**Kac-Moody Groups**

Let $A$ be a generalised Cartan matrix. To any field $\mathbb{K}$ and a root datum $D$ of type $A$ one can combinatorially associate a Kac-Moody group $G_D(\mathbb{K})$. A few topologies can be put on $G_D(\mathbb{K})$. Taking completions, we obtain different types of complete Kac-Moody groups:

- Carbone-Garland completion $G^{cg}$ with respect to the weight topology,
- Caprace-Rémy completion $G^{cr}$ with respect to the building topology,
- Ronan-Rémy completion $G^{rr}$,
- Capdeboscq-Rumynin completion $G^{ca}$ by putting a local $p$-topology,

which are connected by a sequence of continuous open homomorphisms:

$$G^{ca} \to G^{cr} \to G^{cg} \to G^{rr}.$$  

Kac-Moody groups have $(B, N)$-pairs and Bruhat-Tits buildings. There are also **topological groups of Kac-Moody type** - a more general class of groups with a generalised $(B, N)$-pair structure, which resemble complete Kac-Moody groups [1].

**Smooth Representations**

Throughout $G$ is a locally compact totally disconnected topological group and $F$ is a field of characteristic zero. We call a pair $(\pi, V)$ a **smooth representation** of $G$ if:

1. $V$ is a vector space over $F$ and $\pi : G \to \text{Aut}_F(V)$ is a homomorphism,
2. For every $v \in V$, there exists a compact open subgroup $K_v \leq G$, such that $\pi(k)v = v$, for all $k \in K_v$.

Smooth representations of a locally compact totally disconnected group form a category $\mathcal{M}(G)$. It has some particularly nice properties:

- $\mathcal{M}(G)$ is abelian.
- $\mathcal{M}(G)$ has enough projectives.
- $\mathcal{M}(G)$ is Noetherian.

In particular, for each object in $\mathcal{M}(G)$ we can construct a projective resolution.

**Projective Dimension**

**Theorem 1.** Let $G$ be a locally compact totally disconnected group acting on an $n$-dimensional simplicial complex $(\mathcal{X}, \mathbf{x})$ with contractible geometric realisation $|\mathcal{X}|$. Suppose the stabilisers of non-degenerate simplices are compact and open in $G$. Then

$$\text{proj. dim}(\mathcal{M}(G)) \leq n.$$  

Let $(\pi, V) \in \text{Ob}(\mathcal{M}(G))$. The explicit projective resolution of $V$ looks like:

$$X_0 \otimes V \to X_{n-1} \otimes V \to \cdots \to X_0 \otimes V \to V \to 0$$

with

$$X_k \cong \bigoplus_{x \in X_k(G)} \mathbb{F} G \otimes \mathbb{F}[x], \quad \alpha[g \cdot x] \mapsto g \otimes \alpha[x],$$

where $X_k$ is the $\mathbb{F}$-space spanned by non-degenerate $k$-simplices, $\mathbb{F}[x]$ is the space spanned by the non-degenerate $k$-simplex $x$ and $G_x \leq G$ is the stabiliser of $x$.

**Corollary 2.** If $G$ is a complete Kac-Moody group or a topological group of Kac-Moody type, then

$$\text{proj. dim} \mathcal{M}(G) \leq \sup_{\theta \in \Theta(G)} |J|.$$  

**Localisation**

We have two functors:

$$\mathcal{L} : \mathcal{M}(G) \to \text{Csh}_G(\mathcal{X}), \quad (\pi, V) \mapsto V$$

and

$$\mathcal{H} : \text{Csh}_G(\mathcal{X}) \to \mathcal{M}(G), \quad C \mapsto H_0(\mathcal{X}, C).$$  

**Theorem 4.** Let $\Sigma$ be a class of morphisms $f$ in $\text{Csh}_G(\mathcal{X})$, such that $\mathcal{H}(f)$ is an isomorphism. If $|\mathcal{X}|$ is connected, then

$$\mathcal{H}[\Sigma^{-1}] : \text{Csh}_G(\mathcal{X})[\Sigma^{-1}] \to \mathcal{M}(G)$$

is an equivalence of categories, where $\mathcal{H}[\Sigma^{-1}]$ is the functor induced from $\mathcal{H}$ on the category of left fractions $\text{Csh}_G(\mathcal{X})[\Sigma^{-1}]$.

**Conjecture 5.** There is a quotient of categories

$$H_* : \text{SM}(G) \to D^{\text{loc}}(\mathcal{M}(G)),$$

where $D^{\text{loc}}(\cdot)$ is the coderived category and $\text{SM}(G)$ is the category of simplicial representations of $G$.

**Homological Duality**

The space of all locally constant, compactly supported functions $f : G \to F$ with respect to the convolution product is an $F$-algebra $\mathcal{H}_G$, called the Hecke algebra of $G$. An $\mathcal{H}_G$-module $(M, \cdot)$ is called smooth if $\mathcal{H}_G : M \to M$. Denote the category of smooth $\mathcal{H}_G$-modules by $\mathcal{M}(\mathcal{H}_G)$. There is an equivalence of categories:

$$\mathcal{M}(G) \cong \mathcal{M}(\mathcal{H}_G).$$

**Theorem 6.** $\mathcal{H}_G$ is a dualising bimodule.

**Conjecture 7.** Let $G$ be a topological group of Kac-Moody type with $B$ compact open. For a simple module $M \in \mathcal{M}(G)^B$, its dual $M^*$ is a simple module in $\mathcal{M}(G)^B$. Furthermore, when viewed as $\mathcal{H}_{\partial(G)B}$-bimodules, $M$ and $M^*$ are twists of each other under the Toidi-Matsumoto involution, where $\mathcal{M}(G)^B$ is the full subcategory of $\mathcal{M}(G)$ consisting of smooth representations generated by their $B$-invariants and $\mathcal{H}_{\partial(G)B}$ is the subalgebra of $\mathcal{H}_G$ consisting of $B$-invariant functions.

**References**