Projective Resolutions for Smooth $G$-modules

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For a locally compact totally disconnected group $G$ which acts on a simplicial complex $X$ we investigate the following:

- the projective dimension of the category of smooth representations of $G$,
- explicit projective resolutions for each smooth $G$-module $V$,
- explicit finitely generated projective resolutions for each smooth $G$-module $V$.  

Katerina Hristova (University of Warwick) Projective Resolutions for Smooth $G$-modules
A representation \((\pi, V)\) of \(G\) over \(\mathbb{F}\) is called \textit{smooth} if for all \(v \in V\) there exists a compact open subgroup \(K_v\) of \(G\) such that \(\pi(k)v = v\) for all \(k \in K_v\).

Denote by \(\mathcal{M}(G)\) the category of all smooth representations of \(G\). It is abelian and has enough projectives.
Theorem (Rumynin, H.)

Let $G$ be a locally compact totally disconnected group. Suppose $G$ acts on a simplicial complex $X = (X_n)$ such that its geometric realisation $|X|$ is contractible of dimension $n$. Suppose further that the stabiliser $G_x$ of any $x \in X_k$ is open and compact. Then

$$\text{proj. dim}(\mathcal{M}(G)) \leq n.$$
Example

1. $\text{SL}_n(\mathbb{K})$, where $\mathbb{K}$ is a non-archimedean local field, with the action on its Bruhat-Tits building $\mathcal{B}T$.

2. Kac-Moody groups with the action on the Davis realisation $\mathcal{D}$ of their buildings.
For $V \in \mathcal{M}(G)$ the resolution looks like

$$0 \rightarrow X_n \otimes V \xrightarrow{d_n} X_{n-1} \otimes V \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_1} X_0 \otimes V \rightarrow V \rightarrow 0,$$

(1)

where the $X_i$’s are the $\mathbb{F}$-vector spaces formally spanned by the elements of $\mathcal{X}_i$, the set of non-degenerate simplices in $\mathcal{X}_i$.

**Problem:** Even if $V$ is f.g., the modules in (1) do not have to be f.g.
Fix a locally compact totally disconnected group \( G \) which acts on a simplicial complex \( \mathcal{X} = (\mathcal{X}_n) \). New objects are needed:

- an **equivariant cosheaf on** \( \mathcal{X} \) is a cosheaf \( C = (C_x) \) of vector spaces on \( \mathcal{X} \) with additional data: a linear map \( g_x : C_x \to C_{gx} \) for all \( g \in G \) and any simplex \( x \), such that:
  
  (i) \( gh_x \circ h_x = (gh)_x \) for any \( g, h \in G \) and a simplex \( x \).
  
  (ii) \( C_x \) is a smooth representation of the simplex stabiliser \( G_x \) for any simplex \( x \).

  \[
  \begin{array}{ccc}
  C_x & \xrightarrow{g} & C_{gx} \\
  \end{array}
  \]

  (iii) The square  \[
  \begin{array}{ccc}
  C\mathcal{X}(f)_x & \xrightarrow{g\mathcal{X}(f)_x} & C\mathcal{X}(gf)_x \\
  \downarrow{C(f,x)} & & \downarrow{C(gf,gx)} \\
  C\mathcal{X}(f)x & \xrightarrow{g\mathcal{X}(f)x} & C\mathcal{X}(gf)x \\
  \end{array}
  \]

  is commutative for all \( g \in G \), simplices \( x \in \mathcal{X}_n \) and nondecreasing maps \( f : [m] \to [n] \).
A system of subgroups $\mathcal{G}$ of $G$ acting on $\mathcal{X}$ is a datum assigning a subgroup $\mathcal{G}_x$ of the simplex stabiliser $G_x$ to each simplex $x \in \mathcal{X}_n$ such that $gG_xg^{-1} = \mathcal{G}_{g^x}$ for all $g \in G$ and $x \in \mathcal{X}_n$.

We call the system

- **open** if each $\mathcal{G}_x$ is open in $G_x$,
- **compact** if each $\mathcal{G}_x$ is compact,
- **contravariant** if for simplices $x$ and $y$, such that $x \subseteq y$ we have $\mathcal{G}_x \subseteq \mathcal{G}_y$. 
Suppose we are given a compact open subgroup $G_x$ for each vertex $x \in \mathcal{X}_0$ such that

(1) $G_{g_x} = gG_xg^{-1}$ for all $g \in G$, $x \in \mathcal{X}_0$ and

(2) $G_xG_y = G_yG_x$ if $x$ and $y$ are adjacent.

We can extend this to a compact open contravariant system of subgroups by defining:

$$G_x := G_{\chi(f_0^n)x}G_{\chi(f_1^n)x} \cdots G_{\chi(f_n^n)x} \text{ for all } x \in \mathcal{X}_n.$$ 

A system obtained by this construction is called an exquisite system.
The Schneider-Stuhler Resolution

Given a contravariant system of subgroups $G$ on $G$ and $V \in \mathcal{M}(G)$ we define an equivariant cosheaf $\mathcal{V}_G^x \cong \mathcal{V}_G^{G \times x}$ on $\mathcal{X} = (\mathcal{X}_n)$ by

$$V_G^{G \times x} := \{ v \in V \mid g \cdot v = v \text{ for all } g \in G_x \}.$$

We call a finitely generated projective resolution of the form $C_\bullet(\mathcal{X}, \mathcal{V}_G^x \cong)$ a Schneider-Stuhler resolution.
Constructing the resolution for $n=1$

**Theorem (Rumynin, H.)**

Let $G$ be a locally compact totally disconnected group which acts smoothly on a tree $T$ such that $T^{(k)}$, for $k = 0, 1$, has finitely many $G$-orbits. Then for an admissible $V \in \mathcal{M}(G)$ which is generated by invariants $V^G_x$ for some $x \in T_0$ and $G$ a geodesic exquisite system of subgroups, the following complex

$$0 \to C_1(T, \underset{\sim}{V^G}) \xrightarrow{d_1} C_0(T, \underset{\sim}{V^G}) \xrightarrow{w} V$$

is a Schneider-Stuhler resolution.
Conjecture (Rumynin, H.)

Let $G$ be a locally compact totally disconnected group which acts smoothly on a simplicial set $\mathcal{X}_\bullet$ of dimension $n$. Suppose that a face of a non-degenerate simplex in $\mathcal{X}_\bullet$ is non-degenerate and $|\mathcal{X}|$ admits a CAT(0)-metric such that the faces are geodesic. Then for $V \in \mathcal{M}(G)$ with same assumptions as in Theorem 1 and $\mathcal{G}$ a geodesic exquisite system of subgroups, the following complex

$$0 \rightarrow C_n(\mathcal{X}_\bullet, V^G) \xrightarrow{d_n} C_{n-1}(\mathcal{X}_\bullet, V^G) \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_1} C_0(\mathcal{X}_\bullet, V^G) \xrightarrow{w} V$$

is a Schneider-Stuhler resolution.