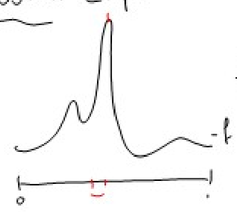
So far $\frac{1}{n}\sum_{i=1}^{n} X_i$ linear

· rate function convex

Varadhan's Lemma (exponential tilting)

1st lesson: "Laplace method"



$$\frac{1}{n}\log\int e^{nf(x)}\,dx$$

$$= -\inf_{x\in[0,1]} f(x)$$

Question μ_n on E (satisfy LDP with rate fct. I)

what is $\underbrace{\frac{1}{n}\log\int e^{-nf(x)}\mu_n(dx)}_{H}$ for $n\to\infty$?

Naively: LD $\rightsquigarrow$ $\mu_n \sim e^{-nI(x)}$ "dx",

$H \rightsquigarrow \frac{1}{n}\log$ "$\int e^{-nf(x)}e^{-nI(x)}$ dx"

$\rightsquigarrow$ one would guess limit of this is $-\inf_x (f+I)$

Thm (Varadhan's Lemma)

μ_n measures on E (= polish space)

$I: E \longrightarrow [0,\infty]$ fct.

(i) If F is u.s.c. & LD lowerbound

$$\underline{\lim}_{n\to\infty}\frac{1}{n}\log\int e^{-nF(x)}\mu_n(dx) \geq -\inf(F+I)$$

(ii) If F is l.s.c. & I compact sublevelsets

& LD upperbound

& ⊛

$$\overline{\lim}\,\frac{1}{n}\log\int e^{-nF(x)}\mu_n(dx) \leq -\inf(F+I)$$

$\lim_{M}$ ⊛ $\overline{\lim}_{n\to\infty}\frac{1}{n}\log\int e^{-nF(y)}\mathbb{1}_{\{F(x)\geq M\}}\mu_n(dy) = -\infty$

Proof: (i) Lowerbound:

Fix $x\in E$. There exists a neighborhood O_x of x such that on O_x

$$F \leq F(x)+\delta$$

$$\Rightarrow \int_E e^{-nF(y)}\mu_n(dy)$$

$$\geq \int_{O_x} e^{-nF(y)}\mu_n(dy)$$

$$\geq e^{-n(F(x)+\delta)}\mu_n(O_x)$$

$\overset{\text{LD lower bd.}}{\Longrightarrow} \underline{\lim}\frac{1}{n}\log\int_E e^{-nF(y)}dy \geq -(F(x)+\delta) - \inf_{y\in O_x} I(y)$

$$\geq -(F(x)+\delta+I(x))$$

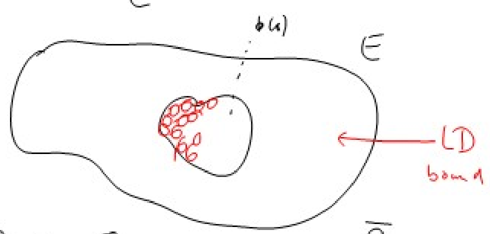
Take inf over x and $\delta\downarrow 0$ $\square$

Upperbound:

First assume $F \geqslant -M$.

$\forall s \quad \Phi(s) = \{ x \in E,\ I(x) \le s \}$

Idea.



$\forall x \in \Phi(s)\ \exists\ \mathcal{O}_x$ such that on $\bar{\mathcal{O}}_x$

$I \geqslant I(x) - \delta \qquad F \geqslant F(x) - \delta$

By compactness $\mathcal{O}_{x_1} \dots \mathcal{O}_{x_N}$ cover $\phi(s)$

$$\Rightarrow \int_E e^{-nF(x)} \mu_n(dx) \leqslant \sum_{k=1}^{N} \int_{\mathcal{O}_{x_i}} e^{-nF(x)} \mu_n(dx) \quad \text{(II)}$$

$$+ \underbrace{\int_{(\cup \mathcal{O}_{x_i})^c} e^{-nF(x)} \mu_n(dx)}_{\text{(I)}}$$

$$\text{(I)} \le e^{+nM} \mu_n\Big(\underbrace{(\cup \mathcal{O})^c}_{\text{closed set}} \Big) \quad \text{closed set} \subseteq \Phi(s)^c$$

$\Rightarrow$ by LD upper bd

$$\overline{\lim}\ \tfrac{1}{n} \log \text{(I)} \le M - s$$

for s large enough $\leqslant -\inf(I+F)$

(II) for a given i consider

$$\int_{\mathcal{O}_{x_i}} e^{-nF(y)} d\mu_n(y) \le e^{-n(F(x)-\delta)} \mu_n(\bar{\mathcal{O}}_{x_i})$$

By LD upper bd

$$\overline{\lim}\ \frac{1}{n} \log (\ \text{"}\ ) \leqslant -(F(x)-\delta) - \inf_{\bar{\mathcal{O}}_{x_i}} I(y)$$

$$\le -(F(x_i)-\delta) - (I(x_i)-\delta)$$

$$\Rightarrow \overline{\lim}\ \frac{1}{n} \log \sum \int_{\mathcal{O}_{x_i}}$$

$$\le \min_{i=1\dots N} -(I(x_i) - F(x_i)) + 2\delta$$

$$\le \inf_{x \in E} -I(x) - F(x) + 2\delta$$

$\delta \downarrow 0 \quad \square$

The general case fix a large M

$$\int_E e^{-nF(x)} \mu_n(dx)$$

apply assumption.

$$\leq \int 1\!\!1_{\{F \leq -M\}} e^{-nF(x)} \mu_n(dx)$$

$$+ \int 1\!\!1_{\{F \geq -M\}} e^{-nF(x)} \mu_n(dx)$$

apply first part.

$\square$

<u>Corollary</u> Suppose that μ_n satisfies LDP with rate I. Then the tilted measure

$$\nu_n(dx) = \frac{1}{Z_n} e^{-nF(y)} \mu_n(dy)$$

for F continuous & bdd from below satisfy a LDP with rate

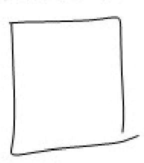
$$J(x) = I(x) + F(x) - \inf_y \left(I(y) + F(y)\right).$$

<u>proof</u>. Left as exercise.

2 Applications

① Curie-Weiss Model.

Spontaneous Magnetisation.



Metal

for $T < T_c$ "Curie Temperature"

the sample turns into a magnet.

Atoms $1 \ldots\ldots N$

Configuration: $(\sigma_i)_{i \in \{1, \ldots, N\}}$

$\sigma_i \in \{\pm 1\}$

mean magnetisation $\bar{\sigma} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \in [-1, 1]$

Energy: E $+\frac{1}{2N} \sum_{i,j} \sigma_i \sigma_j$ "everything interacts with everything"

$\beta = \frac{1}{T}$ inverse temperature

random configuration of spins

$$\mu_N(\sigma) = \frac{1}{Z_{N,\beta}} e^{-\beta E(\sigma)} \left(\tfrac{1}{2}\right)^N$$

$\beta = +\infty$ only conf. of lowest energy.

$\beta = 0$ ∞ temperature, every conf has the same proba.

μ_N distribution of $\bar{\sigma}$ (without density)

μ_N measures on $\mathbb{R}$ satisfy (by Cramér) an LDP with rate fct.

$$I(m) = \frac{1-m}{2}\log(1-m) + \frac{1+m}{2}\log(1+m)$$

As everything interacts with everything we get

$$E(\sigma) = \frac{-1}{2N}\sum_{i,j}\sigma_i\sigma_j$$
$$= \frac{-1}{2N}\left(\sum \sigma_i\right)^2$$
$$= \frac{-N}{2}\bar{\sigma}^2$$

$\bar{\nu}_N$ distribution of $\bar{\sigma}$ with finite temperature

$$\nu_N = \frac{1}{Z_{N,\beta}} e^{+\frac{N}{2}\beta\bar{\sigma}^2}\mu_N$$

By Varadhan we have LDP with rate.

$$\frac{1-m}{2}\log(1-m) + \frac{1+m}{2}\log(1+m)$$

micro states that correspond to a given macro state

$$+\frac{1}{2}\beta m^2 \; !$$

Energy

Result

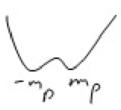
Competition between energy and entropy!

If one is interested in "typical behavior"

What are the zeros of rate fct.

$\beta < 1$ minimum of rate is 0.

for $\beta > 1$

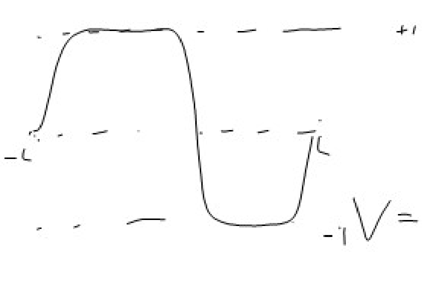


Example. Stoch. Allen-Cahn equation

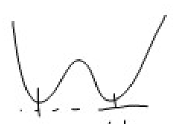
t time $x \in (-L, L)$

$$\frac{du}{dt}(t,x) = \partial_{xx}u(t,x) - V'(u(t,x)) + \sqrt{2\varepsilon}\,\eta(t,x)$$

temp. space time white noise



$V =$



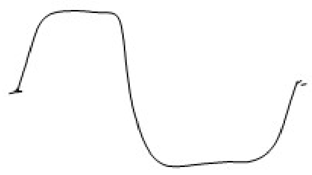
$$V(u) = \frac{(u^2-1)^2}{4}$$

invariant measure

$\sqrt{\varepsilon}\,W_x$ BB from $(-L, L)$ distribution μ_ε

$$\nu_\varepsilon = \frac{1}{Z_\varepsilon}\exp\left(-\frac{1}{\varepsilon}\int_{-L}^{L} V(u(x))\right)\mu_\varepsilon(du)$$

Question: How likely are transitions??



We know that (kind of)

μ_ε satisfy LDP with rate fct

$$\frac{1}{2}\int_{-L}^{L} u'(x)^2 \, dx \quad + \text{b.c.}$$

ν_ε satisfy (by Varadhan) an LDP with rate

$$\underbrace{\frac{1}{2}\int_{-L}^{L} u'(x)^2 dx + \int_{-L}^{L} V(u(x))\, dx}_{\text{Ginzburg-Landau Energy}}$$

Energy difference.

$$\int_{x_-}^{x_+} \frac{1}{2} u'(x)^2 + V(u(x)) \, dx$$

$$= \int_{x_-}^{x_+} \frac{1}{2}\overbrace{\left[u'(x)^2 - 2u'(x)\sqrt{2V(u)} + \left(\sqrt{2V(u(x))}\right)^2\right]}^{\geq 0} + \int_{x_-}^{x_+} u'(x)\sqrt{2V(u)}$$

$$= \int_{u(x_-) \approx -1}^{u(x_+) \approx 1} W(s)\, ds = c_0$$

//

Modica-Mortola trick

$$W' = \sqrt{2V(u)}$$

in standard case $\frac{2\sqrt{2}}{3}$

$\Rightarrow$ Proba of extra transition

$$\approx e^{-\frac{c_0}{\varepsilon}} .$$

$\square$

Inverse of Varadhan's Lemma

Thm: Fix μ_n on E. If I has compact sublevelsets and for all bdd continuous $F: E \to \mathbb{R}$

$$\lim \log \frac{1}{n} \int e^{-nF(x)} \mu_n(dx) = -\inf_x \left(I(x) + F(x)\right)$$

$\Rightarrow \mu_n$ satisfy LDP with rate I.

Thm (Bryc's inverse Varadhan Lemma)

E polish space! Assume that μ_n exponentially tight. And for every F the limit $\lim \log \frac{1}{n} \int e^{-nF(x)} \mu_n(dx) =: \Lambda(F)$ exists.

Then μ_n satisfy a LDP with rate

$$I(x) = \inf_{F \in C_b} \left(F(x) + \Lambda(F) \right)$$

(signs to be confirmed)