Plan extra session Wednesday 14 ? Mon 4 p.m.

Last time: μ_n proba. meas. on E

Varadhan – Bryc: I compact sublevelsets

μ_n satisfies LDP with rate I $\Longleftrightarrow$ $\lim \frac{1}{n} \log \int e^{nF(x)} \mu_n(dx) = \sup_x (F(x) - I(x))$ $\forall F \in C_b(E)$

Gärtner-Ellis

Assumption: E topological n.v.c (Hausdorff regular) vector space.

μ_n proba meas. on E

$$\underbrace{\frac{1}{n} \log \underbrace{\int e^{n\langle F, x\rangle} \mu_n(dx)}_{\varphi_n(nF)}}_{\Lambda_n(F)} \longrightarrow \Lambda(F) \in (-\infty, \infty]$$

$F \in E^*$

Have seen: Upper bound on compact:

$\mathcal{C}$ compact: $\overline{\lim} \frac{1}{n} \log \mu_n(\mathcal{C}) \leq - \inf_{x \in \mathcal{C}} \Lambda^*(x)$

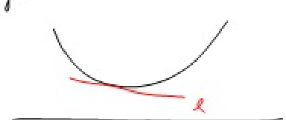
Λ^* = Legendre transform of Λ.

On $\mathbb{R}^n$ this holds for all closed sets as soon as Λ is finite in a neighborhood of 0!

Lower bound: $x \in E$ exposed for Λ^* if

$$\Lambda^*(y) - \Lambda^*(x) > \langle \ell, y - x\rangle$$

for some ℓ (exposing hyperplane)



Lower bound:

$\mathcal{O}$ open. μ_n exponentially tight.

$$\underline{\lim} \frac{1}{n} \log \mu_n(\mathcal{O}) \geq - \inf_{x \in \mathcal{O} \cap \mathcal{E}} \Lambda^*(x)$$

$\mathcal{E} = \{ x : x \text{ exposed by } \ell,\ \Lambda(\ell \gamma) < \infty \text{ for a } \gamma > 1 \}$

proof: $x \in \mathcal{E}$, $\mu_n(B_\delta(x))$

$$\nu_n = \frac{\exp(n\langle \ell, y\rangle)}{\varphi_n(n\ell)} \mu_n(dy)$$

$$\mu_n(B_\delta(x)) = \int_E \mathbb{1}_{B_\delta(x)}(y) \exp(-n(\langle \ell, y\rangle + \Lambda_n(\ell))) \nu_n(dy)$$

$$\frac{1}{n} \underline{\log} \mu_n(B_\delta(x)) \geq (\Lambda_n(\ell) - \langle \ell, x\rangle) - \delta \|\ell\|_{E^*} + \underline{\lim} \frac{1}{n} \log \nu_n(B_\delta(x))$$

Let's proof that $\nu_n(B_\delta^c(x))$ decays to zero exponentially.

For every $N > 0$ we have a compact set $K_N \subset E$ s.t. $\overline{\lim} \frac{1}{n} \log \mu_n(K_N^c) \leq -N$.

We start by bounding

$$\nu_n\big(\underbrace{K_N^c \cap B_\delta^c(x)}_{\text{compact set}}\big)$$

$$\hat{\varphi}_n(nF) := \int_E e^{n\langle F,y\rangle}\,\nu_n(dy)$$

$$= \int_E e^{n\langle F,y\rangle}\frac{e^{n\langle \ell,y\rangle}}{\varphi_n(n\ell)}\,\mu_n(dy).$$

$$= \varphi_n(n\ell)^{-1}\cdot\varphi_n(n(F+\ell))$$

$$\Rightarrow \frac{1}{n}\log\hat{\varphi}_n(nF) = \frac{1}{n}\log\varphi_n(n(F+\ell)) - \frac{1}{n}\log\varphi_n(n\ell)$$

by assumption $\longrightarrow \Lambda(F+\ell) - \Lambda(\ell) = \hat{\Lambda}(F)$

by LD upper bound and compactness of $B_\delta^c(x)\cap K_N$ we know that $\overline{\lim}\,\frac{1}{n}\log\nu_n(B_\delta^c\cap K_N) \leq -\inf_{x\in B_\delta^c\cap K_N}\hat{\Lambda}^*(x)$

$$\hat{\Lambda}^*(y) = \sup_{F\in E^*}\langle F,y\rangle - \hat{\Lambda}(F)$$

$$= \Big[\sup_{F\in E^*}\langle F,y\rangle - \Lambda(F+\ell)\Big] + \Lambda(\ell)$$

$$= \Big(\sup_{F\in E^*}\langle F+\ell,y\rangle - \Lambda(F+\ell)\Big) - \langle \ell,y\rangle + \Lambda(\ell)$$

$$= \Lambda^*(y) - \big(\langle \ell,x\rangle - \Lambda(\ell)\big) + \langle \ell,x-y\rangle$$

$$\geq \Lambda^*(y) - \Lambda^*(x) + \langle \ell,x-y\rangle$$

$> 0 \quad \forall y$ because x is exposed with exposing hyperplane ℓ.

The infimum of rate fct is attained on $B_\delta^c(x)\cap K_N$ (by compactness + l.s.c.) and hence positive. $\Rightarrow$ by LD upper bound the prob. tends to zero exponentially quickly!

It remains to prove that

$$\nu_N(K_N^c) \longrightarrow 0$$

We fix $\rho > 0$. We first bound

$$\nu_n\big(K_N^c \cap \{\langle\ell,y\rangle > \rho\}\big)$$

$$\nu_n\big(K_N^c \cap \{\langle\ell,y\rangle > \rho\}\big)$$

$\gamma n\langle\ell,y\rangle - (\gamma-1)n\langle\ell,y\rangle$

$\gamma > 1$

$$= \int \mathbb{1}_{\{K_N^c\cap\{\langle\ell,y\rangle>\rho\}\}}\frac{e^{n\langle\ell,y\rangle}}{\varphi_n(n\ell)}\,d\mu_n(y)$$

$$\leq e^{-n(\gamma-1)\rho}\,\varphi_n(n\ell)^{-1}\varphi_n(\gamma n\ell)$$

taking $\frac{1}{n}\log$ weget,

$$-(\gamma-1)\rho\underbrace{-\Lambda(\ell)+\Lambda(\gamma\ell)}_{\text{numbers}}$$

by choosing ρ big enough this becomes negative and we are done!

Left. $\nu_n\left(K_N^c \cap \{\langle \ell, y\rangle < \varrho\}\right)$

$$= \int_E \frac{\exp\left(n\langle \ell, y\rangle\right)}{\varphi_n(\ell)} \mathbb{1}_{\{K_N^c \cap \langle \ell, y\rangle < \varrho\}} \, \mu_n(dy)$$

$$\leqslant \varphi_n(\ell)^{-1} \exp(n\varrho)\, \mu_n(K_N^c)$$

$\overline{\lim}\ \frac{1}{n}\log$ " $\leqslant -\Lambda(\ell) + \varrho - N$

by choosing N large enough this is negative

$\Rightarrow$ The proba. decays exp. $\Rightarrow$ we're done $\square$

Try extra session on Thursday 15h $\frac{1}{4}$!

Criterion for $\inf\limits_{O \cap E} = \inf\limits_{O}$!

In $\mathbb{R}^d$: $\cdot\ \Lambda$ defined on neighborhood of 0.

$\cdot\ \Lambda$ differentiable on $\mathcal{D}_\Lambda$.

$\cdot\ \Lambda$ steep at boundary ($x \to \partial\mathcal{D}_\Lambda \Rightarrow |\nabla\Lambda| \to \infty$)

I. E Banach space: Λ defined everywhere + Gateaux diff'ble!

Argument Convex Analysis

differentiability $\xleftrightarrow[\text{Trafo}]{\text{Leg.}}$ strict convexity!

Reference: Rockafellar "Convex Analysis"

Applications: (One can prove Cramér, Sanov, Schilder)

"Notions of Donsker Varadhan theory"

Question: Empirical distributions of Markov processes.

X_n, X_t time discrete / continuous Markov process

Empirical measures:

$$\mu_n(A) = \frac{1}{n}\sum \mathbb{1}_A(X_n)$$

$$\mu_t(A) = \frac{1}{t}\int_0^t \mathbb{1}_A(X_s)\,ds$$

If X_n / X_t ergodic then we know that

$$\frac{1}{N}\sum_{n=1}^{N} F(X_n) \longrightarrow \int F(y)\,d\mu(y)$$

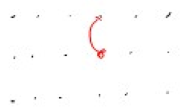
invariant measure.

$$\frac{1}{t}\int_0^t F(X_s)\,ds \longrightarrow \int F(y)\,d\mu(y)$$

In particular $\mu_n / \mu_t \xrightarrow{\text{weakly}} \mu$

Setup (the simplest possible,

Time continuous, Markov chain, on $\mathbb{Z}^d / N\mathbb{Z}^d$



- exponential clock
- jump to neighbor
- wait again ..

$$\Delta_N = \text{discrete Laplacian on } \mathbb{Z}^d / N\mathbb{Z}^d$$

$$\Delta_N(i,j) = \begin{cases} -2d & i = j \\ 1 & i \sim j \\ 0 & \text{else} \end{cases}$$

transition semigroup

$$P_t(i,j) = e^{t \Delta_N}(i,j)$$

proba to jump from i to j in time t.

See that this formula holds,

$$\tilde{\Delta}_N(i,j) = \frac{1}{2d}\left(\Delta_N(i,j) + 2d\, Id\right)$$

$$P_t(i,j) = \sum_{k=0}^{\infty} P(\text{exactly } k \text{ jumps})\, P\left(\begin{array}{l}\text{embedded chain} \\ \text{jumps from } i \text{ to } j \\ \text{in } k \text{ steps}\end{array}\right)$$

$$= \sum_{k \geq 0}^{\infty} \frac{(2dt)^k}{k!} e^{-2dt} \left(\tilde{\Delta}_N\right)^k (i,j)$$

$$= e^{-2dt} e^{2dt \tilde{\Delta}_N}$$

$$= e^{2dt \Delta_N}$$

Aim: look at empirical distribution,

$$\frac{1}{t} \ell_t(i) := \int_0^t \mathbb{1}_{\{i\}}(X_s)\, ds.$$

<u>Thm</u>: $\frac{1}{t} \ell_t$ satisfy an LDP on $\mathcal{M}_1(\mathbb{Z}^d / N\mathbb{Z}^d)$.

The rate function is given by

$$I(\nu) = \left\langle \Delta_N \sqrt{\nu}, \sqrt{\nu} \right\rangle =: \underbrace{\mathcal{E}(\sqrt{\nu}, \sqrt{\nu})}_{\text{Dirichlet form of this jump process.}}$$

<u>proof</u>, We need

1. Convergence of exp. integrals
2. Exponential tightness For free!
3. Λ will be defined everywhere $+ C^1$

We need to calculate

$$\mathbb{E}\left[\exp\left(t \langle \mu_n, F \rangle\right)\right]$$

$$= \mathbb{E}\left[\exp\left(\int_0^t F(X_s)\, ds\right)\right]$$

<u>To treat this</u> look at "Feynman-Kac type" semigroup

$$P_t^F f(x) = \mathbb{E}\left[\exp \int_0^t F(X_s) ds\ f(X_t)\right]$$

P_t^F is a semigroup (Markov property) with generator $\Delta_N + F$.

$$\partial_t P_t^F f(x)\Big|_{t=0} = \mathbb{E}_x\left(\partial_t \exp\left(\int_0^t F(X_s)ds\right) f(X_t)\right) + \mathbb{E}_x\left(\exp(\dots)\,\partial_t f(X_t)\right) \overset{t=0}{=} F(x) + \Delta_N f(x).$$

The largest eigenvalue of $F + \Delta_N$ is isolated!

Argument: $\exists c$ such that $F + \Delta_N + c\,\mathrm{Id}$ has only non-negative entries, & is irreducible.

Perron Frobenius $\Longrightarrow$ $\exists!$ maximal e.v. with multiplicity 1, eigenfunction strictly positive.

$\Longrightarrow$ The same is true for $F + \Delta_N$.

The behaviour of $e^{(\Delta_N + F)t}$ is governed by this largest eigenvalue.

In particular, we will have

$$\mathbb{E}_x\left[e^{\int_0^t F(X_s)ds}\right] = P_t^F \mathbb{1} \sim e^{-\lambda t}.$$

$\Rightarrow \Lambda(F)$ from the theorem is given by $\lambda(F)$ "the biggest EV".

τ' Assumption OK.

$$\Lambda^*(\mu) = \sup_F \langle F, \mu\rangle - \lambda(F)$$

$$\lambda_F = \sup_{\ell \in \ell^2} -\langle \Delta_F \ell, \ell\rangle + \langle F, \ell, \ell\rangle$$

$$= \sup_{\nu \in M_1} -\langle \Delta_F \sqrt{\nu}, \sqrt{\nu}\rangle + \langle F, \nu\rangle$$

Rest next time!