Last time: Varadhan Lemma

- μ_n satisfy LDP with rate fct. I

$$\overline{\lim}\, \frac{1}{n} \log \int_E e^{-nF(x)} \mu_n(dx) = -\inf_x \left(F(x) + I(x)\right)$$

F cont. & bdd

Inverse Varadhan:

① I has compact sublevelsets & if $\forall F \in C_b(E)$
$\lim \frac{1}{n} \log \int_E e^{nF(x)} \mu_n(dx) = \sup_x \left(F(x) - I(x)\right)$
$\Rightarrow$ μ_n satisfy LDP with rate I.

② Bryc's inverse Varadhan: μ_n are exponentially tight

if $\lim_n \frac{1}{n} \log \int_E e^{nF(x)} \mu_n(dx) =: \Lambda(F)$ exists for any F

$\Rightarrow$ LDP with rate

$$I(x) = \sup_F \left(F(x) - \Lambda(F)\right)$$

Remark: looks like Legendre trafo but it's not! In particular I need not be convex!

Topic of today: Gärtner-Ellis Thm

E topological vectorspace (Hausdorff, regular)

E^* dual space

- μ proba measure on E, denote by φ moment generating fct.

$$\varphi(F) = \int e^{\langle F, y\rangle} \mu(dy) \quad \text{fct on } E^* \to [0,\infty]$$

Def. Legendre Trafo $\Lambda : E \to \mathbb{R} \cup \{\pm\infty\}$

$$\Lambda^*(F) = \sup_{x \in E} \left(\langle F, x\rangle - \Lambda(x)\right)$$

Legendre trafo is always convex & lsc.

Assumption: μ_n proba. meas. on E.

$$\underbrace{\frac{1}{n} \log \underbrace{\int_E e^{n\langle F, y\rangle} \mu_n(dy)}_{=:\varphi_n(nF)}}_{\Lambda_n(F)} \xrightarrow{\text{for every } F} \Lambda(F) \in [-\infty, \infty]$$

Question: does this convergence imply an LDP.

Remark 1: Similar to Fourier trafo in theory of weak convergence.

Guessing the rate fct:

if μ_n is distribution of $\frac{1}{n}\sum_{i=1}^{n} X_i$, then

$$\Lambda_n^{(F)} = \frac{1}{n}\log\int e^{n\langle F, y\rangle}\mu_n(dy)$$

$$= \frac{1}{n}\log \underbrace{\mathbb{E}\left[\exp\left(\sum\langle F, X_i\rangle\right)\right]}_{= \mathbb{E}(\exp(\langle F, X_1\rangle))^n}$$

$$= \log \mathbb{E}\left(\exp\langle F, X\rangle\right)$$

"Convergence is a statement about decorrelation"

We would expect LDP with rate given by Legendre trafo of Λ.

Gärtner - Ellis

Thm (1) "Upper bound on compact sets"

Framework as above, E, E^* "reasonable"

$$\Lambda_n(F) \longrightarrow \Lambda \quad \text{for every } F$$

Then for every compact set $\mathcal{C} \subset E$ we have

$$\overline{\lim_n} \frac{1}{n}\log \mu_n(\mathcal{C}) \leq -\inf_{x\in\mathcal{C}} \Lambda^*(x)$$

Exponential tightness / bound for closed sets has to be checked case by case.

Thm (2) (Upper bound for closed sets in $\mathbb{R}^d$)

If $E = \mathbb{R}^d$ and

$$\Lambda_n(F) \longrightarrow \underset{\underset{(-\infty,\infty]}{\cap}}{\Lambda}(F) \text{ for all } F \in \mathbb{R}^d$$

and Λ finite on a neighborhood of 0.

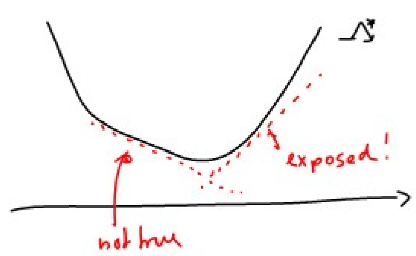
$\Rightarrow$ Λ^* has compact sublevelsets.

& upperbound holds for all closed sets.

Lower bound

Def: We will say that $x \in E$ is exposed for Λ^* if there exists $F \in E^*$ such that for all $y \neq x$

$$\Lambda^*(y) - \Lambda^*(x) > \langle F, y - x\rangle$$



Thm (lower bound)

$$\begin{cases} \cdot\ \Lambda_n(F) \longrightarrow \Lambda(F) \text{ for every } F \\ \cdot\ \mu_n \text{ exponentially tight!} \end{cases}$$

$\Longrightarrow$ for every open set $\mathcal{O}$ we have

$$\varliminf \frac{1}{n} \log \mu_n(\mathcal{O}) \geq - \inf_{x \in \mathcal{O} \cap \mathcal{E}} \Lambda^*(x)$$

$$\mathcal{E} = \left\{ x : x \text{ is exposed for } \Lambda^* \text{ with exposing hyperplane } F ; \ \Lambda(\lambda F) < \infty \text{ for a } \lambda > 1 \right\}$$

Thm 4 (When can we drop $\mathcal{E}$?)

In the case $E = \mathbb{R}^d$, if

- Λ is finite in a neighborhood of 0.
- Λ is C^1 on its domain
- Λ is steep at the boundary

$$\lim_{\substack{x \to \partial D_\Lambda \\ x \in D_\Lambda}} |\nabla \Lambda(x)| = +\infty$$

$\Rightarrow$ then for every open set $\mathcal{O}$

$$\inf_{x \in \mathcal{O} \cap \mathcal{E}} \Lambda^*(x) = \inf_{x \in \mathcal{O}} \Lambda^*(x).$$

Exo ① Verify that this implies Cramér Thm on $\mathbb{R}^d$ under weaker integrability condition.

② $(X_i)_{i \in \mathbb{Z}}$ stationary centered Gaussian process in $\mathbb{R}$.

Let's assume that

$$C_i = \mathbb{E}(X_0 X_i)$$

$$C = \sum_{i \in \mathbb{Z}} C_i < \infty \Rightarrow$$

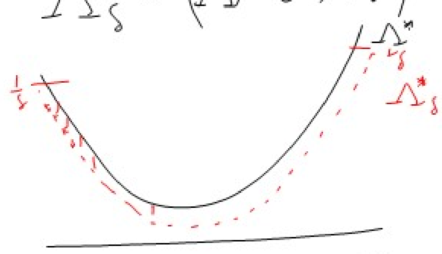
$$Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{satisfy LDP with rate}$$

$$I(x) = \frac{x^2}{2C}$$

Proof: ① Upper bound on compact sets.

Modify Λ^* a bit:

$$\Lambda^*_\delta = \left(\Lambda^* - \delta \wedge \frac{1}{\delta} \right)$$



Fix compact set $\mathcal{C}$. Let $x \in \mathcal{C}$.

Recall that $\Lambda^*(x) = \sup_{F \in E^*} \left(\langle F, x \rangle - \Lambda(F) \right)$

$\Rightarrow \exists F_\delta \in E^*$ such that $\langle F_\delta, x \rangle - \Lambda(F) > \Lambda^*_\delta(x)$

Lets call $\mathcal{O}_x$ the set of points y such that

$$\langle F_x, y\rangle - \Lambda(F_x) > \Lambda^*_\delta(x)$$

this is an open neighborhood of x.

By compactness of $\mathcal{C}$ we have

$$\mathcal{C} \subset \bigcup_{i=1}^{N} \mathcal{O}_{x_i}$$

$$\Rightarrow \quad \mu_n(\mathcal{C}) \leq \sum_{i=1}^{N} \mu_n(\mathcal{O}_{x_i})$$

By the same argument as before, only need to consider $\mu_n(\mathcal{O}_{x_i})$.

$$\mu_n(\mathcal{O}_x) \leq \mu_n\Big(\langle F_x, y\rangle - \Lambda(F_x) > \Lambda^*_\delta(x)\Big)$$

$$\leq \exp(-n\Lambda^*_\delta(x))$$

$$\exp(-n\Lambda(F_x)) \int_E \exp(n\langle F_x, y\rangle)\,\mu_n$$

$$\overline{\lim}\, \frac{1}{n}\log \mu_n(\ldots) \leq -\Lambda^*_\delta(x) - \Lambda(F_x) + \underbrace{\overline{\lim}\, \frac{1}{n}\log \varphi_n(nF_x)}_{\longrightarrow \Lambda(F_x)}$$

$$= -\Lambda^*_\delta.$$

The result follows from letting $\delta \downarrow 0$.

Proof (2) Lets assume that on $B_\delta(0)$ we have

$$\Lambda(x) < c \quad (\text{for some } c)$$

1) Proof compact sublevelsets,

$$\Lambda^*(x) = \sup_{t\in\mathbb{R}^d}\big(t.x - \Lambda(t)\big)$$

$$\geq \sup_{t\in B_\delta(0)} t.x - \Lambda(t)$$

$$\geq \delta|x| - c$$

$$\Rightarrow \text{If } \Lambda^*(x) \leq L \Rightarrow L \geq \delta|x| - c$$

$$\Rightarrow \frac{L+c}{\delta} \geq |x|$$

$\Rightarrow$ Sublevelsets are bdd.

2) We fix L large. We want to estimate

$$\mu_n\big(([-L, L]^d)^c\big)$$

there exist small δ such that for n large enough

$$\frac{1}{n}\log\int \exp(n\delta e_i \cdot y)\,\mu_n(dy) < \tilde{L}$$

$$\Rightarrow \mu_n(y_i \geq L) = \mu_n\big(\exp(n\delta e_i\cdot y) \geq \exp(n\delta L)\big)$$

$$\leq \exp(-n\delta L)\int e^{n\delta e_i y}\,\mu_n$$

for L large enough this kills everything

Intermediate discussion

decorrelation and convexity:

X_i are some r.v. (not nec. independent)
stationary

$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

Formal calculation: $\mathbb{P}\left[Y_n \simeq \lambda x + (1-\lambda) y \right]$

assume $\frac{m}{n} \sim \lambda$

$$Y_n = \underbrace{\frac{m}{n}}_{\lambda} \underbrace{\left(\frac{1}{m}\sum_{i=1}^{m} X_i\right)}_{\overset{d}{=} Y_m} + \underbrace{\frac{n-m}{n}}_{1-\lambda} \underbrace{\left(\frac{1}{n-m}\sum_{i=m+1}^{n} X_i\right)}_{\overset{d}{=} Y_{n-m}}$$

$$\mathbb{P}\left[Y_n \simeq \lambda x + (1-\lambda) y \right] \geq \mathbb{P}\left(\frac{1}{m}\sum_{i=1}^{m} X_i \simeq x \;\&\; \frac{1}{n-m}\sum_{i=m}^{n} X_i \simeq y \right)$$

decorrelation

$$\approx \mathbb{P}\left(\frac{1}{m}\sum_{i=1}^{m} X_i \simeq x\right) \mathbb{P}\left(\frac{1}{n-m}\sum X_i \simeq y\right)$$

given info. if the system

convexity and decorrelation.

Start the proof of LB

$$\mathcal{E} = \{ x : x \text{ is exposed by } F ;\ \Lambda(\gamma F) < \infty \text{ for a } \gamma > 1 \}$$

fix $x \in \mathcal{E}$. We want to prove a lower bound on

$$\mu_n\left(B_\delta(x)\right)$$

Define new measures

$$\nu_n = \frac{\exp\left(n\langle F, y\rangle\right)}{\underbrace{\int \exp\left(n\langle F, y\rangle\right) d\mu_n(y)}_{\varphi_n(nF)}}$$

by ass. $\frac{1}{n}\log \varphi_n(nF) \to \Lambda(F)$

$$\mu_n\left(B_\delta(x)\right) = \int_E \mathbb{1}_{B_\delta(x)}(y) \exp\left(-n\langle F, y\rangle + n\Lambda_n(F)\right) \nu_n(dy)$$

$$\underline{\lim} \frac{1}{n} \log \mu_n(B_\delta(x)) \geq -\langle F, y\rangle - \delta\|F\| + \Lambda(F) + \underline{\lim} \frac{1}{n} \log \nu_n\left(B_\delta(x)\right)$$