PROJECTIVE GEOMETRY

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1. Introduction

Let us consider finite-dimensional vector spaces over a field $\mathbb{F}$, such as the vector space of column vectors $\mathbb{F}^n$. We are now familiar with the machinery of linear algebra as applied to such vector spaces, such as linear transformations, duals and annihilators, bilinear forms etc. In linear algebra, every vector space in particular comes with a distinguished element, the zero vector. Geometrically, we often want to de-emphasise the special role played by the zero vector (or origin). We then refer to n-dimensional space as *affine space* (often the notation $\mathbb{A}^n$ is used in this situation). We have met geometric notions such as lines, planes and hyperplanes in affine geometry, and we know how to compute intersections.

However, doing geometry in vector or affine spaces poses some problems. If, as we usually have done up to now, we work over $\mathbb{R}$ or $\mathbb{C}$, the space has a topology, the Euclidean topology; but this is noncompact. Related to this, intersection theory in affine spaces is complicated due to the presence of special cases. For example, in the plane we have the statement that two distinct lines meet in a unique point *as long as the lines are not parallel.*

Projective geometry is designed to rectify these problems. Roughly speaking, it completes affine space by adding in some points "at infinity". This results in a much nicer intersection theory; for example, we shall see that any two distinct projective lines meet in a unique point in the projective plane. Moreover, if we work over $\mathbb{R}$ or $\mathbb{C}$, there is a natural topology on projective space which makes it *compact*, so we can view it as a compactification of affine space. For these reasons, projective space becomes the natural ambient space in which to consider *algebraic varieties*, sets defined by systems of polynomial equations. The study of such varieties is the focus of the vast subject of algebraic geometry, which underpins much of modern number theory as well as geometry. This course is thus an introduction to some of the basic concepts in algebraic geometry, which are taken much further in the Part B course Algebraic Curves and the Part C course Algebraic Geometry.

In the current course, we shall focus on the more linear aspects of projective geometry, and we shall see that the concepts of linear transformations, duals and bilinear forms you have seen in linear algebra will all find geometric applications here. We refer to [1] as well as [3, Chapter 5] for treatments that go considerably beyond what we have space for in this course. For further work in algebraic geometry, other topics from algebra, notably commutative algebra (including the study of polynomial rings) are fundamental. The book [4] is a good introduction to this subject.

2. Projective space and its linear subspaces

In this section we define the basic objects of study in this course: projective spaces. Let V be a finite-dimensional[1] vector space over a field $\mathbb{F}$. We denote by $\mathbb{F}^*$ the multiplicative group of nonzero elements of $\mathbb{F}$.

Definition 2.1. The *projective space* $\mathbb{P}(V)$ associated to V is the set of 1-dimensional subspaces of V.

We can rephrase this, using the fact that each 1-dimensional subspace L is just the set of multiples of a nonzero vector v, that is[2] $L = \langle v \rangle$. Moreover $\langle v \rangle = \langle w \rangle$ if and only if v is a nonzero scalar multiple of w. This gives us the following equivalent definition.

Definition 2.2. Projective space $\mathbb{P}(V)$ is the quotient of $V \setminus \{0\}$ by the equivalence relation

$$v \sim w \quad \text{iff} \quad v = \lambda w \ \text{ for some } \lambda \in \mathbb{F}^*.$$

Equivalently, in the language of group actions,

$$\mathbb{P}(V) = (V \setminus \{0\})/\mathbb{F}^*$$

is the space of orbits of the $\mathbb{F}^*$-action by scalar multiplication on $V \setminus \{0\}$.

Points of the projective space $\mathbb{P}(V)$ will often denoted $[v]$, for any (nonzero) representing vector $v \in V \setminus \{0\}$, meaning the orbit of v under the $\mathbb{F}^*$-action.

If $\dim V = 2$, then $\mathbb{P}(V)$ is called the *projective line*; if $\dim V = 3$, then $\mathbb{P}(V)$ is called the *projective plane*. This reflects our intuition that factoring out the $\mathbb{F}^*$-action has lowered the dimension by one. We *define* the dimension[3] of $\mathbb{P}(V)$ as a projective space to be $\dim V - 1$. Note that this can take the value -1 if $\dim V = 0$ and therefore $\mathbb{P}(V)$ is *empty*; this convention will be useful below.

If U is a linear subspace of V, then $\mathbb{P}(U)$ is a subset of $\mathbb{P}(V)$ called a *(projective) linear subspace*, of dimension $\dim U - 1$. In particular, if $\dim U = 2$, we obtain the notion of a *projective line* (usually just referred to as a line) in $\mathbb{P}(V)$. If $\dim U = \dim V - 1$, then we call $\mathbb{P}(U)$ a *hyperplane*.

The following statement would also be true in ordinary (affine) geometry.

Lemma 2.3. *Through any two distinct points P, Q in $\mathbb{P}(V)$, there is a unique projective line L.*

Proof. Let $P = [p] \neq Q = [q]$ in $\mathbb{P}(V)$, so the vectors $p, q \in V$ are linearly independent. The unique line containing P, Q is now $L = \mathbb{P}\langle p, q \rangle$. □

[1] The initial definitions also make sense for an infinite-dimensional vector space. However, many of our later results rely on the space being finite-dimensional, so we make this assumption once and for all.

[2] We will denote by $\langle v_1, \ldots, v_n \rangle$ the linear span of the vectors $v_1, \ldots, v_n$ of V, the vector subspace of V of all linear combinations of the v_i.

[3] If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then this is equal to the dimension of projective space as a *manifold*.

We can also see immediately that intersection properties are nicer in projective space than in a vector space; the following statement is of course *false* in affine geometry.

Proposition 2.4. *In the projective plane, any two distinct projective lines meet in a unique point.*

Proof. We can write the projective plane as $\mathbb{P}(V)$ for a 3-dimensional vector space V. The projective lines are $\mathbb{P}(U_1), \mathbb{P}(U_2)$ for two distinct 2-dimensional subspaces U_1 and U_2 of V. Now recall the formula

$$\dim(U_1 + U_2) + \dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2).$$

As U_1, U_2 are distinct 2-dimensional subspaces, the sum $U_1 + U_2$ strictly contains U_1 and hence is of dimension greater than 2, so is the full 3-dimensional space V. Hence the formula shows $U_1 \cap U_2$ is 1-dimensional, and this represents the unique point in projective space where $\mathbb{P}(U_1)$ meets $\mathbb{P}(U_2)$. □

The statements of Lemma 2.3 and Proposition 2.4 can be generalized substantially as follows. First, we define the (projective) span $\langle L_1, L_2\rangle$ of two projective linear subspaces of a projective space $\mathbb{P}(V)$. This is given by the following construction: let $L_i = \mathbb{P}(U_i)$ for linear subspaces $U_i \subset V$. Then

$$\langle L_1, L_2\rangle = \mathbb{P}(U_1 + U_2).$$

See Problem Sheet 1 for a geometric interpretation of the span.

Theorem 2.5. (Projective Dimension of Intersection Formula) *Let L_1, L_2 be two projective linear subspaces of a projective space $\mathbb{P}(V)$. Then we have*

$$\dim(L_1 \cap L_2) = \dim(L_1) + \dim(L_2) - \dim\langle L_1, L_2\rangle.$$

Here the convention $\dim \emptyset = -1$ is in force.

Note what the last pronouncement of the Theorem says: the intersection of two projective linear subspaces is empty if and only if there is a numerical reason for this to be so.

3. Coordinates on projective space

Vector spaces and linear transformations between them may be viewed either abstractly, or else more concretely by choosing a basis and studying the matrix representation of the linear transformation. More concretely, we can think of this process as 'choosing coordinates'.

What is an appropriate coordinate system for projective spaces? Given an ordered basis $\{e_0, \ldots, e_n\}$ for an $(n+1)$-dimensional vector space V, a vector $v = \sum_{i=0}^n x_i e_i$ is thought of as represented by coordinates $(x_0, x_1, \ldots, x_n)$. The choice of basis has set up an identification of V with the vector space $\mathbb{F}^{n+1}$.

The projective space $\mathbb{P}(V) = \mathbb{P}(\mathbb{F}^{n+1})$ is often denoted by $\mathbb{FP}^n$. In this space $\mathbb{FP}^n$, each point is represented by an equivalence class of $(n+1)$-tuples, where $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ iff there exists $\lambda \in \mathbb{F}^*$ such that $y_i = \lambda x_i$ for $i = 0, 1, \ldots, n$. Recall also that the x_i are not allowed to be all zero.

We use the notation

$$[x_0 : \ldots : x_n]$$

to represent a point in projective space, so that not all the x_i are zero, and

$$[x_0 : \ldots : x_n] = [\lambda x_0 : \ldots : \lambda x_n]$$

for any $\lambda \in \mathbb{F}^*$. This construction is referred to as a system of *homogeneous coordinates* on $\mathbb{P}(V) = \mathbb{P}(\mathbb{F}^{n+1})$.

Let us consider the simplest case, ie. the projective line $\mathbb{FP}^1$, where points are represented by homogeneous coordinates $[x_0 : x_1]$. If $x_0 = 0$, then we just get one point $[0 : x_1] = [0 : 1]$, since x_1 is a nonzero scalar. If on the other hand, $x_0 \neq 0$, then we may write $[x_0 : x_1] = [1 : t]$ where $t = x_1/x_0$ is an arbitrary element of the field $\mathbb{F}$. So, we have written the projective line as the disjoint union of two sets, one of which is a point $[0 : 1]$ and one of which is a copy of the affine line $\mathbb{F}$. Moreover, as $[1 : t] = [t^{-1} : 1]$ for t nonzero, we can think of the point $[0 : 1]$ as corresponding to letting the coordinate t on the affine line tend to infinity.

Examples 3.1. (i) If we take $\mathbb{F} = \mathbb{R}$, then we can think of the projective line $\mathbb{RP}^1$ as the circle S^1.

(ii) If $\mathbb{F} = \mathbb{C}$, then the above argument shows that projective line $\mathbb{CP}^1$ is the same as the extended complex plane $\mathbb{C} \cup \{\infty\}$ which you have studied in Part A Complex Analysis. As you saw in that course, the extended plane may also be viewed as a 2-dimensional sphere, the Riemann sphere.

More generally, we may decompose n-dimensional projective space $\mathbb{FP}^n$ as the union of 2 sets

$$S_\infty = \{[x_0 : \ldots : x_n] \mid x_0 = 0\}$$

and

$$S_{\text{aff}} = \{[x_0 : \ldots : x_n] \mid x_0 \neq 0\}.$$

Clearly, S_∞ may be identified with projective space $\mathbb{FP}^{n-1}$ of dimension one lower. In S_{aff}, every point may be written as $[1, t_1, \ldots, t_n]$ where $t_i = x_i/x_0$, and this sets up an identification of S_{aff} with $\mathbb{F}^n$. So we have a decomposition

$$\mathbb{FP}^n = \mathbb{F}^n \bigsqcup \mathbb{FP}^{n-1}. \tag{3.2}$$

Intuitively, we are adding some points at infinity to affine space to obtain projective space. As we mentioned in the Introduction, this ensures that projective space has nicer properties than affine space, especially as regards intersection theory, as in Theorem 2.5 above, or its special case Proposition 2.4. Indeed, we can see that parallel lines in affine space $\mathbb{F}^2$ generate projective lines in $\mathbb{FP}^2$ that meet in the line $S_\infty = \mathbb{FP}^1$ at infinity.

It is important to realise that the decomposition (3.2) is not canonical. We could, for example, choose any other coordinate x_i and decompose projective space according to whether x_i is zero or nonzero.

In fact, it is often useful to consider the subsets $\mathcal{U}_i$ of $\mathbb{FP}^n$ called *affine patches*, given by

$$\mathcal{U}_i = \{[x_0, \ldots, x_n] : x_i \neq 0\}.$$

The sets $\mathcal{U}_i$ cover $\mathbb{FP}^n$, as every point in $\mathbb{FP}^n$ has *some* coordinate x_i nonzero. As above, each $\mathcal{U}_i$ may be identified with $\mathbb{F}^n$. So we have covered[4] projective space by open sets each with an identification with affine space.

Let us return now to linear subspaces $\mathbb{P}(U)$ where U is a subspace of a vector space V of dimension $n+1$. The subspace U may be viewed as defined by the vanishing of a system of some m linear equations

$$\sum_{j=0}^{n} a_{ij}x_j = 0 \text{ for } i = 1, \dots, m.$$

These equations are *homogeneous* in the x_i variables of degree 1: denoting

$$f_i(x_0, \dots, x_n) = \sum a_{ij}x_j,$$

we have

$$f_i(\lambda x_0, \dots, \lambda x_n) = \lambda f_i(x_0, \dots, x_n).$$

This means that the equations $f_i(x_0, \dots, x_n) = 0$ are well defined on projective space. The locus defined by the equations

$$f_i(x) = 0$$

in projective space $\mathbb{FP}^n$ is of course just the linear subspace $\mathbb{P}(U)$.

Example 3.3. Consider the line in *affine* space $\mathbb{R}^2$ with equation $y = 2x$. We can complete this to a *projective* line in $\mathbb{RP}^2$ by embedding $\mathbb{R}^2$ in $\mathbb{RP}^2$ via $(x, y) \mapsto [1 : x : y]$. Now the projective line is given by the projectivisation of the 2-dimensional subspace of $\mathbb{R}^3$ spanned by $(1, 1, 2)$ and $(0, 1, 2)$. The latter represents the 'point at infinity' $[0 : 1 : 2]$ that we add to the affine line to get the projective line. In terms of homogeneous coordinates $[x_0 : x_1 : x_2]$, the projective line has equation $x_2 - 2x_1 = 0$, so is defined by the vanishing of a single homogeneous degree 1 polynomial. ♢

In general projective lines in $\mathbb{FP}^2$ will be given by equations

$$a_0x_1 + a_1x_1 + a_2x_2 = 0,$$

where (a_0, a_1, a_2) are not all zero. Scaling (a_0, a_1, a_2) by a nonzero λ leaves the line unchanged. In this way, lines in $\mathbb{FP}^2$ correspond to points $[a_0 : a_1 : a_2]$ in a different projective plane! We shall return to this idea in §5 when we discuss duality.

Remark 3.4. The idea of defining subsets of projective space by homogeneous polynomials can also be applied to higher degree polynomials. We say that a polynomial $P(x_0, \dots, x_n)$ is *homogeneous* of degree d, if there exists a positive integer d such that

$$P(\lambda x_0, \dots, \lambda x_n) = \lambda^d P(x_0, \dots, x_n)$$

for all $(x_0, \dots, x_n)$; equivalently, all the terms in $P(x_i)$ are of total degree d. Homogeneity is the condition that ensures that the equation

$$P(x_0, \dots, x_n) = 0$$

[4]If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then this endows $\mathbb{RP}^n$ and $\mathbb{CP}^n$ with the structure of an n-dimensional *manifold* and n-dimensional *complex manifold* respectively. More on this in the Part C course Differentiable Manifolds.

is well-defined on projective space. A *projective algebraic variety* is a subset of projective space defined by a system of homogeneous polynomial equations. If the equations are all of degree 1, then we recover linear subspaces. We shall later investigate the case of *quadrics*, which are defined by a single homogeneous quadratic polynomial.

Remark 3.5. (Non-examinable) If we take the field $\mathbb{F}$ to be $\mathbb{R}$ or $\mathbb{C}$ then in fact we can put a topology on projective space, related to the Euclidean topology[5] on $\mathbb{R}^n$ or $\mathbb{C}^n$.

For $\mathbb{RP}^n$, this proceeds by observing that Definition 2.2 is equivalent to

$$\mathbb{RP}^n = \{v \in S^n \subset \mathbb{R}^{n+1}\}/(v \sim -v),$$

where S^n is the unit sphere in $\mathbb{R}^{n+1}$. This is because every nonzero vector $v = (x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ may be scaled by $\mathbb{R}^*$ to an element of length one, which is unique up to replacing v by $-v$. So we have exhibited real projective space as the quotient of the sphere by an action of the finite group C_2. We can thus endow real projective space with the quotient topology, which is *compact* (as the sphere is a compact subset of Euclidean space) and *Hausdorff* (as it is the quotient of a Hausdorff space by the action of a finite group). Similar ideas may be used to topologise complex projective space.

Real and complex projective spaces may thus be viewed as compactifications of the corresponding affine spaces. In particular, the projective lines over these fields are the one-point compactifications of $\mathbb{R}$ and $\mathbb{C}$ respectively. In the complex case, as we alluded to above, we may view the projective line as the Riemann sphere.

4. Projective transformations

Whenever we introduce a class of mathematical objects, we are also interested in the transformations between them. We have defined projective spaces in terms of quotients of vector spaces. It is therefore natural to consider maps of projective spaces induced by linear maps of vector spaces. The obvious definition is

$$\tau : [v] \mapsto [Tv]$$

where $[v]$ is the point of projective space represented by $v \in V - \{0\}$, and $T \colon V \to W$ is a linear map.

There are two potential problems we must consider first, however. One, as always with defining maps on quotient spaces, is to check that the map is welldefined. That is, we must check that if $[v] = [w]$ then $[Tv] = [Tw]$. In our situation this is clear from the linearity of T, and the fact that $[v] = [w]$ if and only if v is a nonzero scalar multiple of w. The second problem is that only nonzero vectors represent points of projective space, so we need Tv to be nonzero whenever v is, that is, we need T to be injective.

[5]For general fields $\mathbb{F}$, we do not have an analogue of the Euclidean topology on $\mathbb{F}^n$, so these ideas are not applicable. In algebraic geometry there is a standard topology for projective spaces over general fields, the *Zariski topology*, but it has very different properties; in particular it has fewer open sets and is not Hausdorff.

Definition 4.1. If $T : V \to W$ is an injective linear transformation, we define the associated *projective linear transformation*

$$\tau \colon \mathbb{P}(V) \to \mathbb{P}(W)$$

by

$$[v] \mapsto [Tv].$$

We are generally interested in the case when $V = W$ and thus T is invertible.

Note that any nonzero scalar multiple of T represents the same projective transformation as does T. In fact the assignment $T \mapsto \tau$ defines a homomorphism from $GL(V)$, the group of invertible linear transformations of V, onto the group of projective linear transformations of $\mathbb{P}(V)$. The kernel of this map is the (normal) subgroup of scalar invertible linear transformations, that is, nonzero scalar multiples of the identity. Therefore, using the first isomorphism theorem for groups, we can make the definition:

Definition 4.2. The group of projective linear transformations of $\mathbb{P}(V)$ is

$$PGL(V) = GL(V)/\{\lambda I : \lambda \in \mathbb{F}^*\}.$$

More concretely, if we identify V with $\mathbb{F}^{n+1}$, then we write the group $PGL(V)$ as $PGL(n+1, \mathbb{F})$, the quotient of the group of size $n+1$ invertible matrices over $\mathbb{F}$ by the subgroup of nonzero scalar matrices.

Of course we can write projective transformations in terms of homogeneous coordinates. We illustrate this in the case of the projective line.

Example 4.3. Consider an invertible linear map $T : \mathbb{F}^2 \to \mathbb{F}^2$ given by

$$T : (x, y) \mapsto (ax + by, cx + dy)$$

with $ad - bc \neq 0$. Then in projective coordinates, the effect of the corresponding projective transformation τ is

$$\tau : [x : y] \mapsto [ax + by : cx + dy].$$

Working on an affine patch $y \neq 0$, we can rewrite the associated projective linear transformation of $\mathbb{FP}^1$ as:

$$\left[\frac{x}{y} : 1\right] \mapsto \left[\frac{ax + by}{cx + dy} : 1\right],$$

so in terms of the affine coordinate $t = \frac{x}{y}$ as

$$t \mapsto \frac{at + b}{ct + d}.$$

In the case $\mathbb{F} = \mathbb{C}$, we have encountered these transformations before: they are the *Möbius transformations* of the Riemann sphere $\mathbb{CP}^1$. The point at infinity ∞ in the Riemann sphere is just identified with $[1 : 0]$. ♢

We may recall from complex analysis the result that given an ordered triple of distinct points in the Riemann sphere, there is a unique Möbius transformation sending the triple to $(0, 1, \infty)$. Hence the group $PGL(2, \mathbb{C})$ of Möbius transformations acts transitively on the set of ordered triples of distinct points in the projective line.

What does the condition that the points are distinct mean in terms of projective geometry? Well, two points in projective space are equal if and only if their representative vectors are proportional, which for two vectors is equivalent to saying they are dependent. This motivates the following definition, which generalises the $n = 1$ case just discussed.

Definition 4.4. In the n-dimensional projective space $\mathbb{P}(V)$ for an $(n+1)$-dimensional vector space V over $\mathbb{F}$, we say that $n+2$ points are *in general position*, if each subset of $n+1$ of these points is represented by linearly independent representative vectors.

We have the following theorem, which generalises the above result about Möbius transformations.

Theorem 4.5. *(General Position Theorem) Let $X_0, \ldots, X_{n+1}$, respectively $Y_0, \ldots, Y_{n+1}$ be two $(n+2)$-tuples of points in n-dimensional projective space $\mathbb{P}(V)$, such that each $(n+2)$-tuple is in general position. Then there exists a unique projective linear transformation τ such that $\tau(X_i) = Y_i$ for each i.*

Proof. Let $X_i = [v_i]$ for $i = 0, \ldots, n+1$, that is, $v_i \in V$ are representative vectors for X_i. The general position hypothesis implies that $v_0, \ldots, v_n$ form a basis for the vector space V. Then for the last point X_{n+1}, we have

$$v_{n+1} = \sum_{i=0}^{n} \lambda_i v_i$$

for some scalars λ_i.

Now, all λ_i are nonzero, again using the general position hypothesis: if one were to be zero, then we would get a dependency relation between v_{n+1} and n of the other v_i. So we may in fact replace v_i by $\lambda_i v_i$ and take

$$v_{n+1} = \sum_{i=0}^{n} v_i$$

as representative vector for our last point. Again using the general position hypothesis, this representation of v_{n+1} is unique.

Similarly we can take $Y_i = [w_i]$ for $i = 0, \ldots, n+1$, with $w_{n+1} = \sum_{i=0}^{n} w_i$, where $w_0, \ldots, w_n$ is another basis of V.

Now there exists an invertible linear transformation T of V with $T(v_i) = w_i$ for $i = 0, \ldots, n$. Linearity and the formulae for v_{n+1}, w_{n+1} imply that $T(v_{n+1}) = w_{n+1}$ also, as required.

If S is another linear transformation inducing a projective transformation with the required property, then $Sv_i = \mu_i w_i$ for $i = 0, \ldots, n+1$, where μ_i are nonzero scalars. Now

$$\mu_{n+1} w_{n+1} = S v_{n+1} = \sum_{i=0}^{n} S v_i = \sum_{i=0}^{n} \mu_i w_i,$$

so $w_{n+1} = \sum_{i=0}^{n} (\mu_i / \mu_{n+1}) w_i$ and by uniqueness of this representation we see all the μ_i are equal. Hence $S = \mu T$ and they induce the same projective map. □

Remark 4.6. A coordinate-based rephasing of the result of the first part of the argument above is that if $P_0, \ldots, P_{n+1}$ are points in the n-dimensional projective space $\mathbb{P}(V)$ in general position, then there is a unique coordinate system in which the points are represented by the projective coordinates

$$P_0 = [1 : 0 : \ldots : 0], \quad P_1 = [0 : 1 : 0 : \ldots : 0], \quad \ldots, \quad P_n = [0 : 0 : \ldots : 0 : 1]$$

and

$$P_{n+1} = [1 : 1 : \ldots : 1].$$

Example 4.7. In the projective plane, 4 points are in general position if and only if no 3 are collinear. We see that any two such quadruples in the plane are projectively equivalent: any two quadrilaterals in the projective plane are projectively equivalent. ♢

5. Some classical theorems of projective geometry

We are now in a position to prove a celebrated classical result of projective geometry, Desargues's Theorem. We are going to give a slick argument which is an application of the General Position Theorem. Other proofs are possible, some more geometric than the one given here.

Theorem 5.1. *(Desargues) Let P, A, A', B, B', C, C' be seven distinct points in a projective space such that the lines AA', BB' and CC' are distinct and concurrent at P. Then the points of intersection $AB \cap A'B', BC \cap B'C'$, $CA \cap C'A'$ are collinear.*

Proof. As in the proof of the General Position Theorem, above we can choose representative vectors p, a, a', b, b', c, c' for our points such that

$$\begin{aligned} p &= a + a', \\ p &= b + b', \\ p &= c + c'. \end{aligned}$$

Now these equations imply $a - b = b' - a'$, so $a - b$ is a representative vector for $AB \cap A'B'$. Similarly $b - c$ and $c - a$ are representative vectors for $BC \cap B'C'$ and $CA \cap C'A'$ respectively.

But $(a - b) + (b - c) + (c - a) = 0$, so these three representative vectors are linearly dependent, hence the points they represent are collinear. □

The Theorem of Pappus is another classical result that may be proved using general position arguments, in this case by standardising the points A, B, C', B' (say) into a simple form, and then explicitly calculating intersections.

Theorem 5.2. *(Pappus) Let A, B, C and A', B', C' be two collinear triples of distinct points in the projective plane. Then the three points $AB' \cap A'B$, $BC' \cap B'C$ and $CA' \cap C'A$ are collinear.*

Proof. Exercise on Problem Sheet. □

6. The axiomatics of projective planes

This section is off-syllabus.

As an alternative to our construction through linear algebra, projective planes can also be introduced using an axiomatic approach. In this approach, an *abstract projective plane* consists of collections $\mathcal{P}$ of points and $\mathcal{L}$ of lines, as well as an incidence relation $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$, describing which point lies on which line. These sets should satisfying the following:

- given two distinct points $P, Q \in \mathcal{P}$, there is a unique line $L \in \mathcal{L}$ containing them;
- any two lines $L_1, L_2 \in \mathcal{L}$ have at least one point $P \in \mathcal{P}$ in common;
- any line $L \in \mathcal{L}$ contains at least three points;
- there are at least two distinct lines $L_1, L_2 \in \mathcal{L}$.

Then it is clear from our discussions that for $\mathbb{F}$ a field and V a three-dimensional vector space over $\mathbb{F}$, the projective plane $\mathbb{F}\mathbb{P}^2 = \mathbb{P}(V)$, with its standard points, lines and their incidence relation, gives an abstract projective plane. The smallest abbstract projective plane is the order 2 *Fano plane*, which is the same as $\mathbb{F}_2\mathbb{P}^2$. (The *order* of abstract projective plane is one less than the number of points in any projective line.) Conversely, it can be proved that any order 2 abstract projective plane is (isomorphic to) the Fano plane.

However, it turns out that in general, projective planes over fields do not exhaust the set of all abstract projective planes. Indeed, there are four non-isomorphic abstract projective planes of order 9, the usual $\mathbb{F}_9\mathbb{P}^2$ and three further planes in which Desargues' Theorem does not hold. Moreover, the only *known* finite abstract projective planes have an order which is the power of a prime. There is no abstract projective plane of order 10, but this is only known to be true using lengthy computer elimination. It is still an open problem as to whether there is a abstract projective plane of order 12.

Hilbert was the first to appreciate that abstract projective planes need not be Desarguesian. It turns out that Desargues theorem holds in an abstract projective plane if and only if it is (isomorphic to) $\mathbb{P}(V)$, where V is a "vector space" over a division ring R. A division ring satisfies all the axioms of a field, other than the commutativity of multiplication. In turn, a Desarguesian projective plane can be expressed as a projective space of a vector space over a field if and only if Pappus' Theorem holds.

7. Cross-ratio

Let us return to the case of the projective line. We know that any two triples of distinct points are equivalent under the action of the projective linear group. What can we say about quadruples?

It turns out that there is a single numerical invariant which distinguishes orbits of quadruples of distinct points in the projective line under the projective group.

Definition 7.1. Let $P_i = [\xi_i : \eta_i]$ for $i = 0, \dots, 3$ be four distinct points in the projective line $\mathbb{F}\mathbb{P}^1$. The *cross-ratio* of the ordered quadruple is

$$(P_0P_1 : P_2P_3) = \frac{(\xi_0\eta_2 - \xi_2\eta_0)(\xi_1\eta_3 - \xi_3\eta_1)}{(\xi_0\eta_3 - \xi_3\eta_0)(\xi_1\eta_2 - \xi_2\eta_1)}$$

We can observe that if we scale any pair (ξ_i, η_i) then the numerator and denominator both scale by the same factor, so the quotient on the right hand side is unchanged. The cross-ratio is therefore well-defined.

Moreover, under projective transformations

$$\begin{pmatrix} \xi_i & \xi_j \\ \eta_i & \eta_j \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_i & \xi_j \\ \eta_i & \eta_j \end{pmatrix}$$

so the cross terms $\xi_i\eta_j - \xi_j\eta_i$ scale by the (nonzero!) determinant $ad - bc$, and hence the cross-ratio is invariant. We have shown

Proposition 7.2. *Cross-ratio is a projective invariant.*

So any two quadruples that are projectively equivalent must have the same cross-ratio. What about the converse? We would like to show that any two quadruples with the same cross-ratio are projectively equivalent. Stated like that, it seems like an involved calculation, but we can greatly simplify it by using the action of the projective group (which leaves the cross-ratio unchanged!) to standardise three of the points. Explicitly, we can move P_0, P_1, P_2 to $[1:0], [0:1], [1:1]$ by a projective transformation by Remark 4.6. Now

$$(P_0P_1 : P_2P_3) = \frac{(1.1 - 0.1)(0.\eta_3 - \xi_3.1)}{(1.\eta_3 - 0.\xi_3)(0.1 - 1.1)} = \frac{\xi_3}{\eta_3}$$

As the points are distinct, we may write $P_3 = [\lambda : 1]$ for a unique $\lambda \neq 0, 1$, and now the cross-ratio $(P_0P_1 : P_2P_3)$ is simply equal to λ. So any quadruple $\{P_0, P_1, P_2, P_3\}$ of distinct points of the projective line $\mathbb{FP}^1$ is projectively equivalent to the quadruple

$$\{[1:0], [0:1], [1:1], [\lambda:1]\},$$

where λ is the value of the cross-ratio. We have proved the following result.

Theorem 7.3. *Two quadruples of distinct points in the projective line* $\mathbb{FP}^1$ *are projectively equivalent if and only if their cross-ratios are equal.*

Notice that the cross-ratio does not take the values 0 or 1. The cross-ratio thus sets up a bijection between

(i) quadruples of distinct points in $\mathbb{FP}^1$ modulo the action of the projective linear group $PGL(2, \mathbb{F})$, and
(ii) the affine line $\mathbb{F}$ with points $0, 1$ removed (or equivalently, the projective line $\mathbb{FP}^1$ with points $0, 1, \infty$ removed[6]).

We conclude by remarking that the cross-ratio has some interesting symmetries.

Theorem 7.4. *The cross-ratio obeys the following equations:*

$$(P_0P_1 : P_2P_3) = (P_1P_0 : P_3P_2) = (P_2P_3 : P_0P_1),$$

$$(P_0P_1 : P_2P_3) = (P_1P_0 : P_2P_3)^{-1},$$

$$(P_0P_2 : P_1P_3) = 1 - (P_0P_1 : P_2P_3).$$

Proof. See the Problem Sheet. □

[6]The question of whether this bijection can be completed to include the points $0, 1, \infty$ by allowing members of the quadruple to coincide suitably is a subtle one that leads into the branch of algebraic geometry known as Geometric Invariant Theory.

8. Duality

We shall now apply some more linear algebra technology to projective geometry. We recall that to any vector space V over $\mathbb{F}$ we can associate the *dual space* V^* of linear maps $f : V \to \mathbb{F}$. In the finite dimensional case, V and V^* are isomorphic, since they are of equal dimension; however this isomorphism depends on a choice of basis and so is not canonical. However the double dual V^{**}, that is, the dual of V^*, *is* canonically isomorphic to V. Explicitly, the map

$$\phi : V \to V^{**}$$

defined by

$$\phi(v) : f \mapsto f(v) \quad (f \in V^*)$$

defines an isomorphism between V and V^{**}.

We have an inclusion-reversing correspondence between subspaces of V and subspaces of V^*, given by associating to $U \leq V$ its annihilator

$$U^\circ = \{f \in V^* : f(u) = 0 \text{ for all } u \in U\}.$$

We recall the following results from part A linear algebra

Proposition 8.1. *For subspaces U, U_1, U_2 of a finite-dimensional vector space V, we have*

- *(i) if $U_1 \leq U_2$, then $U_2^\circ \leq U_1^\circ$; that is, taking the annihilator reverses inclusion;*
- *(ii) $(U_1 + U_2)^\circ = U_1^\circ \cap U_2^\circ$;*
- *(iii) $(U_1 \cap U_2)^\circ = U_1^\circ + U_2^\circ$;*
- *(iv) $\dim U + \dim U^\circ = \dim V$;*
- *(v) $(U^\circ)^\circ = \phi(U)$.*

The last statement follows from the obvious fact that $\phi(U) \leq (U^\circ)^\circ$, and the dimension formula (iv).

We shall usually use the canonical isomorphism ϕ to identify spaces with their double duals, and subspaces with their double annihilators, without further comment.

Turning to projective spaces, we obtain an inclusion-reversing *duality correspondence*

$$\{\text{linear subspaces } \mathbb{P}(U) \subset \mathbb{P}(V)\} \longleftrightarrow \{\text{linear subspaces } \mathbb{P}(U^\circ) \subset \mathbb{P}(V^*)\}.$$

By the dimension formula, if $\mathbb{P}(U)$ is an m-dimensional linear subspace of $\mathbb{P}^n = \mathbb{P}(V)$, then U has dimension $m+1$, so U° has dimension $(n+1) - (m+1) = n-m$, and hence $\mathbb{P}(U^\circ)$ is a linear subspace of $\mathbb{P}(V^*)$ of dimension $n - m - 1$.

In particular, with $\dim V = n + 1$, points of $\mathbb{P}(V^*)$, which represent 1-dimensional subspaces of V^*, correspond to hyperplanes in $\mathbb{P}(V)$, which represent n-dimensional subspaces of V. This is just the assigment to $\langle f \rangle$, where $f \in V^* - \{0\}$, of the hyperplane $\mathbb{P}(\ker(f))$ in $\mathbb{P}(V)$. In terms of homogeneous coordinates, the point $[a_0 : \ldots : a_n]$ in the dual projective space $\mathbb{P}(V^*)$ corresponds to the hyperplane $a_0x_0 + \ldots a_nx_n = 0$ in $\mathbb{P}(V)$; note that scaling all the a_i does not alter the hyperplane. Conversely, hyperplanes in $\mathbb{P}(V^*)$ correspond to points in $\mathbb{P}(V^{**})$ and thus to points in $\mathbb{P}(V)$.

For the projective plane, the duality interchanges points and lines. If $P = [p], Q = [q]$ are two distinct points on the line $L = \mathbb{P}U$ with $U = \langle p, q \rangle$, then the lines $\mathbb{P}\langle p \rangle^\circ, \mathbb{P}\langle q \rangle^\circ$ meet at the point $\mathbb{P}U^\circ$. More generally, a set of collinear points corresponds under duality to a set of concurrent lines. We can interpret $\mathbb{P}\langle x \rangle^\circ$ as the locus in the dual plane parametrising lines through $[x]$ in the original plane.

Notice that for the projective plane, Lemma 2.3 and Proposition 2.4 are dual to each other, in the sense that we get one from the other via duality. In general, each theorem in projective geometry will have a dual version. Moreover, having proved the theorem in all projective spaces $\mathbb{P}(V)$, the result applies equally well to the dual projective space $\mathbb{P}(V^*)$ and so the dual theorem is a free biproduct of the original theorem.

Example 8.2. The dual of Desargues's Theorem in the plane is as follows. Let $\pi, \alpha, \alpha', \beta, \beta', \gamma, \gamma'$ be seven distinct lines in a projective plane such that the points $\alpha \cap \alpha', \beta \cap \beta'$ and $\gamma \cap \gamma'$ are distinct and all lie on π. Then the lines joining $\alpha \cap \beta, \alpha' \cap \beta'$ and $\beta \cap \gamma, \beta' \cap \gamma'$ and $\gamma \cap \alpha, \gamma' \cap \alpha'$ are concurrent.

The principle of duality says that we do not need to prove this result separately; it simply follows from the original result!

$\diamondsuit$

9. Bilinear forms and quadrics

The next piece of algebra we consider in the context of projective geometry is the theory of bilinear forms.

Definition 9.1. A *symmetric bilinear form* on a vector space V over $\mathbb{F}$ is a map $B : V \times V \to \mathbb{F}$ such that

(i) $B(v, w) = B(w, v)$; (ii) B is linear in v (and hence, by (i), in w).

If an addition we have the property

(iii) if $B(v, w) = 0$ for all w, then $v = 0$,

then we say the form is *nondegenerate* or *nonsingular*.

More concretely, if we choose a basis $\{e_0, \ldots, e_n\}$ of V, then a bilinear form is given by $B(v, w) = v^t X w$ for a symmetric matrix X given by $X_{ij} = B(e_i, e_j)$. Nondegeneracy of the form is equivalent to nonsingularity (invertibility) of the matrix X. Symmetric matrices form a vector space of dimension $\frac{1}{2} \dim V(\dim V + 1)$, so we can form linear combinations of bilinear forms.

Remark 9.2. In part A linear algebra we focused particularly on *inner products*. Over $\mathbb{R}$, these are symmetric bilinear forms which satisfy the extra condition of positive definiteness (that is $B(v, v) > 0$ for $v \neq 0$). Over $\mathbb{C}$, positive definiteness requires the form to be conjugate symmetric rather than symmetric and sesquilinear rather than bilinear: the form is linear in one variable and conjugate linear in the other. Here we shall focus instead on bilinear forms and drop the positive definiteness property. In fact, for most purposes nondegeneracy is a good replacement for positive definiteness. In particular, nondegeneracy is actually equivalent to the statement that the map from V to V^* defined by $v \mapsto B(v, .)$ is an isomorphism.

A bilinear form is determined (if the characteristic of $\mathbb{F}$ is $\neq 2$), by the associated *quadratic form*

$$Q(v) = B(v, v),$$

for we can recover B via the polarisation identity

$$B(v, w) = \frac{1}{4}(B(v+w, v+w) - B(v-w, v-w)) = \frac{1}{4}(Q(v+w) - Q(v-w))$$

Over $\mathbb{R}$ or $\mathbb{C}$ we can diagonalise quadratic forms.

Theorem 9.3. *If $v \mapsto Q(v) = B(v, v)$ is a quadratic form defined on a vector space V, then*

(i) if the field $\mathbb{F} = \mathbb{C}$, there is a basis $\{e_0, \ldots, e_n\}$ of V, with respect to which

$$Q(v) = \lambda_0^2 + \ldots \lambda_r^2$$

where $v = \sum_{i=0}^n \lambda_i e_i$;

(ii) if $\mathbb{F} = \mathbb{R}$, there is a basis $\{e_0, \ldots, e_n\}$ of V, with respect to which

$$Q(v) = \lambda_0^2 + \ldots \lambda_r^2 - \lambda_{r+1}^2 - \ldots - \lambda_{r+s}^2$$

where $v = \sum_{i=0}^n \lambda_i e_i$.

Proof. Write $B(v, v) = v^t X v = \sum_{i,j} X_{ij} v_i v_j$ in some basis, where X is a symmetric matrix. We can assume that some X_{ii} is nonzero, because if X_{ij} is nonzero we can introduce new variables $y_i = \frac{1}{2}(v_i + v_j)$, $y_j = \frac{1}{2}(v_i - v_j)$ and now $v_i v_j = y_i^2 - y_j^2$.

Now we complete the square.

$$\frac{1}{X_{ii}} \left(\sum_j X_{ij} v_j \right)^2 = X_{ii} v_i^2 + 2 \sum_{j \neq i} X_{ij} v_j v_i + \text{terms in } v_j (j \neq i)$$

so by introducing the new variable $\tilde{y}_i = \sum X_{ij} v_j$ we can put B into the form

$$B(v, v) = \frac{1}{X_{ii}} y_i^2 + \text{terms in } v_j (j \neq i)$$

Now we repeat the process until we have diagonalised B: rescaling the variables appropriately now brings it into the desired form. (Note that over $\mathbb{R}$ we cannot change the sign of y_i^2 by rescaling). □

Notice that the form is nondegenerate exactly when $r = n$ (in the complex case) and $r + s = n$ (in the real case).

Example 9.4. Consider the form on $\mathbb{R}^3$ given by

$$(x_1, x_2, x_3) \mapsto x_1 x_2 + x_2 x_3 + x_3 x_1.$$

We change variables to

$$y_1 = \frac{1}{2}(x_1 + x_2), \quad y_2 = \frac{1}{2}(x_1 - x_2), \quad y_3 = x_3$$

to generate some nonzero diagonal terms. The form is now

$$y_1^2 - y_2^2 + 2 y_3 y_1.$$

We complete the square, writing this as

$$(y_1 + y_3)^2 - y_3^2 - y_2^2$$

so on putting $z_1 = y_1 + y_3$ and $z_2 = y_2,\ z_3 = y_3$ we get the required form

$$(z_1, z_2, z_3) \mapsto z_1^2 - z_2^2 - z_3^2$$

over the reals. If we work over $\mathbb{C}$ then scaling z_2, z_3 by i brings us into the standard form

$$(z_1, z_2, z_3) \mapsto z_1^2 + z_2^2 + z_3^2$$

of a nondegenerate quadratic form over $\mathbb{C}$. ♢

Remark 9.5. In the real case, we can also prove our result using the theorem from linear algebra that given a real symmetric matrix B, there exists an orthogonal P such that

$$PBP^t = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

Now letting Q be the diagonal matrix with entries $(\sqrt{|\lambda_i|})^{-1}$ if $\lambda_i \neq 0$ and 1 if $\lambda_i = 0$, we see that $(QP)B(QP)^t$ has the desired form (see the comment below about how quadratic forms transform under projective transformations).

We have seen that linear subspaces of projective space $\mathbb{P}(V)$ are projectivisations of subspaces of V, and hence are determined by systems of homogeneous linear equations. The next simplest subsets of projective space defined by polynomial equations are the quadrics, which are defined by the vanishing of a quadratic form.

Definition 9.6. A *quadric* is the locus of points in a projective space $\mathbb{P}(V)$ defined by an equation $Q(v) = 0$, where $v \mapsto Q(v) = B(v, v)$ is a (not identically zero) quadratic form on V.

We remark that this does indeed define a subset of projective space, as $Q(v)$ is homogeneous of degree 2 in v (cf. the remarks at the end of §2).

It is easy to see that projective transformations send quadrics to quadrics. If we write the quadratic form in terms of a symmetric matrix X, then its image under a projective transformation is the form defined by the symmetric matrix MXM^t where M defines the projective transformation. Note also that if Q and Q' are proportional, that is $Q'(v) = \lambda Q(v)$ for all v, then they define the same quadric.

Definition 9.7. We say a quadric is *nonsingular* if the associated symmetric bilinear form is nondegenerate.

On choosing a basis, this is equivalent to the symmetric matrix defining the form being invertible.

The lowest-dimensional nontrivial quadrics are the *conics*, that is, the quadrics in the projective plane. Over $\mathbb{C}$, our diagonalisation theorem tells us that a conic can be put into one of the following three forms:

(i) $z_0^2 + z_1^2 + z_2^2 = 0$;
(ii) $z_0^2 + z_1^2 = 0$;
(iii) $z_0^2 = 0$.

Case (i) is the general case, when the conic is nonsingular. The remaining two cases are the two kinds of singular conics. Case (ii) is a pair of distinct lines: on putting the conic in the above form $z_0^2 + z_1^2 = 0$, we see that the lines are $z_0 - iz_1 = 0$ and $z_0 + iz_1 = 0$, which meet at the point $[0, 0, 1]$ in the plane. Case (iii) is the most degenerate: it is a double line, a line with multiplicity two. We can think of Cases (ii) and (iii) as singular limits or degenerations of the generic nonsingular conics[7].

Definition 9.8. The *singular points* of the quadric are those points $[v]$ where $Xv = 0$.

So in case (ii), where $X = \mathrm{diag}(1, 1, 0)$, the unique singular point is $[0, 0, 1]$, the intersection point of the pair of lines. In case (iii) we have $X = \mathrm{diag}(1, 0, 0)$, then the singular points are the points on the line $z_0 = 0$; in other words every point on the conic is singular.

Remark 9.9. The conic is nonsingular if and only if X is invertible, which is equivalent to the only solution to $Xv = 0$ being $v = 0$. So the conic is nonsingular if and only it has no singular points in projective space. This justifies the terminology in the definition above.

If we work over $\mathbb{C}$ or $\mathbb{R}$, then we may further understand the notion of singular point using some ideas from the Part A course Introduction to Manifolds. The conic is defined by the equation $f = 0$ where $f : v \mapsto v^t X v$. Now, the derivative of f at v in the sense of multivariable calculus is $df_X : h \mapsto 2h^t X v$, which has maximal rank one unless $Xv = 0$. So the singular points are the points where df has less than maximal rank, and hence where the manifold structure on the conic breaks down. In the example above, this happens in case (ii) exactly where the lines intersect.

Nonsingular conics actually have a very nice description. If we fix a point x on the conic, and take a projective line not containing x, then projection from x onto the line actually sets up a bijection between the conic and the line. (If $\mathbb{F} = \mathbb{C}$, this in fact defines a homeomorphism between the conic and the projective line, and hence the Riemann sphere, though of course this is not a projective equivalence.)

Theorem 9.10. *Let C be a nonsingular conic in the projective plane $\mathbb{P}(V) = \mathbb{F}\mathbb{P}^2$, and let X be a point of C. Let $L = \mathbb{P}(U)$ be a projective line in the plane not containing X. Then there is a bijection $\alpha : \mathbb{P}(U) \to C$ such that $X, Y, \alpha(Y)$ are collinear, for $Y \in \mathbb{P}(U)$.*

Proof. Let B denote the nondegenerate bilinear form whose quadratic form Q defines the conic C. Let $X = [x]$ be a point on C, so that $B(x, x) = 0$.

For each $Y \in \mathbb{P}(U)$, we want to see where (other than at X) the projective line XY meets the conic. We will find that there is a unique such point and this will be $\alpha(Y)$.

Let $Y \in \mathbb{P}(U)$ have representative vector $y \in U$, so that x, y are linearly independent, as we are assuming $X \notin \mathbb{P}(U)$. Consider the 2-dimensional

[7]In fact the Theorem of Pappus we saw in Chapter 3 concerning six points on a line-pair generalises to the situation where the six points lie on a conic. This result is often called called Pascal's Mystic Hexagon; a proof may be found in [2].

subspace $W_y = \langle x, y\rangle$ of V, so the projective line we are considering is $XY = \mathbb{P}(W_y)$. Observe that the bilinear form B cannot be identically zero on the space W_y . For we could extend to a basis $\{x, y, z\}$ for $V = \mathbb{F}^3$ and the orthogonal complement of z would meet W_y in at least a 1-dimensional subspace, elements of which would now be orthogonal to the whole of V, thus contradicting nondegeneracy.

With respect to the basis $\{x, y\}$, the form Q restricted to W_y is

$$Q(\lambda_0 x + \lambda_1 y) = 2\lambda_0\lambda_1 B(x, y) + \lambda_1^2 B(y, y)$$

and $B(x, y), B(y, y)$ are not both zero. So the projective line $\mathbb{P}(W_y)$ meets the conic at two points. One is the basepoint $X = [x]$, corresponding to $(\lambda_0, \lambda_1) = (1, 0)$. The other, corresponding to

$$(\lambda_0, \lambda_1) = (B(y, y), -2B(x, y)),$$

is defined to be $\alpha(Y)$. Note that α is injective as given any point $Z \neq Y$ on the conic, the projective line YZ meets the line $\mathbb{P}(U)$ in a unique point Y. Moreover, $\alpha(Y) = X$ exactly when $B(x, y) = 0$, which defines a unique point in $\mathbb{P}(U)$. □

We have set up a bijection between a nonsingular conic and the projective line. This kind of bijection is called a *rational parametrisation*.

The existence of a rational parametrisation for the conic has some nice applications in the theory of Diophantine equations. These are polynomial equations where we are primarily interested in rational or integral solutions.

Example 9.11. Consider the equation

$$x_0^2 - x_1^2 - x_2^2 = 0$$

whose solutions are Pythagorean triples. As our basepoint on the conic defined by the above equation we may take $X = [1 : 1 : 0]$. We can take $x_0 = 0$ as the projective line L, which does not contain the basepoint X. So if Y is a point on the projective line with representative vector $y = (0, \lambda_1, \lambda_2)$ then

$$\begin{aligned}\alpha(Y) &= B(y, y)x - 2B(x, y)y \\ &= -(\lambda_1^2 + \lambda_2^2)(1, 1, 0) + 2\lambda_1(0, \lambda_1, \lambda_2) \\ &= (-(\lambda_1^2 + \lambda_2^2), \lambda_1^2 - \lambda_2^2, 2\lambda_1\lambda_2)\end{aligned}$$

It is clear that this does indeed give solutions to the equations. Replacing x_2 by its negative, we obtain the familiar formula for Pythagorean triples

$$x_0 = s^2 + t^2, \quad x_1 = s^2 - t^2, \quad x_2 = 2st.$$

By taking s, t to be rational (respectively, integral), we get the solution in rational numbers (respectively, integers). For example, $(s, t) = (2, 1)$ gives $(x_0, x_1, x_2) = (5, 3, 4)$, while $(s, t) = (3, 2)$ and $(4, 3)$ give the triples $(13, 5, 12)$ and $(25, 7, 24)$ respectively. ♢

In the Part B course Algebraic Curves you will see that nonsingular curves of higher degree in the projective plane do not admit rational parametrisations. Indeed, over $\mathbb{C}$ such curves are not homeomorphic to the Riemann sphere. The genus of a degree d nonsingular curve in the complex projective plane is $\frac{1}{2}(d-1)(d-2)$ which is only zero for $d = 1, 2$ ie the case of lines and conics. We refer, for example, to [2] for more on this subject.

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