,\varepsilon}) \leq \det(I_{N,\varepsilon} - \Delta_{N,\varepsilon}) \leq (1 + C\varepsilon^{-\gamma})^{2N-1} \det(-\Delta_{N,\varepsilon}).$$

This finishes the proof. \square

As a next step an upper bound on $\mu^{N,\varepsilon}(A^c)$ is derived:

Proposition 4.8. *Choosing γ_1 and γ_2 according to (4.5) one has for $\delta \leq \delta_0$:*

$$(4.27) \quad \limsup_{\varepsilon \downarrow 0} \varepsilon \log \left(Z^{N,\varepsilon} \mu^{N,\varepsilon}(\text{dist}_{H^1}(u, M) \geq \delta) \right) \leq -(C_* + c_0 \delta^2).$$

In particular, setting $\delta = 0$ one obtains

$$(4.28) \quad \limsup_{\varepsilon \downarrow 0} \varepsilon \log Z^{N,\varepsilon} \leq -C_*.$$

Proof. Denote by $A^\delta := \{u : \text{dist}_{H^1}(u, M) \geq \delta\}$. Then one has

$$(4.29) \quad \begin{aligned} Z^{N,\varepsilon} \mu^{N,\varepsilon}(A^\delta) &= \exp\left(-\frac{C_*}{\varepsilon}\right) \frac{1}{Z_1^{N,\varepsilon}} \int_{A^\delta} \exp\left(-\frac{1}{\varepsilon} \mathcal{E}(u)\right) \mathcal{L}^{N,\varepsilon}(du) \\ &\leq \exp\left(-\frac{C_* + c_0 \delta^2}{\varepsilon}\right) \frac{1}{Z_1^{N,\varepsilon}} \int_{A^\delta} \exp\left(-\frac{1}{\varepsilon} (\mathcal{E}(u) - c_0 \delta^2)\right) \mathcal{L}^{N,\varepsilon}(du). \end{aligned}$$

Note that by (2.9) $\mathcal{E}(u) - c_0 \delta^2 \geq 0$ on A^δ . So on this set one gets

$$\exp\left(-\frac{1}{\varepsilon} (\mathcal{E}(u) - c_0 \delta^2)\right) \leq \exp\left(-(\mathcal{E}(u) - c_0 \delta^2)\right).$$

Therefore, one gets

$$(4.30) \quad \begin{aligned} \int_{A^\delta} \exp\left(-\frac{1}{\varepsilon} (\mathcal{E}(u) - c_0 \delta^2)\right) \mathcal{L}^{N,\varepsilon}(du) &\leq \int_{A^\delta} \exp\left(-(\mathcal{E}(u) - c_0 \delta^2)\right) \mathcal{L}^{N,\varepsilon}(du) \\ &\leq \int_{A^\delta} Z_3^{N,\varepsilon} \exp\left(\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} -F(u(s)) ds + c_0 \delta^2\right) \nu^{1,N}(du), \end{aligned}$$

where $\nu^{1,N}$ is the discretized Brownian bridge without rescaling and

$$Z_3^{N,\varepsilon} = \int \exp\left(-\int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \frac{1}{2}|u'(s)|^2 ds\right) \mathcal{L}^{N,\varepsilon}(du)$$

is the appropriate normalization constant. Using the positivity of F the last term in (4.30) can thus be bounded by

$$Z_3^{N,\varepsilon} \exp(c_0 \delta^2).$$

Plugging this into (4.29) yields

$$Z^{N,\varepsilon} \mu^{N,\varepsilon}(A^\delta) \leq \exp\left(-\frac{C_* + c_0 \delta^2}{\varepsilon}\right) \frac{1}{Z_1^{N,\varepsilon}} Z_3^{N,\varepsilon} \exp(c_0 \delta^2).$$

This finishes the proof together with the following bound on the normalization constants $Z_1^{N,\varepsilon}$ and $Z_3^{N,\varepsilon}$. \square

Lemma 4.9. *One has*

$$\frac{Z_3^{N,\varepsilon}}{Z_1^{N,\varepsilon}} = \varepsilon^{-\frac{2N-1}{2}}.$$

Proof. This is a direct consequence of the fact that for matrices $A \in \mathbb{R}^{n \times n}$ and $\xi \in \mathbb{R}$

$$\det(\xi A) = \xi^n \det(A),$$

as well as the explicit formula for the Gaussian normalization constants. \square

One can now summarize the finite-dimensional calculation in the following:

Corollary 4.10. *Choosing the constants γ_1 and γ_2 as in (4.5), (4.7) one obtains for $\delta \leq \delta_0$:*

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log (\mu^{N,\varepsilon}(\text{dist}_{H^1}(u, M) \geq \delta)) \leq -c_0 \delta^2.$$

Note that such a choice is possible for all $\gamma < 1$.

Proof. Dividing and using the estimates from above yields the result. \square

Using the continuous embedding of H^1 into L^∞ one gets:

Corollary 4.11. *Choosing the constants γ_1 and γ_2 as in (4.5), (4.7) one obtains for $\delta \leq \delta_0$:*

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \log (\mu^{N,\varepsilon}(\text{dist}_{L^\infty}(u, M) \geq \delta)) \leq -\tilde{c}_0 \delta^2.$$

Such a choice is possible for all $\gamma < 1$.

As a last step in this section we need to control the deviations from the discretized measure with the help of the Gaussian estimates derived in Section 3. To this end one has to estimate the deviations of the normalization constant Z^ε from $Z^{N,\varepsilon}$. In order to prove the following lemma we will need an additional assumption on the double well potential F .

Assumption 4.12.

$$(4.31) \quad |F'(u)| \text{ is bounded for } u \in \mathbb{R}.$$

In fact one can simply modify the potential F by cutting it off outside of some compact set, such that it satisfies (4.31). We will proceed now by proving Theorem 1.1 under the additional assumption (4.31). The general case will then follow as a corollary.

Proposition 4.13. *Assume that F satisfies (4.31). Then the following holds:*

(i) *For every $\gamma_3 > 0$ we have for ε small enough:*

$$(4.32) \quad \int_{H^\varepsilon} \exp\left(-\frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s) - u^N(s)| ds\right) \nu^\varepsilon(du) \\ \geq \left(\frac{1}{2}\right)^{2N} \exp\left(-\frac{\sqrt{6}\|F'\|_\infty \varepsilon^{-\frac{1}{2}-\frac{3}{2}\gamma-\gamma_3}}{3 N^{\frac{1}{2}}}\right)$$

(ii) *For ε small enough we have:*

$$(4.33) \quad \int_{H^\varepsilon} \exp\left(\frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s) - u^N(s)| ds\right) \nu^\varepsilon(du) \\ \leq \left(\frac{4\|F'\|_\infty^2 \varepsilon^{1-3\gamma}}{\pi N^3}\right)^N \exp\left(\frac{\sqrt{6}\|F'\|_\infty \varepsilon^{-\frac{1}{2}-\frac{3}{2}\gamma-\gamma_3}}{3 N^{\frac{1}{2}}} + \frac{1}{2\pi^2}\|F'\|_\infty^2 \frac{\varepsilon^{-1-3\gamma}}{N^2}\right).$$

(iii) *For $\gamma < \frac{2}{3}$ the the normalization constant satisfies:*

$$(4.34) \quad \liminf_{\varepsilon \downarrow 0} \varepsilon \log Z^\varepsilon = -C_*.$$

Recall that u^N denotes the discretization of the Brownian bridge as introduced in Section 3.

Proof. Applying Cauchy-Schwarz inequality and the independence of the Brownian bridges on the short intervals derived in Lemma 3.2 one gets:

$$(4.35) \quad \int_{H^\varepsilon} \exp\left(-\frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s) - u^N(s)| ds\right) \nu^\varepsilon(du) \\ \geq \int_{H^\varepsilon} \exp\left(-\frac{1}{\varepsilon} \left(\frac{\varepsilon^{-\gamma}}{N}\right)^{\frac{1}{2}} \|F'\|_\infty \sum_{k=-N}^{N-1} \|u - u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]}\right) \nu^\varepsilon(du) \\ = \prod_{k=-N}^{N-1} \int_{H^\varepsilon} \exp\left(-\frac{1}{\varepsilon} \left(\frac{\varepsilon^{-\gamma}}{N}\right)^{\frac{1}{2}} \|F'\|_\infty \|u - u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]}\right) \nu^\varepsilon(du).$$

In the same way one can see that

$$(4.36) \quad \int_{H^\varepsilon} \exp\left(\frac{1}{\varepsilon}\|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s) - u^N(s)| ds\right) \nu^\varepsilon(du) \\ \leq \prod_{k=-N}^{N-1} \int_{H^\varepsilon} \exp\left(\frac{1}{\varepsilon} \left(\frac{\varepsilon^{-\gamma}}{N}\right)^{\frac{1}{2}} \|F'\|_\infty \|u - u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]}\right) \nu^\varepsilon(du).$$

Thus in order to prove (4.32) and (4.33) we have to bound the integrals over the Brownian bridges on the short intervals. To simplify the equations we introduce

the following notation:

$$\begin{aligned}\alpha_\varepsilon &= \sqrt{\varepsilon \frac{\varepsilon^{-2\gamma}}{6N^2}} \\ \beta_\varepsilon &= \frac{1}{\varepsilon} \left(\frac{\varepsilon^{-\gamma}}{N} \right)^{\frac{1}{2}} \|F'\|_\infty \\ \delta_\varepsilon &= \frac{\pi^2 N^2}{\varepsilon^{1-2\gamma}}.\end{aligned}$$

In this notation the concentration inequality (3.8) reads

$$(4.37) \quad \nu^\varepsilon \left(\|u - u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]} \geq \alpha_\varepsilon + r \right) \leq \exp(-r^2 \delta_\varepsilon).$$

Now let us proceed to prove (4.32): Using the formula

$$(4.38) \quad \mathbb{E}[e^{\beta X}] = 1 + \beta \int_0^\infty e^{\beta x} \mathbb{P}[X \geq x] dx,$$

which holds for every non-negative random variable X and every $\beta \in \mathbb{R}$ one obtains:

$$\begin{aligned}(4.39) \quad & \int_{H^\varepsilon} \exp\left(-\beta_\varepsilon \|u - u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]}\right) \nu^\varepsilon(du) \\ &= 1 - \beta_\varepsilon \int_0^\infty e^{-\beta_\varepsilon x} \mathbb{P}\left[\|u - u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]} \geq x\right] dx \\ &\geq 1 - \beta_\varepsilon \int_0^{\varepsilon^{-\gamma_3} \alpha_\varepsilon} e^{-\beta_\varepsilon x} dx - \beta_\varepsilon \int_{\varepsilon^{-\gamma_3} \alpha_\varepsilon}^\infty e^{-\beta_\varepsilon x} \exp\left(- (x - \alpha_\varepsilon)^2 \delta_\varepsilon\right) dx \\ &= e^{-\beta_\varepsilon \varepsilon^{-\gamma_3} \alpha_\varepsilon} - e^{-\beta_\varepsilon \varepsilon^{-\gamma_3} \alpha_\varepsilon} \beta_\varepsilon \int_0^\infty e^{-\beta_\varepsilon x} \exp\left(- (x + \alpha_\varepsilon(\varepsilon^{-\gamma_3} - 1))^2 \delta_\varepsilon\right) dx.\end{aligned}$$

Using the elementary inequality

$$\int_0^\infty \exp(-\beta x - \delta x^2) dx \leq \int_0^\infty \exp(-\beta x) dx \leq \frac{1}{\beta},$$

which holds for $\beta, \delta > 0$ the last integral in (4.39) can be bounded by

$$\begin{aligned}& \int_0^\infty e^{-\beta_\varepsilon x} \exp\left(- (x + \alpha_\varepsilon(\varepsilon^{-\gamma_3} - 1))^2 \delta_\varepsilon\right) dx \\ & \leq \exp\left(-\delta_\varepsilon \alpha_\varepsilon^2 (\varepsilon^{-\gamma_3} - 1)^2\right) \frac{1}{\beta_\varepsilon + 2\delta_\varepsilon \alpha_\varepsilon (\varepsilon^{-\gamma_3} - 1)}\end{aligned}$$

Noting that $\delta_\varepsilon \alpha_\varepsilon^2$ is a quantity of order $O(1)$ one sees that this term decays to zero exponentially in ε . In particular for ε small enough

$$\beta_\varepsilon \int_0^\infty e^{-\beta_\varepsilon x} \exp\left(- (x + \alpha_\varepsilon(\varepsilon^{-\gamma_3} - 1))^2 \delta_\varepsilon\right) dx \leq \frac{1}{2}.$$

Plugging this into (4.39) and using (4.35) finishes the proof of (4.32).

To derive the lower bound in (4.34) using (4.38) one gets

$$\begin{aligned}
(4.40) \quad & \int_{H^\varepsilon} \exp\left(\beta_\varepsilon \|u - u^N\|_{L^2[s_k^{N,\varepsilon}, s_{k+1}^{N,\varepsilon}]}\right) \\
& \leq 1 + \beta_\varepsilon \int_0^{\alpha_\varepsilon} e^{\beta_\varepsilon x} dx + \beta_\varepsilon \int_{\alpha_\varepsilon}^\infty e^{\beta_\varepsilon x} \exp\left(-\delta_\varepsilon (x - \alpha_\varepsilon)^2\right) dx \\
& = e^{\beta_\varepsilon \alpha_\varepsilon} + e^{\beta_\varepsilon \alpha_\varepsilon} \beta_\varepsilon \int_0^\infty \exp\left(\beta_\varepsilon x - \delta_\varepsilon x^2\right) dx.
\end{aligned}$$

Completing the squares the last integral can be bounded by

$$\begin{aligned}
(4.41) \quad & \int_0^\infty \exp\left(\beta_\varepsilon x - \delta_\varepsilon x^2\right) dx = \exp\left(\frac{\beta_\varepsilon^2}{4\delta_\varepsilon}\right) \int_0^\infty \exp\left(-\delta_\varepsilon \left(x - \frac{\beta_\varepsilon}{2\delta_\varepsilon}\right)^2\right) \\
& \leq \exp\left(\frac{\beta_\varepsilon^2}{4\delta_\varepsilon}\right) \sqrt{\frac{\pi}{\delta_\varepsilon}}.
\end{aligned}$$

Plugging this into (4.40) and using (4.36) yields the desired result.

To see (4.33) write:

$$\begin{aligned}
(4.42) \quad & Z^\varepsilon = \int_{H^\varepsilon} \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^\varepsilon(du) \\
& = \int_{H^\varepsilon} \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u^N(s)) ds\right) \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} (F(u(s)) - F(u^N(s))) ds\right) \nu^\varepsilon(du) \\
& \geq \int_{H^\varepsilon} \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u^N(s)) ds\right) \exp\left(-\frac{1}{\varepsilon} \|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s) - u^N(s)| ds\right) \nu^\varepsilon(du).
\end{aligned}$$

Using the independence of the discretized Brownian bridge and the bridges on the small intervals derived in Lemma 3.2 the last term can be rewritten as

$$(4.43) \quad Z^{N,\varepsilon} \int_{H^\varepsilon} \exp\left(-\frac{1}{\varepsilon} \|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s) - u^N(s)| ds\right) \nu^\varepsilon(du).$$

If one chooses $N = N(\varepsilon)$ growing like $\varepsilon^{-\gamma_2}$ according to (4.32) the exponent in the error terms scales like

$$\varepsilon^{-\frac{1}{2} - \frac{3}{2}\gamma - \gamma_3 + \frac{\gamma_2}{2}},$$

so that if one chooses the γ_i such that (4.5),(4.7) hold and in addition

$$(4.44) \quad -\frac{1}{2} - \frac{3\gamma}{2} - \gamma_3 + \frac{\gamma_2}{2} > -1$$

$$(4.45) \quad \gamma_3 > 0$$

the result follows from the bound (4.8) on the discretized normalization constant. Note that such a choice is possible for $\gamma < \frac{2}{3}$.

For the upper bound in (4.34) similar to (4.42) and (4.43) one can write

$$(4.46) \quad Z^\varepsilon \leq Z^{N,\varepsilon} \int_{H^\varepsilon} \exp\left(\frac{1}{\varepsilon} \|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s) - u^N(s)| ds\right) \nu^\varepsilon(du).$$

For $\gamma < \frac{2}{3}$ one can chose the parameters γ_i such that (4.7)-(4.5) and (4.44)-(4.45) hold and in addition

$$(4.47) \quad -3\gamma + 2\gamma_2 > -1.$$

Then the bound (4.33) implies

$$(4.48) \quad \limsup_{\varepsilon \downarrow 0} \varepsilon \log \int_{H^\varepsilon} \exp\left(\frac{1}{\varepsilon} \|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s) - u^N(s)| ds\right) \nu^\varepsilon(du) = 0.$$

Thus (4.28) implies the upper bound in (4.34). \square

In the sequel we will always assume that the γ_i satisfy (4.7)-(4.5), (4.44)-(4.45) and (4.47). Now one can conclude:

Proposition 4.14. *The statement of Theorem 1.1 holds under the additional assumption (4.31).*

Proof. One can estimate

$$(4.49) \quad \begin{aligned} \mu^\varepsilon\left(\text{dist}_{L^2}(u, M) \geq \delta\right) &= \frac{1}{Z^\varepsilon} \int_{\{\text{dist}_{L^2}(u, M) \geq \delta\}} \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^\varepsilon(du) \\ &\leq \frac{1}{Z^\varepsilon} \int_{A_1} \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^\varepsilon(du) \\ &\quad + \frac{1}{Z^\varepsilon} \int_{A_2} \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^\varepsilon(du), \end{aligned}$$

where

$$\begin{aligned} A_1 &= \left\{ \|u - u^N\|_{L^2[-\varepsilon^{-\gamma}, \varepsilon^{-\gamma}]} \geq \frac{\delta}{2} \right\} \\ A_2 &= \left\{ \text{dist}_{L^2}(u^N, M) \geq \frac{\delta}{2} \right\}. \end{aligned}$$

The concentration inequality (3.7) implies together with the lower bound on the normalization constant (4.34) that the first integral decays to zero on a quicker exponential scale than ε^{-1} . For the integral over A_2 we can write using the independence of the discretized Brownian bridges and the bridges on the intermediate intervals again:

$$(4.50) \quad \begin{aligned} &\frac{1}{Z^\varepsilon} \int_{A_2} \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^\varepsilon(du) \\ &\leq \frac{Z^{N, \varepsilon}}{Z^\varepsilon} \frac{1}{Z^{N, \varepsilon}} \int_{A_2} \exp\left(-\frac{1}{\varepsilon} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(v(s)) ds\right) \nu^\varepsilon(du) \times \\ &\quad \int_{H^\varepsilon} \exp\left(\frac{1}{\varepsilon} \|F'\|_\infty \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} |u(s) - u^N(s)| ds\right) \nu^\varepsilon(du). \end{aligned}$$

The fraction $\frac{Z^{N, \varepsilon}}{Z^\varepsilon}$ can be bounded using (4.32), the integral in the last line is bounded due to (4.48). Thus the statement follows from the result on the discretized measures in Corollary 4.10. This finishes the proof for the L^2 -norm. To see analogue result for the L^∞ -norm repeat the same reasoning with Lemma 4.10 replaced by Lemma 4.11 and the L^2 bound (3.7) replaced by the L^∞ -bound (3.9). \square

Proof. (Of Theorem 1.1 in the general case): Denote by dist either dist_{L^2} or dist_{L^∞} . Assume that F only satisfies assumptions (1.2). By cutting F off outside of $[-2, 2]$ one can choose a function \bar{F} that coincides with F on $[-2, 2]$ that satisfies (1.2) and (4.31) as well as

$$\bar{F}(u) \leq F(u) \quad \text{for } u \in \mathbb{R}.$$

Then one can write

$$(4.51) \quad \begin{aligned} \mu^\varepsilon(\text{dist}(u, M) \geq \delta) &= \frac{\int_{\{\text{dist}(u, M) \geq \delta\}} \exp\left(-\varepsilon^{-1} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^\varepsilon(du)}{\int \exp\left(-\varepsilon^{-1} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^\varepsilon(du)} \\ &\leq \frac{\int_{\{\text{dist}(u, M) \geq \delta\}} \exp\left(-\varepsilon^{-1} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^\varepsilon(du)}{\int_{\{\|u\|_{L^\infty} \leq 2\}} \exp\left(-\varepsilon^{-1} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} F(u(s)) ds\right) \nu^\varepsilon(du)} \end{aligned}$$

The denominator of this fraction coincides with

$$\int_{\{\|u\|_{L^\infty(\mathbb{R})} \leq 2\}} \exp\left(-\varepsilon^{-1} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \bar{F}(u(s)) ds\right) \nu^\varepsilon(du)$$

and the numerator is bounded from above by

$$\int_{\{\text{dist}(u, M) \geq \delta\}} \exp\left(-\varepsilon^{-1} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \bar{F}(u(s)) ds\right) \nu^\varepsilon(du),$$

such that one can write

$$(4.52) \quad \begin{aligned} \mu^\varepsilon(\text{dist}(u, M) \geq \delta) &\leq \frac{\int_{\{\text{dist}(u, M) \geq \delta\}} \exp\left(-\varepsilon^{-1} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \bar{F}(u(s)) ds\right) \nu^\varepsilon(du)}{\int \exp\left(-\varepsilon^{-1} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \bar{F}(u(s)) ds\right) \nu^\varepsilon(du)} \times \\ &\quad \times \frac{\int \exp\left(-\varepsilon^{-1} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \bar{F}(u(s)) ds\right) \nu^\varepsilon(du)}{\int_{\{\|u\|_{L^\infty} \leq 2\}} \exp\left(-\varepsilon^{-1} \int_{-\varepsilon^{-\gamma}}^{\varepsilon^{-\gamma}} \bar{F}(u(s)) ds\right) \nu^\varepsilon(du)}. \end{aligned}$$

Now applying Proposition 4.14 shows that the second factor can be bounded by 2 for ε small enough and thus applying Proposition 4.14 to the first factor yields the desired result. \square

With a similar argument one can see that the bounds on the normalization constant in Proposition 4.13 also holds without assumption (4.31):

Corollary 4.15. *Suppose $\gamma < \frac{2}{3}$. Then one has the following bound:*

$$(4.53) \quad \liminf_{\varepsilon \downarrow 0} \varepsilon \log Z^\varepsilon = -C_*.$$

It remains to prove Corollary 1.2:

Proposition 4.16. *The family of measures $\tilde{\mu}^\varepsilon$ is tight. All points of accumulation are concentrated on functions of the type*

$$(4.54) \quad \tilde{m}_\xi(s) = -\mathbf{1}_{[-1, \xi]}(s) + \mathbf{1}_{[\xi, 1]}(s).$$

Proof. Denote by $\tilde{M} = \{\tilde{m}_\xi : \xi \in [-1, 1]\}$ and $\text{dist}(\tilde{u}, \tilde{M}) = \inf_{\xi \in [-1, 1]} \|\tilde{u} - \tilde{m}_\xi\|_{L^2[-1, 1]}$. Furthermore, denote by $\tilde{m}_\xi^\varepsilon(s) = m\left(\frac{s-\xi}{\varepsilon}\right)$. Note that for all $\xi \in [-1, 1]$ $\tilde{m}_\xi^\varepsilon$ converges to \tilde{m}_ξ in L^2 . Now choose $\delta > 0$ and ε_0 such that $\|\tilde{m}_\xi^\varepsilon - \tilde{m}_\xi\|_{L^2} \leq \frac{\delta}{2}$ for all $\varepsilon \leq \varepsilon_0$. Then Theorem 1.1 implies that

$$(4.55) \quad \begin{aligned} \tilde{\mu}^\varepsilon \left(\text{dist}_{L^2}(\tilde{u}, \tilde{M}) \geq \delta \right) &\leq \tilde{\mu}^\varepsilon \left(\inf_{\xi} \|\tilde{u} - \tilde{m}_\xi^\varepsilon\|_{L^2[-1, 1]} \geq \frac{\delta}{2} \right) \\ &\leq \mu^\varepsilon \left(\text{dist}_{L^2}(T^\varepsilon(\tilde{u}), M) \geq \frac{\delta}{2\sqrt{\varepsilon^{-\gamma}}} \right) \downarrow 0. \end{aligned}$$

This is sufficient to show the tightness of the measures $\{\tilde{\mu}^\varepsilon\}$. In fact fix a small constant $\kappa > 0$. Let us construct a precompact set K such that $\tilde{\mu}^\varepsilon(K^C) \leq \kappa$. For a fixed $N \in \mathbb{N}$ due to (4.55) there exists ε^N such that for all $\varepsilon \leq \varepsilon^N$

$$\tilde{\mu} \left(\text{dist}(\tilde{u}, \tilde{M}) \geq \frac{1}{2N} \right) \leq \frac{\kappa}{2^N}.$$

In particular, there exist finitely many $\xi_i^N \in [-1, 1]$ $i = 1, \dots, i_N$ such that for all $\varepsilon \leq \varepsilon^N$

$$\tilde{\mu}^\varepsilon \left(\cup_i B \left(\tilde{m}_{\xi_i^N}, \frac{1}{N} \right) \right) \geq 1 - \frac{\kappa}{2^N}.$$

Furthermore, due to tightness of the measures $(\tilde{\mu}^\varepsilon, \varepsilon \in [\varepsilon_N, 1])$ there exist finitely many balls \tilde{B}_i^N of radius $\frac{1}{N}$ such that for all $\varepsilon \in [\varepsilon_N, 1]$ one has

$$\tilde{\mu}^\varepsilon(\cup_i \tilde{B}_i) \geq 1 - \frac{\kappa}{2^N}.$$

Set $K^N = \left(\cup_i \tilde{B}_i \right) \cup \left(\cup_i B \left(\tilde{m}_{\xi_i^N}, \frac{1}{N} \right) \right)$ and $K = \cap_N K^N$. Then K is precompact and for all ε has measure $\geq 1 - \kappa$. This shows tightness. The concentration follows from (4.55). \square

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