Dimensions and Embeddings in Analysis
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1 Lebesgue covering dimension

Let X be a topological space.

A *covering* α of X is a finite collection $\{U_j\}_{j=1}^r$ of open sets such that

$$X = \bigcup_{j=1}^{r} U_j.$$

The *order* of a covering is n if the largest number of elements in the cover that have non-empty intersection is $n+1$.

Another covering β of X is a *refinement* of α if every element of β is contained some element of α.

Definition 1. *We say that* $\dim(X) \leq n$ *if every covering has a refinement of order* $\leq n$*, and set*

$$\dim(X) = \min\{n : \dim(X) \leq n\}.$$

Note that $\dim(X)$ is integer valued and is a topological property, i.e. invariant under homeomorphisms.

For on this definition and its elementary properties see Hurewicz & Wallman (1941) and Munkres (2000).

We now restrict our attention to a compact metric space (X, d); we prove our first embedding result in this context. We use an equivalent definition of the dimension, but first we need an auxiliary definition.

Given $U \subset X$ the diameter of U is given by

$$\operatorname{diam}(U) = |U| = \sup_{u,v \in U} d(u, v).$$

The *mesh size* of a covering is the largest diameter of elements of the covering.

Lemma 2. *If* (X, d) *is a compact metric space then* $\dim(X) \leq n$ *if and only if there exists coverings of order* $\leq n$ *with arbitrarily small mesh size.*

Proof. (Exercise) $\Rightarrow$ use compactness of X and cover with balls of radius $\epsilon/2$, take a finite sub-cover, and then a refinement.

$\Leftarrow$ Show that any covering α has a 'Lebesgue number': there exists $\eta > 0$ such that any subset A of X with $|A| < \eta$ is entirely contained in some element of α. $\square$

We will use the Baire Category Theorem to prove the following result. We say that a map from (X, d) into $\mathbb{R}^k$ is an embedding if it is a homeomorphism between X and its image.

Theorem 3. *Let (X, d) be a compact metric space with $\dim(X) \leq d$. Then a residual set of maps in $C(X; \mathbb{R}^{2d+1})$ are embeddings of X into $\mathbb{R}^{2d+1}$.*

Before we begin the proof, another definition. We say that $g \in C(X; \mathbb{R}^k)$ is an ε-mapping if

$$\operatorname{diam}(g^{-1}(y)) < \varepsilon \qquad \text{for every} \quad y \in g(X),$$

i.e. if $g(x) = g(x')$ implies that $d(x, x') < \varepsilon$.

Lemma 4. *If $g \in C(X; \mathbb{R}^k)$ is a $1/n$-mapping for every n then g is a homeomorphism.*

Proof. We give what is essentially a topological proof.

First, observe that in this case g is injective. So it is an injective map from the compact space X into $\mathbb{R}^k$. So we can define $g^{-1}: g(X) \to X$, and g is continuous iff g maps closed sets to closed sets.

Let K be a closed subset of X. Since X is compact, K is compact. Then $g(K)$ is compact, since K is continuous, and so $g(K)$ is closed. So g^{-1} is continuous and g is a homeomorphism. $\square$

Our strategy of proof is therefore to show that for k sufficiently large the set

$$\mathscr{F}_\varepsilon^k = \{f \in C(X; \mathbb{R}^k) : \ f \text{ is an } \varepsilon\text{-mapping}\}$$

is open and dense in $C(X; \mathbb{R}^k)$; we use the sup metric on $C(X; \mathbb{R}^k)$, i.e.

$$\rho(f, g) = \|f - g\|_\infty = \sup_{x \in X} |f(x) - g(x)|.$$

Lemma 5. *The set $\mathscr{F}_\varepsilon^k$ is open in $C(X;\mathbb{R}^k)$ for every $\varepsilon > 0$ and for any $k \in \mathbb{N}$.*

Proof. Take $g \in \mathscr{F}_\varepsilon^k$. Noting that

$$Z = \{(x, x') \in X \times X : \ d(x, x') \geq \varepsilon\}$$

is compact and that $h\colon Z \to \mathbb{R}$ defined by

$$h(x, x') = |g(x) - g(x')|$$

is continuous, since $h > 0$ on Z it follows that $h(x, x') \geq \delta$ for some $\delta > 0$, for every $(x, x') \in Z$.

Now suppose that $f \in C(X;\mathbb{R}^k)$ with $\|f - g\|_\infty < \delta/2$. If $f(x) = f(x')$ then

$$|g(x) - g(x')| < \delta \qquad \Rightarrow \qquad d(x, x') < \varepsilon,$$

and so f is an ε-mapping. This shows that $\mathscr{F}_\varepsilon^k$ is open, as claimed. □

To show that $\mathscr{F}_\varepsilon^k$ is dense when k is large enough we need more ideas/definitions.

Definition 6. *We say that a set of points $\{x_0, \ldots, x_N\}$ in $\mathbb{R}^k$ is* geometrically independent *if*

$$\sum_{i=0}^{N} \alpha_i x_i = 0 \qquad \textit{and} \qquad \sum_{i=0}^{N} \alpha_i = 0$$

implies that $\alpha_i = 0$ for $i = 0, 1, \ldots, N$.

This is the same as saying that for any $\{\alpha_i\}_{i=1}^N$ we have

$$\sum_{i=1}^{N} \alpha_i (x_i - x_0) = 0 \qquad \Rightarrow \qquad \alpha_i = 0 \text{ for } i = 1, 2, \ldots N,$$

i.e. the vectors $\{x_i - x_0\}_{i=1}^N$ are linearly independent.

We can use any set of points $\{x_0, \ldots, x_N\}$ to 'generate a hyperplane' P,

$$P = \{x = \sum_{i=0}^{N} t_i x_i \quad \text{where} \qquad \sum_{i=0}^{N} t_i = 1\}$$
$$= \{x = x_0 + \sum_{i=1}^{N} a_i (x_i - x_0) \quad \text{for any } a_i, \ i = 1, \ldots N\}.$$

Note that if $N < k$ then P has empty interior (and zero measure) and if $y \notin P$ then

$$(x_0, \ldots, x_N, y)$$

are geometrically independent.

Definition 7. *A set of points in* $\mathbb{R}^k$ *is in* general position *if any subset of* $\leq k+1$ *points is geometrically independent.*

The following lemma is the key to showing the density of $\mathscr{F}_\varepsilon^k$.

Lemma 8. *Given* $\{x_1, \ldots, x_N\} \in \mathbb{R}^k$ *and* $\delta > 0$*, there exists a collection* $\{y_1, \ldots, y_N\} \subset \mathbb{R}^k$ *that is in general position and* $|y_j - x_j| < \delta$ *for every* $j = 1, \ldots, N$.

Proof. We argue by induction, taking $y_1 = x_1$.

If $(y_1, \ldots, y_m)$ are in general position, consider all the hyperplanes determined by $\leq k$ elements. Each of these has empty interior/zero measure, and so their union has empty interior/zero measure. So certainly there exists y_{m+1} such that $|y_{m+1} - x_{m+1}| < \delta$ and y_{m+1} is not contained in any of these hyperplanes.

We claim that $\{y_1, \ldots, y_{m+1}\}$ are in general position. Take $k+1$ of the $\{y_j\}$: if none of these are y_{m+1} then they are geometrically independent, since they come from the points $\{y_1, \ldots, y_m\}$, which are in general position; if one is y_{m+1} then they are in general position, since y_{m+1} is not in the hyperplane generated by the other k elements. □

We are now in a position to prove the density result we need.

Proposition 9. *If* (X, d) *is compact and* $\dim(X) \leq n$ *then* $\mathscr{F}_\varepsilon^{2n+1}$ *is dense in* $C(X; \mathbb{R}^{2n+1})$.

Proof. Take $f \in C(X; \mathbb{R}^{2n+1})$ and $\eta > 0$.

Since X is compact the function f is uniformly continuous, so there exists $\delta < \varepsilon$ such that

$$d(x, x') < \delta \qquad \Rightarrow \qquad |f(x) - f(x')| < \frac{\eta}{2}. \tag{1}$$

Since $\dim(X) \leq n$, there exists a covering $\{U_j\}_{j=1}^r$ of X with $|U_j| < \delta$ and of order $\leq n$, i.e. any point in X is in at least one, and at most $n+1$, of the $\{U_j\}$.

It follows from (1) that

$$\operatorname{diam}(f(U_j)) < \eta/2 \qquad \text{for all } j = 1, \ldots, r.$$

Now use Lemma to find points $\{p_j\}$ that are in general position in $\mathbb{R}^{2n+1}$ and satisfy

$$\operatorname{dist}(p_j, f(U_j)) < \eta/2.$$

Define $w_i(x) = \operatorname{dist}(x, X \setminus U_i)$, which is continuous for every i; then $w_i(x) > 0$ if and only if $x \in U_i$. Since $w_i(x) > 0$ for at least index i, and at most $n+1$,

$$0 < \sum_i w_i(x) < \infty$$

for every $x \in X$, and so

$$\phi_i(x) := \frac{w_i(x)}{\sum_j w_j(x)}$$

is well defined and satisfies $\sum_j \phi_j(x) = 1$ for every $x \in X$.

Now set

$$g(x) = \sum_{i=1}^r \phi_i(x) p_i,$$

which is an element of $C(X; \mathbb{R}^{2n+1})$ that, for any $x \in X$, satisfies

$$\begin{aligned} |f(x) - g(x)| &= \left| \sum_{i=1}^r \phi_i(x) \left(p_i - f(x)\right) \right| \\ &\leq \sum_{i=1}^r \phi_i(x) |p_i - f(x)| \\ &\leq \left\{ \sum_{i=1} \phi_i(x) \right\} \eta = \eta, \end{aligned}$$

using repeatedly the fact that $\sum_i \phi_i(x) = 1$. It follows that $\|f - g\|_\infty < \eta$.

It remains only to show that g is an ε-mapping. So suppose that $g(x) = g(x')$; then

$$\sum_{i=1}^{r} \phi_i(x)p_i = \sum_{i=1}^{r} \phi_i(x')p_i$$

and so

$$\sum_{i=1}^{r} \underbrace{(\phi_i(x) - \phi_i(x')}_{:=\alpha_i} p_i = 0.$$

Now, recall that there are at most $n+1$ values of i for which $\phi_i(x) \neq 0$; and at most $n+1$ values for which $\phi_i(x') \neq 0$. So at most $2n+2$ of the α_i are non-zero.

Observing that $\sum_i \alpha_i = \sum_i \phi_i(x) - \sum_i \phi_i(x') = 0$, it follows from the fact that the $2n+2$ points $\{p_i\}$ are in general position in $\mathbb{R}^{2n+1}$ that we must have $\alpha_i = 0$ for every i, i.e. $\phi_i(x) = \phi_i(x')$ for every i.

Thus if $g(x) = g(x')$ and $x \in U_i$, so that $\phi_i(x) \neq 0$, it follows that $\phi_i(x') \neq 0$ and $x' \in U_i$; since $\mathrm{diam}(U_i) < \delta < \varepsilon$, we must have $d(x, x') < \varepsilon$ and so g is an ε-mapping. $\square$

Now the proof of Theorem 3 is straightforward: the set

$$\bigcap_{m=1}^{\infty} \mathscr{F}_{1/m}^{2n+1},$$

which we have already observed consists of embeddings of (X, d) into $\mathbb{R}^{2n+1}$, is a countable intersection of open and dense sets, and so forms a residual (in particular dense) subset of $C(X; \mathbb{R}^{2n+1})$, by the Baire Category Theorem.

To conclude this section, observe that any compact metric space (X, d) that is homeomorphic to a subset of $\mathbb{R}^{2n+1}$ must have $\dim(X) \leq 2n+1 < \infty$, so the existence of a homeomorphism onto a subset of some finite-dimensional Euclidean space characterises compact metric spaces with finite Lebesgue covering dimension.

2 Hausdorff dimension

The Hausdorff measure of a metric space (X, d) (or a subset of (X, d)) is defined as follows. First, set

$$\mathscr{H}^s_\delta(X) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \ X \subseteq \bigcup_{i=1}^{\infty} U_i, \ |U_i| \le \delta\right\},$$

and then put

$$\mathscr{H}^s(X) = \lim_{\delta\to 0} \mathscr{H}^s_\delta(X).$$

Defined in this was $\mathscr{H}^s$ is an (outer) measure on subsets of (X, d).

In $\mathbb{R}^n$, $\mathscr{H}^n$ is proportional to $\mathscr{L}^n$ (Lebesgue measure).

The definition of the Hausdorff dimension is based on the following simple observation: if $\mathscr{H}^s(X) < \infty$, then for any $s' > s$ we have $\mathscr{H}^{s'}(X) = 0$. Indeed, for every $\delta > 0$ there is a cover of X by sets $\{U_i\}$ with diameters $\le \delta$ such that

$$\sum_{j=1}^{\infty} |B_j|^s \le \mathscr{H}^s(X) + 1.$$

So for $s' > s$ we can use the same cover to show that

$$\sum_{j=1}^{\infty} |B_j|^{s'} \le \delta^{s'-s} \sum_j |B_j|^s \le \delta^{s'-s}[\mathscr{H}^s(X) + 1],$$

from which it follows that $\mathscr{H}^{s'}(X) = 0$.

Definition 10. *We set*

$$\dim_{\mathrm{H}}(X) = \inf\{s \ge 0 : \ \mathscr{H}^s(X) = 0\}.$$

Because of the way it is defined, we do not actually need to have a definition of the Hausdorff measure to define the Hausdorff dimension, apart from the fact given in the following lemma.

Lemma 11. $\mathscr{H}^s(X) = 0$ *if and only if for every* $\varepsilon > 0$ *there exists a countable covering* $\{U_i\}$ *of* X *such that*

$$\sum_{j=1}^{\infty} |U_j|^s < \varepsilon. \tag{2}$$

Proof. $\Rightarrow$ is immediate from the definition of $\mathscr{H}^s$. Conversely, given any $\delta > 0$ choose $\varepsilon > 0$ such that $\varepsilon^{1/s} < \delta$ and find a covering that satisfies (2); then every element in the cover satisfies $|U_j|^s < \varepsilon$ and so $|U_j| < \varepsilon^{1/s} < \delta$. $\square$

We can now prove some elementary properties of the Hausdorff dimension.

Proposition 12. *(i) If $A \subset X$ then $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{H}}(X)$;*

(ii) if $\{A_k\}_{k=1}^{\infty} \subset X$ then

$$\dim_{\mathrm{H}}\left(\bigcup_{k=1}^{\infty} A_k\right) = \sup_k \dim_{\mathrm{H}}(A_k);$$

(iii) if $f\colon (X,d) \to (Y,\rho)$ is θ-Hölder continuous, i.e.

$$\rho(f(x), f(y)) \leq C d(x,y)^{\theta}$$

then $\dim_{\mathrm{H}}(f(X)) \leq \dim_{\mathrm{H}}(X)/\theta$; and

(iv) if U is a subset of $\mathbb{R}^n$ then $\dim_{\mathrm{H}}(U) \leq n$; if U contains an open subset of $\mathbb{R}^n$ then $\dim_{\mathrm{H}}(U) = n$.

Proof. (i) is immediate from the definition.

(ii) If the supremum is ∞ there is nothing to prove. Otherwise let $\sigma = \sup_k \dim_{\mathrm{H}}(A_k)$; then for any $s > \sigma$ we can find countable covers $\{U_j^{(k)}\}$ of the A_k such that

$$\sum_{j=1}^{\infty} |U_j^{(k)}|^s < 2^{-k}\varepsilon \qquad \Rightarrow \qquad \sum_{j,k} |U_j^{(k)}| < \varepsilon,$$

and so $\mathscr{H}^s(\text{union}) = 0$. It follows that $\dim_{\mathrm{H}}(\text{union}) \leq \sigma$, and it follows from (i) that $\dim_{\mathrm{H}}(\text{union}) \geq \sigma$.

(iii) Take $s > \dim_{\mathrm{H}}(X)$; the for every $\varepsilon > 0$ there exists $\{U_j\}$ such that

$$\sum_j |U_j|^s < \varepsilon.$$

Under f, we have $|f(U_j)| \leq C|U_j|^\theta$, and $\{f(U_j)\}$ cover $f(X)$, such that

$$\sum_j |f(U_j)|^{s/\theta} < C^s\varepsilon.$$

It follows $\mathscr{H}^{s/\theta}(f(X)) = 0$, and therefore $\dim_{\mathrm{H}}(f(X)) \leq \dim_{\mathrm{H}}(X)/\theta$.

(iv) Any subset of $\mathbb{R}^n$ is contained in the union of a countable number of unit cubes; by parts (i) and (ii) it is therefore enough to show that $\dim_{\mathrm{H}}(Q) \leq n$, where $Q = [0,1]^n$.

For any $\varepsilon > 0$, the cube Q can certainly be covered by ε^{-n} cubes of sides 2ε; for this cover

$$\sum_{j=1}^{\varepsilon^{-n}} |U_j|^s \leq \varepsilon^{-n}(2\sqrt{n}\varepsilon)^s = (2\sqrt{n})^s\varepsilon^{s-n},$$

and so $\mathscr{H}^s(Q) = 0$ for every $s > n$. It follows that $\dim_{\mathrm{H}}(Q) \leq n$.

If U contains an open set, then it contains an open ball B. If $\dim_{\mathrm{H}}(U) < n$ then $\mathscr{H}^n(U) = 0$, and so $\mathscr{H}^n(B) = 0$. Then for any $\varepsilon > 0$, B can be covered by sets U_j such that

$$\sum |U_j|^n < \varepsilon.$$

Since any set U can be covered by a ball of radius $2|U|$, it follows that B can be covered by balls B_j such that

$$\sum \mu(B_j) < C \sum 2^n |U_j|^n < C2^n\varepsilon$$

for any ε, i.e. $\mu(B) = 0$. But this is impossible, so $\dim_{\mathrm{H}}(U) = n$. □

We now want to show that $\dim_{\mathrm{H}}(X) \geq \dim(X)$. To do this we first need another result about the covering dimension.

Lemma 13. *Suppose that for every open cover $\{U_1, \ldots, U_{n+2}\}$ of X there is a cover of X by closed sets $\{F_1, \ldots, F_{n+2}\}$ such that $F_j \subseteq U_j$ and $\cap_j F_j = \emptyset$. Then* $\dim(X) \leq n$.

Proof. First we show that the same holds if we replace the closed sets F_j by open sets V_j.

Observe that

$$F_1 \subset U_1 \cap \left\{X \setminus \cap_{i=2}^{n+2} F_i\right\}, \text{ which is open.}$$

It follows that there is an open set V_1 such that

$$F_1 \subseteq V_1 \subseteq \overline{V_1} \subseteq U_1 \cap \left\{X \setminus \cap_{i=2}^{n+2} F_i\right\}.$$

Clearly $\{V_1, U_2, \ldots, U_{n+2}\}$ is still a cover; $V_1 \subseteq \overline{V_1} \subseteq U_1$; and

$$V_1 \cap \bigcap_{i=2}^{n+2} F_i = \overline{V_1} \cap \bigcap_{i=2}^{n+2} F_i = \emptyset.$$

From this it follows that

$$F_2 \subseteq U_2 \cap \left\{X \setminus \left(\overline{V_1} \cap \bigcap_{i=3}^{n+2} F_i\right)\right\},$$

and so we can find an open set V_2 such that

$$F_2 \subseteq V_2 \subseteq \overline{V_2} \subseteq U_2 \cap \left\{X \setminus \left(\overline{V_1} \cap \bigcap_{i=3}^{n+2} F_i\right)\right\}.$$

Now $\{V_1, V_2, U_3, \ldots, U_{n+2}\}$ is still a cover; $V_2 \subset \overline{V_2} \subset U_2$; and

$$\overline{V_1} \cap \overline{V_2} \cap \bigcap_{i=3}^{n+2} F_i = \emptyset.$$

We can continue in this way to obtain $\{V_1, \ldots, V_{n+2}\}$, open sets with the required property.

We now show that this implies that $\dim(X) \leq n$.

Take an open cover $\{U_1, \ldots, U_k\}$ of X. If $k \geq n+1$ then this is a cover of order $\leq n$; so assume that $k \geq n+2$.

We will apply the following construction to every $n+2$-element subset.

Set

$$W_j = \begin{cases} U_j & j = 1, \ldots, n+1 \\ \cup_{i=n+2}^{k} U_i & j = n+2. \end{cases}$$

These sets $\{W_j\}_{j=1}^{n+2}$ cover X, and so there exists a cover of X by open sets $\{V_i\}_{i=1}^{n+2}$ such that $V_i \subseteq W_i$ and $\cap_{i=1}^{n+2} V_i = \emptyset$.

Now set

$$U'_j = \begin{cases} V_j & j = 1, \ldots, n+1 \\ U_j \cap V_{n+2} & j \geq n+2. \end{cases}$$

Then $\{U'_j\}$ is a new cover of X by open sets, such that

$$U'_j \subseteq U_j \qquad \text{and} \qquad \cap_{i=1}^{n+1} U'_j \cap U'_i = \emptyset$$

for every $i \geq n+2$.

In this way we end up with a refinement of the original covering of order $\leq n$, whence $\dim(X) \leq n$. □

Theorem 14. *Let* (X, d) *be a compact metric space. Then* $\dim(X) \leq \dim_{\mathrm{H}}(X)$.

Proof. Let $\dim(X) = n$. Then it is not true that $\dim(X) \leq n-1$.

So there exists an open cover $\{U_i\}_{i=1}^{n+1}$ of X such that any family of closed sets $\{F_i\}$, $F_i \subseteq U_i$, that cover X, must have non-empty intersection, $\cap_{i=1}^{n+1} F_i \neq \emptyset$.

Give this cover, let $w_i(x) = \mathrm{dist}(x, X \setminus U_i)$, so that (as in the embedding proof) $w_i(x) \neq 0$ if and only if $x \in U_i$. For each $x \in X$ we know that $w_i(x) \neq 0$ for at least one index i, since the $\{U_i\}$ cover X.

Note that w_i is Lipschitz continuous with

$$|w_i(x) - w_i(y)| \leq d(x, y),$$

and so $\sum_i w_i(x)$ is also Lipschitz continuous,

$$\left| \sum_i w_i(x) - \sum_i w_i(y) \right| \leq (n+1) d(x, y).$$

Since $\sum_i w_i(x) > 0$ for all $x \in X$ and X is compact, it follows that there is some $\gamma > 0$ such that $\sum_i w_i(x) \geq \gamma$ for all $x \in X$. It follows that

$$x \mapsto \phi_i(x) := \frac{w_i(x)}{\sum_j w_j(x)}$$

is Lipschitz continuous for each i. Note that $\sum_i \phi_i(x) = 1$ for every $x \in X$.

Since each $\phi_i(x)$ is Lipschitz continuous, the map $\Phi\colon X \to \mathbb{R}^{n+1}$ defined by

$$\Phi(x) = (\phi_1(x), \ldots, \phi_{n+1}(x))$$

is also Lipschitz. It follows that $\dim_{\mathrm{H}}(\Phi(X)) \leq \dim_{\mathrm{H}}(X)$.

We now show that $\Phi(X)$ contains

$$T = \{(t_1, \ldots, t_{n+1}) :\ t_i > 0, \quad \sum_{i=1}^{n+1} t_i = 1\}.$$

Noting that the set T is the image of the open set

$$\{(t_1, \ldots, t_n) \in \mathbb{R}^n :\ 0 < \sum_{i=1}^{n} t_i < 1\}$$

under the Lipschitz map

$$(t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_n, 1 - \sum_{i=1}^{n} t_i)$$

it follows that $\dim_{\mathrm{H}}(T) = n$. So if we can show that $T \subset \Phi(X)$ we will have shown that $\dim_{\mathrm{H}}(X) \geq \dim_{\mathrm{H}}(\Phi(X)) \geq n = \dim(X)$.

To show this, we take $\underline{t} \in T$ and show that $\underline{t} \in \Phi(X)$. We consider the sets

$$F_i = \{x \in X :\ \phi_i(x) \geq t_i\}.$$

These sets are closed, $F_i \subseteq U_i$, and $\cup_i F_i = X$. To see this final equality, suppose not: then there would exist some $x \in X$ such that $\phi_i(x) < t_i$ for every i; but then, since $\sum_i t_i = 1$, we would have $\sum_i \phi_i(x) < 1$, a contradiction.

Now, we chose our initial open cover $\{U_i\}$ such that any such collection of closed sets must have non-empty intersection, so $\cap F_i \neq \emptyset$. In particular, there exists a point $x \in X$ such that

$$\phi_i(x) \geq t_i \qquad i = 1, \ldots, n+1.$$

Since $\sum_i \phi_i(x) = 1$ and $\sum_i t_i = 1$ it follows that $\phi_i(x) = t_i$ for every i, which shows that $\Phi(x) = \underline{t}$ as required. $\square$

By combining this result with our first embedding theorem (Theorem 3) we immediately obtain an embedding result in terms of this Hausdorff dimension.

Theorem 15. *If (X, d) is a compact metric space with $\dim_{\mathrm{H}}(X) \leq n$ then a residual set of maps in $C(X; \mathbb{R}^{2n+1})$ provide embeddings of X into $\mathbb{R}^{2n+1}$.*

With a little more work one can obtain the following interesting corollary of Theorem 14; for a proof see Theorem VII.4 in Hurewicz & Wallman or Theorem 2.12 in Robinson.

Theorem 16. *If (X, d) is a compact metric space and $\dim(X) = n$ then $\dim_{\mathrm{H}}(h(X)) = n$ for a residual set of maps $h \in C(X; \mathbb{R}^{2n+1})$.*

3 Isometric embeddings into Banach spaces and linear embeddings into Euclidean spaces

For the next few lectures we will switch from looking at arbitrary metric spaces to subsets of Banach and/or Hilbert spaces.

3.1 Embedding metric spaces into Banach spaces

In some ways, the study of subsets of Banach spaces is not a restriction: we now give two results showing that any compact metric space can be isometrically mapped onto a subset of a Banach space.

Lemma 17 (Kuratowksi embedding)**.** *If* (X, d) *is a compact metric space then the mapping*

$$x \mapsto d(x, \cdot)$$

is an isometry of (X, d) *onto a subset of* $L^\infty(X)$.

Proof. Since (X, d) is compact it is bounded, i.e. $\operatorname{diam}(X) < \infty$, and then $d(x, y) \leq \operatorname{diam}(X)$ for every $y \in X$, so $\|d(x, \cdot)\|_{L^\infty} \leq \operatorname{diam}(X)$. To show that this mapping is an isometry, first note that

$$|d(x_1, y) - d(x_2, y)| \leq d(x_1, x_2)$$

for every $y \in X$, while

$$|d(x_1, x_1) - d(x_2, x_1)| = d(x_1, x_2),$$

and so

$$\|d(x_1, \cdot) - d(x_2, \cdot)\|_{L^\infty} = d(x_1, x_2). \quad \square$$

This is a relatively simple isometry, but has the disadvantage that it provides an embedding into a space that depends on X. An only slightly more involved construction yields an embedding into ℓ^∞. Note that any compact metric space is separable, since every cover

$$\bigcup_{x \in X} B(x, 1/n)$$

has a finite subcover.

Lemma 18 (Fréchet embedding)**.** *Let (X, d) be a separable metric space with $\{x_j\}_{j=0}^{\infty}$ a countable dense subset. Then the map*

$$x \mapsto s(x), \qquad s_j(x) = d(x, x_j) - d(x_j, x_0), \; j = 1, 2, \ldots$$

is an isometric embedding of (X, d) into ℓ^∞.

Proof. First note that for every j we have

$$|s_j(x)| = |d(x, x_j) - d(x_j, x_0)| \leq d(x, x_0),$$

and so $\|s\|_{\ell^\infty} \leq d(x, x_0)$.

Now,

$$|s_j(x) - s_j(y)| = |d(x, x_j) - d(y, x_j)| \leq d(x, y),$$

which shows that $\|s(x) - s(y)\|_{\ell^\infty} \leq d(x, y)$. To prove the opposite inequality, given $x, y \in X$, for any $\varepsilon > 0$ use the density of the $\{x_j\}$ in X to find k such that $d(y, x_k) < \varepsilon$ and then

$$|s_k(x) - s_k(y)| = |d(x, x_k) - d(y, x_k)| \geq d(x, y) - 2d(y, x_k) > d(x, y) - 2\varepsilon,$$

which implies that $\|s(x) - s(y)\|_{\ell^\infty} \geq d(x, y)$. □

3.2 Embedding subsets of Banach spaces into Euclidean spaces

We now prove an embedding theorem for subsets of Banach spaces with $\dim_{\mathrm{H}}(X - X) < \infty$; again we use the Baire Category Theorem. The argument is due to Mañé, with some minor simplifications.

We will need the following lemma in the course of the proof.

Lemma 19. *Suppose that K is a compact subset of a Banach space B with $0 \notin K$. Then there exist $\{f_j\}_{j=1}^{\infty} \in B^*$ with $\|f_j\| = 1$ such that $f_j(x) = 0$ for every j implies that $x \notin K$.*

Proof. Use the compactness of K to find a countable dense subset $\{x_j\}$ of K (as above). Let $f_j \in X^*$ be the support functional at x_j, i.e. $\|f_j\| = 1$ and $f_j(x_j) = \|x_j\|$. If $x \in X$ then find k such that $\|x_k - x\| < \|x\|/3$, and then $f_k(x) = f_k(x_k) - f_k(x_k - x) \geq \|x_k\| - \|x\|/3 > \|x\|/3 \neq 0$. □

Theorem 20. *Let X be a subset of a real Banach space B and let*

$$X - X := \{x_1 - x_2 : \; x_1, x_2 \in X\}.$$

If $\dim_{\mathrm{H}}(X - X) < d \in \mathbb{N}$ *then a residual set of maps in* $L(B; \mathbb{R}^{d+1})$ *are embeddings of* X.

Proof. Let $A = [X - X] \setminus \{0\}$. The key point is that if L is one-to-one on X if and only if $0 \notin LA$: if $Lx = Ly$ for some $x \neq y$ in X then $0 \neq x - y \in A$ with $L(x - y) = 0$; if $La = 0$ for some $a \in A$ then by definition there exist $x \neq y$ in X such that $a = x - y$ and $L(x - y) = 0$.

We set

$$A_r = \{x \in A : \; \|x\| \geq 1/r\},$$

so that $A = \bigcup_r A_r$, and for each r use Lemma 19 find a countable set of linear functionals $\{\phi_j^r\}$ such that $\phi_j^r(x) = 0$ for every j implies that $x \notin A_r$. Note that if $x \in A_r$ then $-x \in A_r$.

We now set

$$A_{r,j,n} = \{x \in A_r : \; |\phi_j^r(x)| \geq 1/n\}.$$

Note that each $A_{r,j,n}$ is compact, and since $0 \notin A_r$ for every $x \in A_r$ we must have $\phi_j^r(x) \neq 0$ for some j, we have

$$A_r = \bigcup_{j,n} A_{r,j,n}.$$

Note again that if $x \in A_{r,j,n}$ then $-x \in A_{r,j,n}$.

We set

$$L_{r,j,n} = \{L \in L(B; \mathbb{R}^{d+1}) : \; L^{-1}(0) \cap A_{r,j,n} = \emptyset\};$$

then $\cap_{r,j,n} L_{r,j,n}$ consists precisely of those maps for which $L^{-1}(0) \cap A = \emptyset$, i.e. the embeddings of X.

We show that each $L_{r,j,n}$ is open and dense, and then the Baire Category Theorem will show that the set of embeddings is residual.

Openness is relatively straightforward. Take $L_0 \in L_{r,j,n}$; then since $A_{r,j,n}$ is compact and $L_0 u \neq 0$ for every $u \in A_{r,j,n}$ there exists $\delta > 0$ such that

$$|L_0 u| \geq \delta \qquad u \in A_{r,j,n}.$$

Since $A_{r,j,n}$ is compact it is bounded, and so $\|u\| \leq M$ for all $u \in A_{r,j,n}$. Now if $L \in L(B; \mathbb{R}^{d+1})$ with $\|L - L_0\| < \delta/2M$ we have

$$|Lu| = |L_0 u + (L - L_0)u| \geq \delta - \|L - L_0\| M > \delta/2 \neq 0.$$

We will use the following observation in our density proof.

Let Φ be the map from $\mathbb{R}^{d+1}$ into the unit sphere S_d defined by setting

$$\Phi(x) = \begin{cases} x/|x| & x \neq 0 \\ p, & x = 0, \end{cases}$$

for some p with $|p| = 1$.

Notice that Φ is a Lipschitz map on every set of the form $\{x : \ |x| \geq R\}$, and for any set $W \subset \mathbb{R}^{d+1}$ we can write

$$\Phi(W) = p \cup \bigcup_{m \in \mathbb{N}} \Phi(W \cap \{x : \ |x| \geq 1/m\},$$

where we include p if $0 \in W$. Since the Hausdorff dimension is monotonic, non-increasing under Lipschitz maps, and stable under countable unions, it follows that $\dim_{\mathrm{H}}(\Phi(W)) \leq \dim_{\mathrm{H}}(W)$.

Now pick $L_0 \in L(B; \mathbb{R}^{d+1})$ and $\varepsilon > 0$. Using the above, it follows that

$$\dim_{\mathrm{H}}(\Phi(L_0 A_{r,j,n})) \leq \dim_{\mathrm{H}}(L_0 A_{r,j,n}) \leq \dim_{\mathrm{H}}(A_{r,j,n}) \leq \dim_{\mathrm{H}}(A) < d.$$

Since S_d has dimension d, it follows that there exists a $z \in S_d$ such that $z \notin \Phi(L_0 A_{r,j,n})$.

We now set

$$L = L_0 + \epsilon z \phi_j^r,$$

and show that $L \in L_{r,j,n}$. Clearly $L \in L(B; \mathbb{R}^{d+1})$, and $\|L - L_0\| < \varepsilon$. It only remains to show that $L^{-1}(0) \cap A_{r,j,n} = \emptyset$. If not, there exists $u \in A_{r,j,n}$ such that $Lu = 0$, i.e.

$$L_0 u + \epsilon z \phi_j^r(u) = 0.$$

Since $u \in A_{r,j,n}$, we have $\phi_j^r(u) \geq 1/n$, and in particular $\phi_j^r(u) \neq 0$. We can therefore write

$$z = -\frac{1}{\epsilon \phi_j^r(u)} L_0 u.$$

Now, using the definition of Φ it follows since $z \in S_d$ that

$$z = \Phi(z) = \Phi(\pm L_0 u) \in \Phi(L_0 A_{r,j,n}),$$

but this contradicts the choice of z.

An application of the Baire Category Theorem completes the proof. $\square$

There is an unfortunate issue here. While we can start with a metric space (X, d) and then embed it into a Banach space to yield $\Psi(X) \subset \ell^\infty$, it is not clear how to translate a condition on $\Psi(X) - \Psi(X)$ into an 'intrinsic' condition on (X, d). This is a very interesting open problem.

One can construct a compact subset X of a Hilbert space H such that $\dim_{\mathrm{H}}(X) = 0$ but no linear map from H into any finite-dimensional Euclidean space can be injective; this was done by Ben-Artzi et al. (1993), by adapting an example due to Kan given in the appendix of the paper by Sauer, Yorke, & Casdagli (1991). Because of the embedding result we have just proved, this example also gives a set for which $\dim_{\mathrm{H}}(X) = 0$ but $\dim_{\mathrm{H}}(X - X) = \infty$.

There is also another intriguing shortcoming of the above result, which is shared by the 'Fundamental Embedding Theorem'; one cannot say anything about the smoothness of the (continuous) mapping $L^{-1} \colon LX \to X$, i.e. the parametrisation of X that is a consequence of the above theorem. To prove this we will use the following two simple (but very nice) lemmas about orthogonal projections and linear maps.

Lemma 21. *Let H be a Hilbert space and take $L \in L(H; \mathbb{R}^k)$ such that $LH = \mathbb{R}^k$. Then $U = (\ker L)^\perp$ has dimension k and L can be decomposed uniquely as MP, where P is the orthogonal projection onto U and $M : U \to \mathbb{R}^k$ is an invertible linear map.*

Proof. Let $U = (\ker L)^\perp$. Then U has dimension at least k; if there are $m > k$ linearly independent elements $\{x_j\}_{j=1}^m$ of U for which $Lx_j \neq 0$ then $\{Lx_j\}_{j=1}^m$ are linearly dependent, so there are α_j, not identically zero, such that

$$0 = \sum_j \alpha_j L x_j = L\left(\sum_j \alpha_j x_j\right) \quad \Rightarrow \quad \sum_j \alpha_j x_j \in \ker L \cap (\ker L)^\perp = \{0\}.$$

It follows that the $\{x_j\}$ are linearly dependent, a contradiction.

Now let P denote the orthogonal projection on to U, and M the restriction of L to U. Take $x \in H$, and write $x = u + v$, where $u \in U$ and $v \in \ker L$ (this decomposition is unique). Now

$$Lx = Lu = Mu = M(Px).$$

It remains only to show that M is invertible; but this is clear since $\dim U = \dim \mathbb{R}^k = k$ and M is linear. □

Lemma 22. *Let P be any orthogonal projection in H, and $\{e_j\}_{j=1}^{\infty}$ any orthonormal subset of H. Then*

$$\operatorname{rank} P \geq \sum_{j=1}^{\infty} \|Pe_j\|^2,$$

with equality if $\{e_j\}$ is a basis for H.

Proof. Suppose that P has rank k. Then there exists an orthonormal basis $\{u_1, \ldots, u_k\}$ for PH, so that for any $x \in H$ we have

$$Px = \sum_{j=1}^{k} (x, u_j) u_j.$$

In particular, $Pe_j = \sum_{i=1}^{k} (e_j, u_i) u_i$, so that

$$\|Pe_j\|^2 = (Pe_j, Pe_j) = (Pe_j, e_j) = \sum_{i=1}^{k} (e_j, u_i)(u_i, e_j) = \sum_{i=1}^{k} |(e_j, u_i)|^2.$$

It follows that

$$\sum_{j=1}^{\infty} \|Pe_j\|^2 = \sum_{j=1}^{\infty} \sum_{i=1}^{k} |(e_j, u_i)|^2 = \sum_{i=1}^{k} \sum_{j=1}^{\infty} |(e_i, u_j)|^2 \leq \sum_{j=1}^{k} \|u_j\|^2 = k. \quad \square$$

We now show that we cannot find a modulus of continuity that works in Theorem 15.

Let $f : [0,\infty) \to [0,\infty)$ have $f(0) = 0$. We show that we can find a compact set $X \subset H$ with $0 \in X$ and $\dim_{\mathrm{H}}(X - X) = 0$ but such that the inequality

$$\|Px\| \geq \varepsilon f(\|x\|) \qquad \text{for all } x \in X \tag{3}$$

cannot hold for any $\varepsilon > 0$ or any finite rank projection P. Since $0 \in X$, this shows that there is no 'general' modulus of continuity possible under any condition on the Hausdorff dimension of $X - X$ if we require our embedding map to be linear.

The set X will be an orthogonal sequence of the form $\{\alpha_n e_n\}_{n=1}^{\infty} \cup \{0\}$, where $\{e_n\}$ is an orthonormal sequence in H. Since this is a countable set, so is $X - X$, and so $\dim_{\mathrm{H}}(X - X) = 0$.

If (3) does hold then

$$\|P(\alpha_j e_j)\| = |\alpha_j|\|Pe_j\| \geq \varepsilon f(|\alpha_j|)$$

for each j, so $\|Pe_j\| \geq \varepsilon f(|\alpha_j|)/\alpha_j$, and Lemma 22 implies that

$$\operatorname{rank} P \geq \sum_{j=1}^{\infty} \|Pe_j\|^2 \geq \varepsilon^2 \sum_{j=1}^{\infty} \left(\frac{f(\alpha_j)}{\alpha_j} \right)^2 .$$

Now set $\phi_n = nf(1/n)$, let N_n be the first integer that is greater than $1/\phi_n$, and let $T_j = \sum_{n=1}^{j} N_n$. Then for $T_j < i \leq T_{j+1}$ set $\alpha_j = 1/j$.

This gives an orthogonal sequence for which the right-hand side of the above inequality is infinite for any $\varepsilon > 0$, and shows that (3) cannot hold for any $\varepsilon > 0$ if P has finite rank.

4 The box-counting dimension I

4.1 Definition and properties

The following definition applies to a compact metric space (X, d), or equivalently to any subset of a compact set in a metric space.

Definition 23. *Let X be a subset of a compact set, and let $N(X, \varepsilon)$ denote the minimum number of open ε-balls with centres in X required to cover X. Then we define the (upper) box-counting dimension of X to be*

$$\dim_{\mathrm{B}}(X) := \limsup_{\varepsilon\to 0} \frac{\log N(X, \varepsilon)}{-\log \varepsilon}. \tag{4}$$

One can construct sets where the lim inf of the expression on the right-hand side (which defines the 'lower box-counting dimension') is not equal to the limsup; so the limit need not exist. When it does this is the 'box-counting dimension'.

This dimension is also referred to as the Minkowski dimension in the literature (and, unhelpfully, as the 'fractal dimension').

Note that if $d > \dim_{\mathrm{B}}(X)$ then $N(X, \varepsilon) \le C\varepsilon^{-d}$ for some $C = C(d)$, with $N(X, \varepsilon) \le \varepsilon^{-d}$ for $0 < \varepsilon < \varepsilon_d$ for some $\varepsilon_d > 0$. If $d < \dim_{\mathrm{B}}(X)$ then there exist $\varepsilon_j \to 0$ such that $N(X, \varepsilon_j) > \varepsilon_j^{-d}$.

The following lemma is often useful.

Lemma 24. *Suppose that $\varepsilon_k \to 0$ with $\varepsilon_{k+1} \ge \alpha\varepsilon_k$ for some $\alpha \in (0, 1)$. Then*

$$\dim_{\mathrm{B}}(X) = \limsup_{k\to\infty} \frac{\log N(X, \varepsilon_k)}{-\log \varepsilon_k}. \tag{5}$$

Proof. The limsup in (5) is taken through a subset of ε; so the RHS $\le \dim_{\mathrm{B}}(X)$ as defined in (4). To show the opposite inequality, given $\varepsilon > 0$ choose k such that

$$\varepsilon_{k+1} \le \varepsilon < \varepsilon_k;$$

then

$$\frac{\log N(X, \varepsilon)}{-\log \varepsilon} \le \frac{\log N(X, \varepsilon_{k+1})}{-\log \varepsilon_k} \le \frac{\log N(X, \varepsilon_{k+1})}{\alpha - \log \varepsilon_{k+1}},$$

from which it follows that $\dim_{\mathrm{B}}(X) \leq$ RHS. $\square$

Proposition 25. *Let (X, d) be a metric space, and A and B compact subsets of X.*

(i) If $A \subseteq B$ then $\dim_{\mathrm{B}}(A) \leq \dim_{\mathrm{B}}(B)$;

(ii) if $\overline{U}$ is compact then $\dim_{\mathrm{B}}(U) = \dim_{\mathrm{B}}(\overline{U})$;

(iii) $\dim_{\mathrm{B}}(A \cup B) = \max(\dim_{\mathrm{B}}(A), \dim_{\mathrm{B}}(B))$;

(iv) if $f\colon (X, d) \to (Y, \rho)$ is θ-Hölder, $\theta \in (0, 1]$, then

$$\dim_{\mathrm{B}}(f(X)) \leq \frac{\dim_{\mathrm{B}}(X)}{\theta};$$

(v) $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{B}}(A)$; and

(vi) $\dim_{\mathrm{B}}([0, 1]^d) = d$.

Proof. (i) is immediate from the definition, since $N(A, \varepsilon) \leq N(B, \varepsilon)$.

(ii) Clearly $\dim_{\mathrm{B}}(U) \leq \dim_{\mathrm{B}}(\overline{U})$ since any cover of $\overline{U}$ is a cover of U; if a collection of ε-balls covers U, then the 2ε-balls with the same centres cover an ε-neighbourhood of U, and so cover $\overline{U}$, so $\dim_{\mathrm{B}}(\overline{U}) \leq \dim_{\mathrm{B}}(U)$.

(iii) is immediate from the definition, since $N(A \cup B, \varepsilon) \leq N(A, \varepsilon) + N(B, \varepsilon)$.

(iv) Cover X with $N(X, \varepsilon)$ balls of radius ε. The image of any ε-ball under f is contained in a $C\varepsilon^\theta$-ball; so

$$N(f(X), C\varepsilon^\theta) \leq N(X, \varepsilon).$$

It follows that

$$\dim_{\mathrm{B}}(f(X)) = \limsup_{\varepsilon \to 0} \frac{\log N(f(X), C\varepsilon^\theta)}{-\log C\varepsilon^\theta} \leq \limsup_{\varepsilon \to 0} \frac{\log N(X, \varepsilon)}{0 \log C - \theta \log \varepsilon} = \frac{\dim_{\mathrm{B}}(X)}{\theta}.$$

(v) Take $s > \dim_{\mathrm{B}}(X)$, and choose σ such that $\dim_{\mathrm{B}}(X) < \sigma < s$. It follows that X can be covered by $N_\varepsilon \leq \varepsilon^{-\sigma}$ balls $\{B_j\}$ of radius ε for ε

sufficiently small. Then

$$\sum_{j=1^{N_\varepsilon}} (2\varepsilon)^s \leq 2^s \varepsilon^{s-\sigma},$$

which can be made arbitrarily small by taking ε sufficiently small. It follows that $\mathscr{H}^s(X) = 0$ and so $\dim_{\mathrm{H}}(X) < s$. Since this holds for any $s > \dim_{\mathrm{B}}(X)$ we obtain $\dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{B}}(X)$.

(vi) We know that $\dim_{\mathrm{B}}([0,1]^d) \geq \dim_{\mathrm{H}}([0,1]^d) = d$; for the upper bound we can cover $[0,1]^d$ by 2^{dk} closed cubes of side 2^{-k}, so the same number of balls of radius $\sqrt{d}2^{-k}$, which yields the same upper bound (using Lemma 24). □

The following lemma is also useful.

Lemma 26. *Let (X, d_X) and (Y, d_Y) be compact metric spaces. Then $(X \times Y, \rho_p)$, where*

$$\rho_p = (d_X^p + d_Y^p)^{1/p}, \ 1 \leq p < \infty, \qquad \textit{or} \qquad \max(d_X, d_Y),$$

has

$$\dim_{\mathrm{B}}(X \times Y) \leq \dim_{\mathrm{B}}(X) + \dim_{\mathrm{B}}(Y).$$

Proof. Take $s_X > \dim_{\mathrm{B}}(X)$ and $s_Y > \dim_{\mathrm{B}}(Y)$. For ε small enough, cover X with $\leq \varepsilon^{-s_X}$ d_X-balls X_j of radius ε, and cover Y with $\leq \varepsilon^{-s_Y}$ d_Y-balls Y_k of radius ε. Then $\{X_j \times Y_k\}$ is a cover of $X \times Y$ with $\leq \varepsilon^{-(s_X+s_Y)}$ balls of radius $2^{1/p}\varepsilon$. Therefore $\dim_{\mathrm{B}}(X \times Y) \leq s_X + s_Y$, and the result follows. □

Now observe that this lemma, along with the fact that $\dim_{\mathrm{B}}$ is non-increasing under Lipschitz maps, implies that

$$\dim_{\mathrm{B}}(X - X) \leq 2\dim_{\mathrm{B}}(X);$$

indeed, $X - X$ is the image of $X \times X$ under the Lipschitz map $(x, y) \mapsto x - y$. We therefore obtain the following as a corollary of Theorem 20.

Theorem 27. *Let X be a compact subset of a real Banach space B with $2\dim_{\mathrm{B}}(X) < d \in \mathbb{N}$. Then a residual set of maps in $L(B; \mathbb{R}^{d+1})$ are embeddings of X.*

Actually, since we know that any compact metric space can be embedded into a Banach space using an isometry (which is Lipschitz), we also have the following nice result.

Theorem 28. *Let (X, d) be a compact metric space with $2\dim_{\mathrm{B}}(X) < d \in \mathbb{N}$. Then there is a Lipschitz embedding of X into $\mathbb{R}^{d+1}$.*

The embedding in the theorem is the composition of one of the isometries we discussed at the beginning of the previous chapter, and the linear map from Theorem 27.

4.2 Examples

Lemma 29. *If*

$$X_\alpha = \{n^{-\alpha} : \ n \in \mathbb{N}\} \cup \{0\} \qquad \textit{then} \qquad \dim_{\mathrm{B}}(X_\alpha) = \frac{1}{1+\alpha}.$$

Proof. Note, using the Mean Value Theorem, that

$$\frac{\alpha}{(n+1)^{1+\alpha}} \leq \left|n^{-\alpha} - (n+1)^{-\alpha}\right| \leq \frac{\alpha}{n^{1+\alpha}}.$$

It follows that if n_ε is the smallest integer such that

$$\frac{\alpha}{(n_\varepsilon+1)^{1+\alpha}} \leq \varepsilon$$

then the first n_ε elements of the sequence all lie in distinct $\varepsilon/2$-balls. It follows that

$$N(X_\alpha, \varepsilon/2) \geq n_\varepsilon \geq (\alpha/\varepsilon)^{1/(1+\alpha)} - 1$$

and so $\dim_{\mathrm{B}}(X) \geq 1/(1+\alpha)$.

On the other hand, once we cover the first n_ε elements as above (say with ε-balls, i.e. intervals of length 2ε), all the remaining elements lie in the interval $[0, (n_\varepsilon+1)^{-\alpha}] \subset [0, (\varepsilon/\alpha)^{\alpha/(1+\alpha)}]$, which can be covered by no more than $(\varepsilon/\alpha)^{\alpha/(1+\alpha)}/\varepsilon \sim \varepsilon^{-1/(1+\alpha)}$ intervals of length 2ε. This shows that $\dim_{\mathrm{B}}(X) \leq 1/1(1+\alpha)$ and yields the required equality. □

Question: let $A = \{x_j\} \cup \{0\}$ such that $x_j \to 0$ and $\delta_j := x_j - x_{j+1}$ is decreasing. Can one find an explicit expression for $\dim_{\rm B}(A)$?

We now give an example in infinite-dimensional spaces. Let $\{e_j\}$ be the sequences consisting entirely of zero apart from having a 1 in the jth place.

Lemma 30. *Suppose that $\alpha_j \to 0$ with $\alpha_{j+1} \leq \alpha_j$; then the set*

$$A = \{\alpha_j e_j : \ j \in \mathbb{N}\} \cup \{0\}.$$

considered as a subset of ℓ^p, $1 \leq p < \infty$, or of c_0, has

$$\begin{aligned}\dim_{\rm B}(A) &= \limsup_{n\to\infty} \frac{\log n}{-\log |\alpha_n|} \\ &= \inf\left\{d : \ \sum_{j=1}^{\infty} |\alpha_j|^d < \infty\right\}.\end{aligned}$$

Proof. Take $0 < \varepsilon < |a_1|$ and choose $n = n(\varepsilon)$ such that

$$|\alpha_{n+1}| < \varepsilon \leq |\alpha_n|.$$

It follows that A can be covered by $n(\varepsilon) + 1$ balls of radius ε (one ball for each of the first $n(\varepsilon)$ points, and one ball centred at 0 for the remaining points). It follows that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{-\log \varepsilon} &\leq \limsup_{\varepsilon\to 0} \frac{\log(n(\varepsilon)+1)}{-\log \varepsilon} \\ &\leq \limsup_{\varepsilon\to 0} \frac{\log(n(\varepsilon)+1)}{-\log |\alpha_{n(\varepsilon)}|} \\ &\leq \limsup_{n\to\infty} \frac{\log n}{-\log |\alpha_n|}.\end{aligned}$$

For the opposite inequality, given any n such that $|\alpha_n| < 1$ let n' be the integer such that

$$|\alpha_n| = |\alpha_{n+1}| = \ldots = |\alpha_{n'}| > |\alpha_{n'+1}|,$$

and set $\varepsilon(n) = (|\alpha_n| + |\alpha_{n'+1}|)/4 < |\alpha_n|/2$. Then any two of the first n' elements lie at least $|\alpha_{n'}| > 2\varepsilon(n)$ apart, so all require their own $\varepsilon(n)$-ball in any cover. It follows that $N(A, \varepsilon(n)) \geq n'$, and so

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\log n}{-\log|\alpha_n|} &\leq \limsup_{n\to\infty} \frac{\log n'}{-\log|\alpha_{n'}|} \\
&\leq \limsup_{n\to\infty} \frac{\log N(A, \varepsilon(n))}{-\log 4\varepsilon(n)} \\
&\leq \limsup_{\varepsilon\to 0} \frac{\log N(A, \varepsilon)}{-\log \varepsilon}.
\end{aligned}$$

The equivalent formulation is not a property of dimensions, but an algebraic identity. Call the first expression (the limsup) d_1 and the second (the infimum) d_2. Given $d > d_2$, we have

$$\sum_{j=1}^{\infty} |\alpha_j|^d \leq M$$

for some $M > 0$. Since the $|\alpha_j|$ are non-increasing, it follows that $n|\alpha_n|^d \leq M$, from which it follows that $d_1 \leq d$, and hence $d_1 \leq d_2$.

Now take $d > d_1$; then $|a_n| \leq n^{-1/d}$ for all n sufficient large, and so for any $d' > d$ we have $\sum_{j=1}^{\infty} |\alpha_j|^{d'} \leq \sum_{j=1}^{\infty} n^{-d'/d} < \infty$. So $d_2 \leq d'$ for every $d' > d_1$, i.e. $d_2 \leq d_1$. □

Some examples (note that these are all countable sets so $\dim_{\mathrm{H}}(A) = 0$: if $\alpha_n = \alpha^n$ for some $\alpha \in (0, 1)$ then $\dim_{\mathrm{B}}(A) = 0$; if $\alpha_n = n^{-\beta}$ then $\dim_{\mathrm{B}}(A) = 1/\alpha$; if $\alpha_n = 1/\log(n + 1)$ then $\dim_{\mathrm{B}}(A) = \infty$.

4.3 Embeddings into Hilbert spaces with good inverses

We have seen that we can embed sets with finite box-counting dimension into Euclidean spaces. But we want to do better, and show that we can control the continuity of their inverses. We now prove a first version of this, embedding into a Hilbert space with a Hölder continuous inverse.

We use the following simple observation to build up our embedding map.

Lemma 31. *Suppose that X is a compact subset of a Banach space $\mathcal{B}$ with $\dim_{\mathrm{B}}(X) < \infty$. Then for any $\sigma > 2\dim_{\mathrm{B}}(X)$ there exist $\phi_n \in L(\mathcal{B}; \mathbb{R}^{m_n})$, where $M_n \le C2^{n\sigma}$, such that $\|\phi_n\| \le \sqrt{M_n}$ and*

$$z \in X - X \text{ with } \|z\| \ge 2^{-n} \qquad \Rightarrow \qquad |\phi_n(z)| \ge 2^{-(n+1)}.$$

Proof. Write $Z = X - X$; then $\dim_{\mathrm{B}}(Z) \le 2\dim_{\mathrm{B}}(X)$. Given σ as in the statement of the lemma we can cover Z using no more than $M_n := C2^{n\sigma}$ balls of radius $2^{-(n+2)}$.

Let the centres of these balls be $\{z_j\}$. Using the Hahn–Banach Theorem find $f_j \in \mathcal{B}^*$ such that $\|f_j\| = 1$ and $f_j(z_j) = \|z_j\|$. Define $\phi_n : \mathcal{B} \to \mathbb{R}^{M_n}$ by setting

$$\phi_n(x) = (f_1(x), \ldots, f_{M_n}(x));$$

then $\|\phi_n\| \le \sqrt{M_n}$.

Suppose that $z \in X - X$ with $\|z\| \ge 2^{-n}$. Then there exists a j such that $\|z - z_j\| < 2^{-(n+2)}$ and so

$$\begin{aligned} |\phi_n(z)| &\ge |f_j(z)| = |f_j(z_j) + f_j(z - z_j)| \\ &\ge \|z_j\| - |f_j(z - z_j)| \ge \|z\| - \|z - z_j\| - |f_j(z - z_j)| \\ &\ge \|z\| - 2\|z - z_j\| \ge 2^{-n} - 2 \cdot 2^{-(n+2)} \\ &= 2^{-n} - 2^{-(n+1)} \ge 2^{-(n+1)}. \end{aligned}$$

□

By combining these maps ϕ_n we can obtain a 'nice' embedding into a Hilbert space.

Theorem 32. *Suppose that X is a compact subset of a Banach space $\mathcal{B}$ with $\dim_{\mathrm{B}}(X) < \infty$. Then for any $\theta > 1 + \dim_{\mathrm{B}}(X)$ there exists a linear map $\Phi : \mathcal{B} \to H$, where H is a separable Hilbert space, with*

$$c_\theta^{-1}\|x - y\|^\theta \le |\Phi(x) - \Phi(y)| \le c_\theta\|x - y\|.$$

Proof. Now let $\{e_k\}$ be an orthonormal basis for a Hilbert space H. Set $s = \theta - 1$; then $s > \dim_{\mathrm{B}}(X)$ and we apply Lemma 31 with $2\dim_{\mathrm{B}}(X) < \sigma < 2s$. Now set

$$\Phi(x) = \sum_{k=1}^{\infty} 2^{-ns}\phi_k(x) \otimes e_k.$$

Then Φ is linear,

$$\|\Phi\| \leq \left(\sum_k 2^{-2ns} M_n \right)^{1/2} \leq C^{1/2} \left(\sum_k 2^{-n(2s-\sigma)} \right)^{1/2} < \infty,$$

and if $x, y \in X$ with $2^{-k} \leq \|x - y\| \leq 2^{-(k-1)}$ then

$$\begin{aligned} |\Phi(x) - \Phi(y)| = |\Phi(x - y)| &\geq 2^{-ks} |\phi_k(x - y)| \\ &\geq 2^{-ks} 2^{-(k+1)} \\ &\geq \frac{1}{2} 2^{-k(1+s)} \geq 2^{-(2+s)} \|x - y\|^{1+s}. \qquad \square \end{aligned}$$

5 Embedding subsets of $\mathbb{R}^N$ into $\mathbb{R}^k$

We now show that 'most' linear maps from $\mathbb{R}^N$ into $\mathbb{R}^k$ provide embeddings of sets with $N > k > 2\dim_{\rm B}(X)$ that have Hölder continuous inverses. The argument is a finite-dimensional version of the result of Hunt & Kaloshin (1999) that will form the main subject of the next lecture.

5.1 Linear maps from $\mathbb{R}^N$ into $\mathbb{R}^k$

Any linear map $L\colon \mathbb{R}^N \to \mathbb{R}^k$ we can write as

$$L = (L_1, \ldots, L_k), \qquad \text{where } L_j \in L(\mathbb{R}^N; \mathbb{R}).$$

We can write each L_j in the form

$$L_j = \phi_j^* = (\cdot, \phi_j) \qquad \text{for some } \phi_j \in \mathbb{R}^N.$$

We will consider the set

$$\mathcal{E} = \{L = (\phi_1^*, \ldots, \phi_k^*) : \ \phi_j \in B_N\},$$

where B_N denotes the (closed) unit ball in $\mathbb{R}^N$. Note that $\|L\| \le \sqrt{k}$ for every $L \in \mathcal{E}$.

We put a probability measure μ on $\mathcal{E}$ by choosing each ϕ_j at random from a uniformly distribution μ_0 on B_N, i.e. $\mu_j = |\Omega_N|^{-1}\lambda$, where Ω_N is the volume of the unit ball in $\mathbb{R}^N$ and λ is the Lebesgue measure.

We will need the following two lemmas. The estimate in the next is the key to the proof both in finite and infinite dimensions.

Lemma 33. *For any $\alpha \in \mathbb{R}^k$ and any $x \in \mathbb{R}^N$,*

$$\mu\{L \in \mathcal{E} : \ |\alpha + Lx| \le \varepsilon\} \le c_k N^{k/2} \left(\frac{\varepsilon}{|x|}\right)^k. \tag{6}$$

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$. Then

$$\begin{aligned}
&\mu\{L \in \mathcal{E} : \ |\alpha + Lx| \leq \varepsilon\} \\
&\qquad \leq \prod_{j=1}^{k} \mu\{L \in \mathcal{E} : \ |\alpha_j + \phi_j^* x| \leq \varepsilon\} \\
&\qquad = \prod_{j=1}^{k} \mu_0\{\phi \in B_N : \ |\alpha_j + (\phi, x)| \leq \varepsilon\}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\mu_0\{\phi \in B_N : |\alpha + (\phi, x)| \leq \varepsilon\} \\
&\qquad \leq \mu_0\{\phi \in B_N : \ |(\phi, x)| \leq \varepsilon\} \\
&\qquad \leq \frac{1}{\Omega_N}\left\{\Omega_{N-1} \ \times \ 2\frac{\varepsilon}{|x|}\right\} \\
&\qquad = 2\frac{\Omega_{N-1}}{\Omega_N}\frac{\varepsilon}{|x|}.
\end{aligned}$$

Since $\Omega_n = \pi^{n/2}/\Gamma((n/2)+1)$ it follows that

$$\mu_0\{\phi \in B_N : \ |\alpha + (\phi, x)| \leq \varepsilon\} \leq cN^{1/2}\frac{\varepsilon}{|x|},$$

and (6) follows. □

We will also need the Borel–Cantelli Lemma.

Lemma 34. *Let μ be a probability measure on a set $\mathcal{E}$. If $Q_j \subset \mathcal{E}$ are such that*

$$\sum_{j=1}^{\infty} \mu(Q_j) < \infty$$

then μ-almost every $x \in \mathcal{E}$ is contained in at most finitely many of the Q_j.

Proof. The set

$$\mathcal{Q} = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} Q_j$$

consists precisely of all those $x \in \mathcal{E}$ that lie in infinitely many of the $\{Q_j\}$. But for any n

$$\mu(\mathcal{Q}) \le \mu\left(\bigcup_{j=n}^{\infty} Q_j\right) \le \sum_{j=n}^{\infty} \mu(Q_j),$$

which can be made arbitrarily small since $\sum_{j=1}^{\infty} \mu(Q_j) < \infty$. □

We now use these two results to prove our embedding theorem.

Theorem 35. *Let X be a compact subset of $\mathbb{R}^N$. If $k > 2\dim_{\mathrm{B}}(X)$ then for any α with*

$$0 < \alpha < 1 - \frac{2\dim_{\mathrm{B}}(X)}{k}$$

μ-almost every $L \in \mathcal{E}$ satisfies

$$|x - y| \le c_L |Lx - Ly|^{\alpha}, \qquad x, y \in X.$$

Proof. Choose d such that $k > 2d > 2\dim_{\mathrm{B}}(X)$ and $\alpha < 1 - \frac{2d}{k}$.

We want to make sure that whenever $z \in X - X$ we have $|Lz| \ge |z|^{1/\alpha}$.

We split Z up into sets

$$Z_n = \{z \in X - X : \ |z| \ge 2^{-n}\}$$

and consider the sets

$$Q_n = \{L \in \mathcal{E} : \ |Lz| < 2^{-n/\alpha} \text{ for some} z \in Z_n\},$$

i.e. maps that fail to satisfy $|Lz| \ge |z|^{1/\alpha}$ for some $z \in Z_n$.

Since $\dim_{\mathrm{B}}(Z_n) \le \dim_{\mathrm{B}}(X - X) \le 2\dim_{\mathrm{B}}(X) < 2d$, Z_n can be covered by no more than $M_n := 2^{2nd/\alpha}$ balls of radius $2^{-n/\alpha}$; let these be $B(z_i, 2^{-n/\alpha})$.

Note that if

$$|Lz_i| \ge 2^{-n/\alpha} + \sqrt{k}2^{-n/\alpha}$$

then for every $z \in B(z_i, 2^{-n/\alpha})$ we have

$$\begin{aligned} |Lz| &\ge |Lz_i| - |L(z - z_i)| \ge 2^{-n/\alpha}(1 + \sqrt{k}) - \sqrt{k}|z - z_i| \\ &\ge 2^{-n/\alpha}(1 + \sqrt{k}) - \sqrt{k}(2^{-n/\alpha}) = 2^{-n/\alpha}. \end{aligned}$$

Therefore

$$\begin{aligned}
\mu\{L \in \mathcal{E} : |Lz| < 2^{-n/\alpha} \text{ for some } z \in B(z_j, 2^{-n/\alpha})\} \\
&\leq \mu\{L \in \mathcal{E} : \ |Lz_j| < (1+\sqrt{k})2^{-n/\alpha}\} \\
&\leq c_k N^{k/2} \left(\frac{(1+\sqrt{k})2^{-n/\alpha}}{2^{-n}} \right)^k \\
&= C_{k,N} 2^{nk(1-(1/\alpha))}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mu(Q_n) &\leq M_n C_{k,N} C_{k,N} 2^{nk(1-(1/\alpha))} \\
&= C'_{k,N} 2^{2nd/\alpha} 2^{nk(1-(1/\alpha))} \\
&= C'_{k,N} 2^{n[k-(k-2d)/\alpha]}.
\end{aligned}$$

To ensure that $\mu(Q_n)$ is summable, we need (i) $k > 2d$ and then (ii)

$$k - \frac{1}{\alpha}(k-2d) < 0 \qquad \Leftrightarrow \qquad \alpha < 1 - \frac{2d}{k},$$

which follows from our choice of d.

The Borel–Cantelli Lemma now ensures that μ-almost every L lies in only finitely many of the Q_j. We show that this implies the required Hölder continuity of L^{-1}.

Take L such that $L \notin Q_j$ for all $j \geq j_0$. If $z \in X - X$ with $2^{-(j+1)} \leq |z| \leq 2^{-j}$ for some $j \geq j_0$ then

$$|Lz| \geq 2^{-(j+1)/\alpha} \geq \left(\frac{|z|}{2} \right)^{1/\alpha} = 2^{-1/\alpha} |z|^{1/\alpha}.$$

If $z \in X - X$ with $|z| \geq 2^{j_0}$ then since $L \notin Q_{j_0}$ we certainly have

$$|Lz| \geq 2^{j_0/\alpha}.$$

Since X is compact it is bounded, so $X - X \subset B(0, R)$ for some $R > 0$; then

$$|Lz| \geq 2^{j_0/\alpha} \frac{|z|^\alpha}{R^\alpha}.$$

So we always have

$$|Lz| \geq \min\left(2^{j_0/\alpha} R^{-\alpha}, 2^{-1/\alpha}\right) |z|^{1/\alpha},$$

which implies that $|z| \leq c_L |Lz|^\alpha$, with $c_L = \max(R^{-\alpha} 2^{-j_0/\alpha}, 2^{1/\alpha})$. $\square$

6 Embedding subsets of H into $\mathbb{R}^k$

We now turn to the main result of these lectures, due to Hunt & Kaloshin (1999). We show that 'most' linear maps from H into $\mathbb{R}^k$ provide embeddings with Hölder continuous inverses.

6.1 A measure on a subset of $B(H;\mathbb{R}^k)$

First we have to show how to define a measure on an appropriate collection of linear maps.

Suppose that $\{V_j\}$ is a sequence of finite-dimensional subspaces of H. We let $d_j = \dim(V_j)$, and write B_j for the unit ball in V_j. By identifying this unit ball with the unit ball in $\mathbb{R}^{d_j}$ (e.g. by referring elements in V_j to any set of orthonormal basis elements) we can define a uniform probability measure on B_j, which we will call μ_j.

We now take $\gamma > 1$ and consider the set

$$\mathcal{E} = \{L = (L_1, \ldots, L_k) : \ L_j = \left(\sum_{n=1}^{\infty} n^{-\gamma}\phi_n^j\right)^*, \ \phi_n^j \in B_j\},$$

where for any $u \in H$, u^* is the linear functional on H given by $x \mapsto (x, u)$.

The choice $\gamma > 1$ ensures that the sum in the expression for L_j converges in H. We define a probability measure on H by choosing all of the ϕ_n^j at random according to the distribution μ_j on B_j.

Note that this is the same as choosing each L_j at random from the space

$$\mathcal{E}_0 := \{L_0 = \left(\sum_{n=1}^{\infty} n^{-\gamma}\phi_n^j\right)^*, \ \phi_n^j \in B_j\},$$

equipped with the measure $\mu_0 := \otimes_{j=1}^{\infty}\lambda_j$.

The following result is an immediate corollary of Lemma 33.

Corollary 36. *For any $\alpha \in \mathbb{R}$ we have*

$$\mu_j\{\phi \in B_j : \ |\alpha + (\phi, x)| < \varepsilon\} \leq c d_j^{1/2} \frac{\varepsilon}{\|P_j x\|}, \tag{7}$$

where P_j denotes the orthogonal projection onto V_j.

Proof. Noting that $(\phi, x) = (\phi, P_j x)$, the expression in (7) is precisely (6) in the case $k = 1$. □

We now prove the equivalent of Lemma 33 that will be the crucial estimate in our Hilbert-space version of Theorem 35.

Lemma 37. *For any $x \in H$ and any $\varepsilon > 0$,*

$$\mu\{L \in \mathcal{E} : \ |Lx| \leq \varepsilon\} \leq c \left(j^\gamma d_j^{1/2} \frac{\varepsilon}{\|P_j x\|} \right)^k, \tag{8}$$

for any choice of $j = 1, 2, \ldots$.

Proof. We have

$$\mu\{L \in \mathcal{E} : \ |Lx| \leq \varepsilon\} \leq \mu\{L \in \mathcal{E} : \ |L_j x| \leq \varepsilon, \text{ for all } j = 1, \ldots, k\}$$
$$\prod_{j=1}^{k} \mu_0\{L_0 \in \mathcal{E}_0 : \ |L_0 x| \leq \varepsilon\}.$$

We have

$$\begin{aligned} &\mu_0\{L_\in \mathcal{E}_0 : |L_0 x| \leq \varepsilon\} \\ &\qquad = \otimes_{n=1}^{\infty} \lambda_n \left\{ \left| \sum_{n=1}^{\infty} n^{-\gamma} (\phi_n, x) \right| \leq \varepsilon \right\} \\ &\qquad = \otimes_{n=1}^{\infty} \lambda_n \left\{ \left| \sum_{n \neq j} n^{-\gamma} (\phi_n, x) + j^{-\gamma} (\phi_j, x) \right| \leq \varepsilon \right\} \\ &\qquad \leq c j^\gamma d_j^{1/2} \frac{\varepsilon}{\|P_j x\|}, \end{aligned}$$

using Lemma 36 and the fact that the bound on the right-hand side of (7) does not depend on α. The estimate in (8) now follows. □

6.2 The thickness exponent

We now introduce a quantity that measures how well X can be approximated by finite-dimensional subspaces. We let $d(X,\varepsilon)$ denote the smallest dimension of a linear subspace V such that $\mathrm{dist}(X,V) \le \varepsilon$, and then define the thickness exponent $\tau(X)$ by setting

$$\tau(X) := \limsup_{\varepsilon \to 0} \frac{\log d(X,\varepsilon)}{-\log \varepsilon}.$$

Lemma 38. *Suppose that X is a compact subset of a Banach space B. Then $\tau(X) \le \dim_{\mathrm{B}}(X)$.*

Proof. Cover X with $N(X,\varepsilon)$ balls $\{B(x_j,\varepsilon)\}$ of radius ε, and then take $V = \mathrm{span}\{x_j\}$. It follows that $d(X,\varepsilon) \le \dim_{\mathrm{V}} \le N(X,\varepsilon)$. □

Clearly one can have $\tau(X) = 0$ but $\dim_{\mathrm{B}}(X) \neq 0$; just take a subset $\hat{X}$ of $\mathbb{R}^k$ with $\dim_{\mathrm{B}}(\hat{X}) \neq 0$ and include this in a Hilbert space by augmenting $\mathbb{R}^k$ with a countable orthonormal set of vectors.

[There are situations in which better bounds on the thickness are available: for example, Friz & Robinson (1999) showed that if $X \subset L^2(\Omega)$, where $\Omega \subset \mathbb{R}^d$ and X is bounded in $H^s(\Omega)$ then $\tau(X) \le d/s$.]

We can now prove the Hilbert space version of Theorem 35.

Theorem 39. *Let X be a compact subset of a Hilbert space H with $\dim_{\mathrm{B}}(X)$ finite. Then for any $k > 2\dim_{\mathrm{B}}(X)$ and any θ with*

$$0 < \theta < \frac{k - 2\dim_{\mathrm{B}}(X)}{k(1 + \tau(X)/2)}$$

μ-almost every $L \in \mathcal{E}$ (a subset of $L(H;\mathbb{R}^k)$ that will be defined in the proof) is injective on X with

$$\|x - y\| \le c_L |Lx - Ly|^{\theta}, \qquad x, y \in X.$$

Proof. Take $d > \dim_{\mathrm{B}}(X)$ and $t > \tau(X)$ such that

$$0 < \theta < \frac{k - 2d}{k(1 + t/2)}.$$

We now construct an appropriate set of linear maps $\mathcal{E} \subset L(H; \mathbb{R}^k)$, following the procedure earlier in this chapter. We let $\{V_j\}$ be a finite-dimensional linear subspace such that

$$\operatorname{dist}(X, V_j) \leq 2^{-(j+1)};$$

then by the definition of the thickness exponent we have

$$\dim(V_j) = d_j \leq c2^{t(j+1)} = c2^{tj}.$$

We construct $\mathcal{E}$ using this choice of $\{V_j\}$. Observe that every $L \in \mathcal{E}$ satisfies

$$\|L\|_{B(H;\mathbb{R}^k)} \leq C := \sqrt{k}\zeta(\gamma).$$

We follow very similar lines to the proof of Theorem 35. We let

$$Z_j := \{z \in X - X : \ \|z\| \geq 2^{-j}\}$$

and we want to ensure that $|Lz| \geq 2^{-j/\theta}$. So we let

$$Q_j := \{L \in \mathcal{E} : \ |Lz| < 2^{-j/\theta} \text{ for some } z \in Z_j\}.$$

Since $\dim_{\mathrm{B}}(Z_j) \leq \dim_{\mathrm{B}}(X - X) \leq 2\dim_{\mathrm{B}}(X) < 2d$ we can cover Z_j with $M_j \leq c2^{2dj/\theta}$ balls of radius $2^{-j/\theta}$. We consider for now one of these balls $B(z_0, 2^{-j/\theta})$, and let $Y = Z_j \cap B(z_0, 2^{-j/\theta})$. Note that if

$$|Lz_0| \geq (1 + C)2^{-j/\theta}$$

then

$$|Lz| \geq |Lz_0| - |L(z - z_0)| \geq (C + 1)2^{-j/\theta} - C2^{-j/\theta} = 2^{-j/\theta}.$$

So

$$\begin{aligned}
\mu\{L \in \mathcal{E} : |Lz| < 2^{-j/\theta} \text{ for some } z \in Y\} \\
&\leq \mu\{L \in \mathcal{E} : \ |Lz_0| < (1 + C)2^{-j/\theta}\} \\
&\leq c\left(j^\gamma d_j^{1/2} \frac{c2^{-j/\theta}}{\|P_j z_0\|}\right)^k.
\end{aligned}$$

Now note that since $\mathrm{dist}(X, V_j) \leq 2^{-(j+1)}$ we have $\|P_j z_0\| \geq 2^{-(j+1)}$; indeed

$$\|P_j z_0\| = \|P_j z_0 - z_0 + z_0\| \geq \|z_0\| - \|z_0 - P_j z_0\| \geq 2^{-j} - 2^{-(j+1)} = 2^{-(j+1)}.$$

Therefore

$$\mu\{L \in \mathcal{E} : \ |Lz| < 2^{-j/\theta} \text{ for some } z \in Y\} \leq c_k \left(j^\gamma 2^{jt/2} \frac{2^{-j/\theta}}{2^{-j}} \right)^k,$$

and so

$$\begin{aligned} \mu\{L \in \mathcal{E} : \ |Lz| < 2^{-j/\theta} \text{ for some } z \in Q_j\} &\leq M_j c_k \left(j^\gamma 2^{jt/2} \frac{2^{-j/\theta}}{2^{-j}} \right)^k \\ &= c_k 2^{2dj/\theta} \left(j^\gamma 2^{jt/2} \frac{2^{-j/\theta}}{2^{-j}} \right)^k \\ &= c_k j^{2\gamma k} 2^{j\{-(k-2d)/\theta + k(1+t/2)\}}. \end{aligned}$$

So $\sum \mu(Q_j) < \infty$ provided that

$$k > 2d \qquad \text{and} \qquad \frac{k-2d}{\theta} > k(1+t/2),$$

which were our initial assumptions.

The proof concludes as for that of Theorem 35. $\square$

We now have two immediate corollaries, obtained by combining this theorem with Theorem 32 and Lemma 17 or 18. This gives a Lipschitz-Hölder embedding of subsets of Banach spaces in Euclidean spaces.

Corollary 40. *Let X be a compact subset of a Banach space $\mathcal{B}$ with $\dim_{\mathrm{B}}(X)$ finite. Then if $k > 2\dim_{\mathrm{B}}(X)$ and*

$$0 < \alpha < \frac{1}{1 + \dim_{\mathrm{B}}(X)} \left\{ \frac{k - 2\dim_{\mathrm{B}}(X)}{k(1 + \tau(X)/2)} \right\}$$

there exists a linear map $\Lambda \in B(\mathcal{B}; \mathbb{R}^k)$ such that such that

$$\|x - y\| \leq c|\Lambda x - \Lambda y|^\alpha \qquad x, y \in X \tag{9}$$

and in particular Λ is injective on X.

Proof. Choose $t_1 > 1+\dim_{\mathrm{B}}(X)$ and $0 < t_2 < (k-2\dim_{\mathrm{B}}(X))/k(1+\tau(X)/2)$ such that

$$\alpha = \frac{t_2}{t_1}.$$

We use the linear map $\Phi\colon \mathcal{B} \to H$ from Theorem 32, for which

$$c_\theta^{-1}\|x-y\|_{\mathcal{B}}^{t_1} \le \|\Phi(x)-\Phi(y)\|_H \le c_\theta\|x-y\|_{\mathcal{B}}, \qquad x,y\in X.$$

Now let $\hat{X} = \Phi(X)$; since Φ is Lipschitz we have $\dim_{\mathrm{B}}(\hat{X}) \le \dim_{\mathrm{B}}(X)$; since Φ is linear and continuous, if V is any linear subspace of $\mathcal{B}$ then $\Phi(V)$ is a linear subspace of H of the same dimension, and so $d(\hat{X}, \|\Phi\|\varepsilon) \le d(X,\varepsilon)$, which shows that $\tau(\hat{X}) = \tau(X)$. We now use Theorem 39 to find a map $L \in B(H,\mathbb{R}^k)$ such that

$$\|x-y\|_H \le c_L|Lx-Ly|^\theta \qquad x,y\in\hat{X}.$$

Finally we set $\Lambda = L\circ\Phi$ to obtain (9). □

Since $\tau(X) \le \dim_{\mathrm{B}}(X)$, there is a corresponding result for metric spaces (cf. Foias & Olson, 1996); we set $\psi = \Lambda\circ\phi$, where ϕ is one of the isometric embeddings from Lemmas 17 and 18.

Corollary 41. *Suppose that* (X,d) *is a compact metric space with* $\dim_{\mathrm{B}}(X) < \infty$*. Then for any* $k > 2\dim_{\mathrm{B}}(X)$ *and*

$$0 < \alpha < \frac{1}{1+\dim_{\mathrm{B}}(X)}\left\{\frac{k-2\dim_{\mathrm{B}}(X)}{k(1+\dim_{\mathrm{B}}(X)/2)}\right\}$$

there exists a map $\psi\colon (X,d)\to\mathbb{R}^k$ *such that*

$$c^{-1}|\psi(x)-\psi(y)| \le \|x-y\| \le c|\psi(x)-\psi(y)|^\alpha \qquad x,y\in X.$$

6.3 Optimality of the Hölder exponent

We now show that the Hölder exponent in Theorem 39 is optimal as $k\to\infty$; in this limit we can achieve any α in the range

$$0 < \alpha < \frac{1}{1+\tau(X)/2} = \frac{2}{2+\tau(X)}.$$

We show that this cannot be improved by returning again to the orthogonal sequence $A = \{\alpha_j e_j\} \cup \{0\}$, where $\alpha_j \to 0$ and $|\alpha_{j+1}| \leq |\alpha_j|$. We show that this set has thickness exponent equal to its box-counting dimension, which (recall Lemma 30) is given by

$$\limsup_{n\to\infty} \frac{\log n}{-\log |a_n|}.$$

We first need a lemma.

Lemma 42. *Let $\{e_j\}_{j=1}^n$ be orthonormal vectors in a Hilbert space, and let $X = \{a_1 e_1, \ldots, a_n e_n\}$, where $|a_{i+1}| \leq |a_i|$. Then*

$$d(X, \varepsilon) \geq n\left(1 - \frac{\varepsilon^2}{|a_n|^2}\right).$$

Proof. If $d(X, \varepsilon) = d$ then there are $\{v_i\}_{i=1}^d$ such that $\|a_i e_i - v_i\| < \varepsilon$ and $\dim(\mathrm{span}\{v_i\}) = d$.

Let $U = \mathrm{span}\{e_i\}$, which has dimension n, and let P be the orthogonal projection onto U.

Set $v_i' = Pv_i$; then we have

$$\|a_i e_i - v_i'\| = \|P(a_i e_i - v_i)\| \leq \|a_i e_i - v_i\| < \varepsilon,$$

so the space spanned by the v_i' still approximates X within ε. Since $v_i' \in U$ for every i, clearly

$$n = \dim(U) = \dim(\mathrm{span}\{e_i\}) \geq \dim(\mathrm{span}\{v_i'\}) =: n - r,$$

for some $0 \leq r < n$.

Let $\{u_1, \ldots, u_r\}$ be an orthonormal basis for the orthogonal complement of span$\{v_i'\}$ in U; then we have

$$
\begin{aligned}
n\varepsilon^2 &\geq \sum_{i=1}^{n} \|a_i e_i - v_i'\|^2 \\
&\geq \sum_{i=1}^{n}\sum_{j=1}^{r} |(a_i e_i - v_i', u_j)|^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{r} |(a_i e_i, u_j)|^2 \\
&= \sum_{j=1}^{r}\sum_{i=1}^{n} |a_i|^2 |(u_j, e_i)|^2 \\
&\geq |a_n|^2 \sum_{j=1}^{r}\sum_{i=1}^{n} |(u_j, e_i)|^2 \\
&= |a_n|^2 r,
\end{aligned}
$$

since $\|u_j\|^2 = 1$. This gives the required inequality. □

Lemma 43. *The set* $A = \{\alpha_j e_j\}_{j=1}^{\infty} \cup \{0\}$ *as above has*

$$\tau(A) = \dim_{\mathrm{B}}(A) = \limsup_{n\to\infty} \frac{\log n}{-\log|a_n|} = \inf\{d : \sum_{j=1}^{\infty} |\alpha_j|^d < \infty\}.$$

Proof. We know that in general $\tau(A) \leq \dim_{\mathrm{B}}(A)$, so we only need show that $\limsup_{n\to\infty} \log n/(-\log|\alpha_n|)$ is a lower bound for $\tau(A)$.

Take n large enough that $|\alpha_n| < 1$ and choose n' such that

$$|\alpha_n| = \cdots = |\alpha_{n'}| > |\alpha_{n'+1}|;$$

set $\varepsilon_n^2 = (|\alpha_{n'}|^2 + |\alpha_{n'+1}|^2)/4$ so that

$$|\alpha_n|^2 = |\alpha_{n'}|^2 > 2\varepsilon_n^2 \qquad \Rightarrow \qquad 1 - \frac{\varepsilon_n^2}{|\alpha_{n'}|^2} > \frac{1}{2}.$$

Then, using Lemma 42 we have

$$d(A, \varepsilon_n) \geq n'\left(1 - \frac{\varepsilon_n^2}{|\alpha_{n'}|^2}\right) > \frac{n'}{2},$$

from which it follows that

$$\begin{aligned}\limsup_{n\to\infty}\frac{\log(n'/2)}{-\log|a_{n'}|} &\leq \limsup_{n\to\infty}\frac{\log(n/2)}{-\log|a_n|}\\ &\leq \limsup_{n\to\infty}\frac{\log d(A,\varepsilon_n)}{-\log\varepsilon_n}\\ &\leq \limsup_{\varepsilon\to 0}\frac{\log d(A,\varepsilon)}{-\log\varepsilon} = \tau(A). \qquad \square\end{aligned}$$

We now show that the Hölder exponent is optimal, following Pinto de Moura & Robinson (2010). We will use the result of Lemma 22 that

$$\operatorname{rank} P \geq \sum_{j=1}^{\infty}\|Pe_j\|^2,$$

for any orthogonal projection P and orthonormal set $\{e_j\}$.

Lemma 44. *If* $L\colon H\to\mathbb{R}^k$ *is a linear map such that*

$$\|x-y\| \leq c|Lx-Ly|^{\theta}, \qquad x,y\in A$$

then $\theta < 2/(2+\tau(A))$.

Proof. First, observe that if we decompose L as in Lemma 21 as $L = UP$, where P is an orthogonal projection onto a k-dimensional subspace and $U\colon PH\to\mathbb{R}^k$ is an invertible linear map, it follows that

$$\|x-y\| \leq c'\|Px-Py\|^{\theta}, \qquad x,y\in A.$$

Now, since P is linear and $0\in A$, it follows that we must have

$$|\alpha_j| = \|\alpha_j e_j\| \leq c'\|P(\alpha_j e_j)\|^{\theta} = c'|\alpha_j|^{\theta}\|Pe_j\|^{\theta} \qquad \text{for every } j\in\mathbb{N},$$

i.e.

$$\|Pe_j\| \geq c''|\alpha_j|^{(1-\theta)/\theta}.$$

Lemma 22 now implies that

$$\operatorname{rank} P \geq \sum_{j=1}^{\infty}\|Pe_j\|^2 \geq c''\sum_{j=1}^{\infty}|\alpha_j|^{2(1-\theta)/\theta}.$$

It follows that if the rank of P is finite we must have $\sum_j|\alpha_j|^{2(1-\theta)/\theta} < \infty$, from which it follows that $2(1-\theta)/\theta > \tau(A)$, and hence $\theta < 2/(2+\tau(A))$. $\square$

7 Assouad dimension and (almost) bi-Lipschitz embeddings

The main open problem in the theory of embeddings is what conditions are required on a metric space to ensure that there is a bi-Lipschitz embedding into some Euclidean space, i.e. a map $\phi\colon (X,d) \to \mathbb{R}^k$ (for some k) such that

$$\frac{1}{L} d(x,y) \le |\phi(x) - \phi(y)| \le L d(x,y), \qquad x, y \in X,$$

for some $L > 0$. (We say that such a ϕ is L-bi-Lipschitz.)

Note that any properties of subsets of $\mathbb{R}^k$ that are invariant under bi-Lipschitz maps must be shared by such a metric space. One such property is homogeneity.

Definition 45. *A metric space* (X,d) *is* (M,s)*-homogeneous if*

$$N\left(X \cap B(x,r), \rho\right) \le M \left(\frac{r}{\rho}\right)^s \qquad \text{for every} \quad x \in X,\ 0 < \rho < r.$$

Lemma 46. *Any subset of* $\mathbb{R}^n$ *is* $(2^n n^{n/2}, n)$*-homogeneous.*

Proof. Any cube of side $2r$ can be covered by $[(r/\rho) + 1]^n$ cubes of side 2ρ. So any ball of radius r, which is a subset of a cube of side $2r$, can be covered by

$$\left[\left(\frac{r}{\rho/\sqrt{n}}\right) + 1\right]^n$$

balls of radius ρ, since every cube of side $2\rho/\sqrt{n}$ is contained in a ball of radius ρ. It follows that

$$N(B(x,r),\rho) \le 2^n n^{n/2}. \quad \square$$

Lemma 47. *If* (X, d_X) *is* (M,s)*-homogeneous and* $f\colon (X,d_X) \to (Y,d_Y)$ *is* L*-bi-Lipschitz then* $f(X)$ *is* $(L^{2s}M, s)$*-homogeneous.*

Note that while the constant M changes under this mappping, the exponent s does not.

Proof. Take $y \in f(X)$, and consider $B_Y(y,r) \cap f(X)$. Then $y = f(\xi)$ for some $\xi \in X$, and

$$f^{-1}[B_Y(y,r) \cap f(X)] \subset B_X(\xi, Lr) \cap X,$$

which can be covered by $\leq M(Lr/(\rho/L))^s$ X-balls of radius ρ/L. The image of these balls are contained in ρ balls in Y, and so

$$N(B_Y(y,r) \cap f(X), \rho) \leq ML^{2s}(r/\rho)^s,$$

as required. □

We can now define the Assouad dimension. Note that the previous lemma shows that this dimension is invariant under bi-Lipschitz embeddings; we also have $\dim_{\mathrm{A}}(U) \leq n$ for any subset of $\mathbb{R}^n$.

Definition 48. *The* Assouad dimension *of a metric space* (X,d) *is*

$$\dim_{\mathrm{A}}(X) = \inf\{s : X \text{ is } (M,s)\text{-homogeneous for some } M > 0\}.$$

If (X,d) has a bi-Lipschitz embedding into $\mathbb{R}^k$, it must be (M,k)-homogeneous for some $M > 0$, i.e. it must have Assouad dimension $\leq k$. However, this is not sufficient.

Lemma 49. *Properties of the Assouad dimension:*

(i) *If* $A \subseteq B$ *then* $\dim_{\mathrm{A}}(A) \leq \dim_{\mathrm{A}}(B)$*;*

(ii) $\dim_{\mathrm{A}}(A \cup B) = \max(\dim_{\mathrm{A}}(A), \dim_{\mathrm{A}}(B))$*;*

(iii) $\dim_{\mathrm{A}}(X) = n$ *is* X *is an open subset of* $\mathbb{R}^n$*; and*

(iv) *if* X *is compact then* $\dim_{\mathrm{B}}(X) \leq \dim_{\mathrm{A}}(X)$.

We only prove (iv): take $R > \mathrm{diam}(X)$, and then $X \cap B(x_0, R) = X$ for any $x_0 \in X$, and for any $s > \dim_{\mathrm{A}}(X)$ there exists $M > 0$ such that

$$N(X,r) = N(X \cap B(x_0,R), r) \leq M(R/r)^s = [MR^s]r^{-s},$$

and so $\dim_{\mathrm{B}}(X) \leq \dim_{\mathrm{A}}(X)$.

A simple example shows that this inequality can be strict.

Lemma 50. *Let $A = \{n^{-1} : n \in \mathbb{N}\} \cup \{0\}$. Then* $\dim_A(A) = 1$.

Proof. Note that $N([0, 1/n] \cap A, 1/n^2) \simeq n$. □

Another example showing that the Assouad dimension is very sensitive is $A = \{n^{-\alpha} e_n\} \cup \{0\}$ which has $\dim_A(A) = \infty$ (consider the number of balls of radius $m^{-\alpha}/2$ required to cover $B(0, m^{-\alpha}) \cap A$.

7.1 Assouad's embedding theorem

Theorem 51. *Let (X, d) be a homogeneous metric space. Then for every $0 < \alpha < 1$ the space (X, d^α) can be bi-Lipschitz embedded into $\mathbb{R}^{N(\alpha)}$, for some $N(\alpha)$.*

Proof. An ε-net A is a collection of points such that

$$\cup_{x \in A} B(x, \varepsilon) = X$$

An ε-net is maximal if

$$d(x, y) \geq \varepsilon \qquad x, y \in A, \ x \neq y.$$

Note that if X is (M, s)-homogeneous and A_1 is a maximal 1-net then

$$|A_1 \cap B(x, 12)| \leq M' \qquad x \in X;$$

note that every point in a maximal 1-net requires a distinct 1/2-ball to cover it, so

$$|A_1 \cap B(x, 12)| \leq N(X \cap B(x, 12), 1/2) \leq 24^s M =: M'.$$

We say that $\kappa : A \to \mathbb{N}$ is a (K, δ)-colouring map if $\kappa : A \to \{1, \ldots, K\}$ is such that $\kappa(x) \neq \kappa(y)$ if $x \neq y$ and $d(x, y) \leq \delta$. We now show that there is a $(M', 12)$ colouring map κ of A_1.

Let $Y = \{y_1, y_2, y_3, \ldots\}$ be a denumeration of A_1, and suppose that we have found a map

$$\kappa_i : \{y_1, \ldots, y_i\} \to \{1, \ldots, M'\}$$

that satisfies the required 'colouring' property. We can extend this to a map $\kappa_{i+1}\colon \{y_1 \dots, y_{i+1}\}$ provided we can choose $\kappa_{i+1}(y_{i+1})$ appropriate: but we can do this, since

$$|\{y_1, \dots, y_i\} \cap B(y_{i+1}, 12)| \leq M - 1,$$

so there is an allowable choice.

Now define a map

$$\phi_1\colon X \to \mathbb{R}^{M'}$$

by setting

$$\phi_1(x) = \sum_{a_i \in A_1} \max\{2 - d(x, a_i), 0\}\hat{\kappa}(a_i),$$

where $\hat{\kappa}(a_i) = w_{\kappa(a_i)}$, with $(w_1, \dots, w_{M'})$ an orthonormal basis for $\mathbb{R}^{M'}$.

If we set $\Delta_i(x) = \max\{2 - d(x, a_i), 0\}$ then

(i) $0 \leq \Delta_i(x) \leq 2$;

(ii) if $d(x, y) > 4$ then at least one of $\Delta_i(x)$ and $\Delta_i(y)$ is zero;

(iii) $|\{i : \ \Delta_i(x) \neq 0\}| \leq M'$;

(iv) $D_{ij} := |\Delta_i(x) - \Delta_i(y)| \leq \min(d(x, y), 2)$, which follows from a case-by-case analysis:

1. if $d(x, a_i) \geq 2$ and $d(y, a_i) \geq 2$ then $D_{ij} = 0$;
2. if $d(x, a_i) \geq 2$ and $d(y, a_i) < 2$ then

   $$D_{ij} = 2 - d(y, a_i) \leq d(x, a_i) - d(y, a_i) \leq d(x, y)$$

   but also $D_{ij} \leq 2$;
3. if $d(x, a_i) < 2$ and $d(y, a_i) < 2$ then

   $$D_{ij} = d(y, a_i) - d(x, a_i) \leq d(x, y)$$

   but also $D_{ij} < 2$.

From this it follows easily that

(i) $\phi_1(x) \leq 2M'$ (from (i) and (iii) above);

(ii) $|\phi_1(x) - \phi_1(y)| \leq 2M' \min(d(x,y), 2)$ (from (iii) and (iv) above);

(iii) if $\frac{1}{2}8 < d(x,y) \leq 8$ then

$$|\phi_1(x) - \phi_1(y)| \geq 1.$$

Note that $d(x, a_i) < 1$ for at least one $a_i \in A_1$, so $\Delta_i(x) \geq 1$; but if $d(x,y) > 4$ then $d(y, a_i) > 2$ and $\Delta_i(y) = 0$; we also know that $x, y \in B(a_i, 10)$ and if j is such that $d(y, a_j) < 2$ we have $d(a_i, a_j) < 12$, and hence $\kappa(a_j) \neq \kappa(a_i)$.

Applying the same construction to the metric space $(X, 2^{-j}d)$ yields a map $\phi_j\colon X \to \mathbb{R}^{M'}$ such that

$$|\phi_j(x) - \phi_j(y)| \leq 2M' \min(2^j d(x,y), 2)$$

and

$$2^{-(j+1)}8 < d(x,y) \leq 2^{-j}8 \qquad \Rightarrow \qquad |\phi_j(x) - \phi_j(y)| \geq 1.$$

Fix some $x_0 \in X$, and let $\phi'_j(x) = \phi_j(x) - \phi_j(x_0)$; then ϕ'_j has the two properties above, but now $\phi'_j(x_0) = 0$. We drop the prime.

Now let $(e_0, \ldots, e_{2N-1})$ be an orthonormal basis for $\mathbb{R}^{2N}$ and let $\hat{e}_j = e_{(j \bmod 2N)+1}$ be a 'cyclic extension' of this basis. We define

$$\phi(x) = \sum_{j \in \mathbb{Z}} 2^{-\alpha j} \phi_j(x) \otimes \hat{e}_j,$$

where by $\xi \otimes \eta$ we denote the tensor product of $\xi \in \mathbb{R}^{M'}$ and $\eta \in \mathbb{R}^{2N}$ (an element of a $2M'N$-dimensional Euclidean space). Note that $\phi(x_0) = 0$.

We now show that ϕ is bi-Lipschitz from (X, d^α) into $\mathbb{R}^{2M'N}$ if N is chosen sufficiently large.

Suppose that

$$2^{-(k+1)}8 < d(x,y) \leq 2^{-k}8;$$

then

$$\begin{aligned}
|\phi(x)-\phi(y)| &\le \sum_{m\le k} 2^{-\alpha j}|\phi_m(x)-\phi_m(y)| + \sum_{m\ge k+1} 2^{-\alpha m}|\phi_m(x)-\phi_m(y)| \\
&\le C\sum_{m\le k} 2^{-\alpha m}2^m d(x,y) + C\sum_{m\ge k+1} 2^{-\alpha m} \\
&= C\left[\sum_{m\le k} 2^m(1-\alpha)\right] d(x,y) + C'2^{-\alpha k} \\
&\le C'2^k(1-\alpha)d(x,y) + C'2^{-\alpha k} \\
&\le C'd(x,y)^\alpha.
\end{aligned}$$

Since $\phi(x_0)=0$ this shows in particular that $\phi(x)$ converges for every $x\in X$.

For the lower bound we have

$$\begin{aligned}
|\phi(x)-\phi(y)| \ge &\left|\sum_{-N+k<j\le N+k} 2^{-\alpha j}(\phi_j(x)-\phi_j(y))\otimes\hat{e}_j\right| \\
&- \sum_{j\le -N+k} 2^{-\alpha j}|\phi_j(x)-\phi_j(y)| - \sum_{j>N+k} 2^{-\alpha j}|\phi_j(x)-\phi_j(y)| \\
\ge\, & 2^{-\alpha k}|\phi_k(x)-\phi_k(y)| - c2^{-\alpha(k-N)}2^{k-N}d(x,y) - c2^{-\alpha(N+k)}.
\end{aligned}$$

Note that for the second term on the right-hand side we have

$$c2^{-\alpha(k-N)}2^{k-N}d(x,y) \le c2^{-(1-\alpha)N}2^{k(1-\alpha)}2^{-k} \le c2^{-(1-\alpha)N}2^{-k\alpha},$$

and so choosing N sufficiently large we can ensure that

$$|\phi(x)-\phi(y)| \ge 2^{-\alpha k} - \frac{1}{2}2^{-\alpha k} = \frac{1}{2}2^{-\alpha k} \ge cd(x,y)^\alpha,$$

as required. □

A simple result weakening the requirement of a Lipschitz inverse, and embedding into a Hilbert space, uses a similar idea (Olson & Robinson, 2010).

Proposition 52. *Suppose that (X,d) is a compact metric space. Then for any $\gamma>1/2$ there exists a map $\phi\colon (X,d)\to H$, where H is a separable Hilbert space, such that*

$$\frac{1}{c_\gamma}\frac{d(x,y)}{|\log_2 d(x,y)|^\gamma} \le \|\phi(x)-\phi(y)\| \le c_\gamma d(x,y) \qquad x,y\in X.$$

Proof. Define the maps ϕ_k as in the proof of the above theorem, and set

$$\phi(x) = \sum_{k=0}^{\infty} k^{-\gamma} 2^{-k} \phi_k \otimes e_k,$$

where $\{e_k\}$ is an orthonormal set in a Hilbert space $\hat{H}$. Then

$$\begin{aligned} \|\phi(x) - \phi(y)\|^2 &\leq \sum_{k=0}^{\infty} k^{-2\gamma} 2^{-2k} |\phi_k(x) - \phi_k(y)|^2 \\ &\leq c\zeta(2\gamma) d(x,y)^2, \end{aligned}$$

and for $2^{-(k+1)} \leq d(x,y) \leq 2^{-k}$

$$\|\phi(x) - \phi(y)\| \geq k^{-\gamma} 2^{-k} |\phi_k(x) - \phi_k(y)| \geq k^{-\gamma} 2^{-k} \geq \frac{1}{c} \frac{d(x,y)}{|\log_2 d(x,y)|^{\gamma}},$$

as claimed. □

By combining these 'homogeneity' ideas with the approach of the Hunt & Kaloshin proof, Robinson (2009), following Olson & Robinson's Hilbert space proof, also proved the following about embeddings into Euclidean spaces.

Theorem 53. *Let X be a compact subset of a Banach space such that $\dim_{\rm A}(X - X) < s < N$. If*

$$\gamma > \frac{N+1}{N-s}$$

then 'many' bounded linear maps $L\colon B \to \mathbb{R}^N$ are injective on X with

$$\frac{1}{c_L} \frac{\|x-y\|}{|\log \|x-y\||^{\gamma}} \leq |Lx - Ly| \leq c_L \|x-y\| \qquad x, y \in X,$$

whenever $\|x - y\| < \rho_L$.