Four dimensions

1. Topological dimension n
   homeomorphism $C(X; \mathbb{R}^{2n+1})$
2. Hausdorff
   Mañé — linear maps
3. Box-counting dimension
   Hunt & Kaloshin — Hölder AS parametrisation

4. Assouad dimension

(X, d)

(X, d^α) bi-Lipschitz embeddings into $\mathbb{R}^N$ $\alpha \in (0,1)$

Lebesgue covering dimension

(X,d) compact

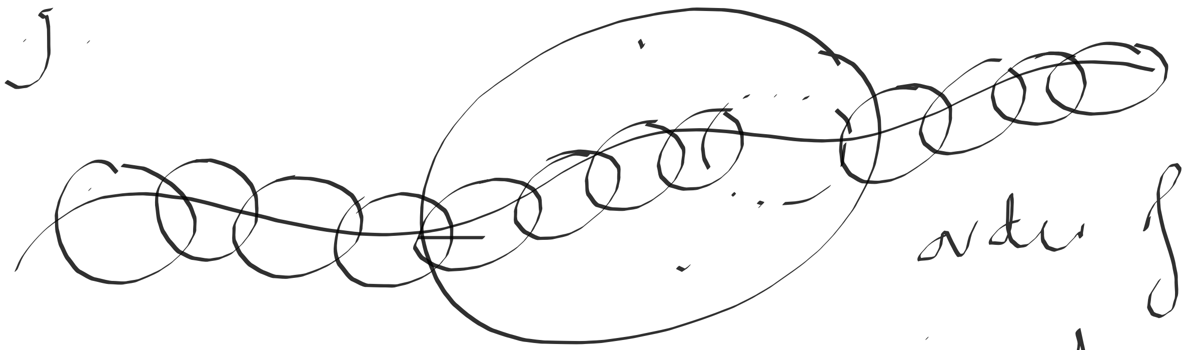
(open) covering of X is a collection $\{U_j\}_{j=1}^{l}$ open sets s.t.

$$X = \bigcup_{j=1}^{l} U_j$$

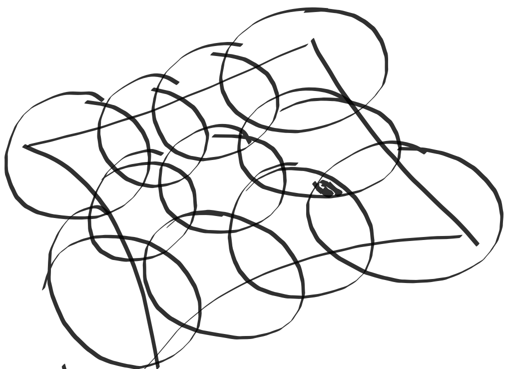
A refinement β of a cover α is another covering s.t. every element of β is contained in an element of α

The order of covering is n if the largest number of elements in the cover that have non-empty intersection is $n+1$

e.g.



order of refinement is 1.



order of covering is 2

<u>Definition</u>

$\dim(X) \leq n$ if every covering has a refinement of order n.

$$\dim(X) = \min\{n : \dim(X) \leq n\}$$

This is a topological property
i.e. invariant under homeomorphisms.

- This is integer-valued.

Hurewicz & Wallman (1941)
Munkres (2000)

Theorem

(X, d) compact metric space with $\dim(X) = n$.

Then a residual set of $f \in C(X, \mathbb{R}^{2n+1})$ are homeomorphisms between X and its image.

$f \longrightarrow$ embedding $\qquad f^{-1} \longrightarrow$ parametrisation

<u>We</u> will use the Baire Category theorem

Given $U \subset X$

define $\operatorname{diam} U = |U|$

$$= \sup_{u_1, u_2 \in U} d(u_1, u_2)$$

<u>Mesh size</u> of a covering is the largest diameter of elements of the covering (must be in a metric space)

Lemma

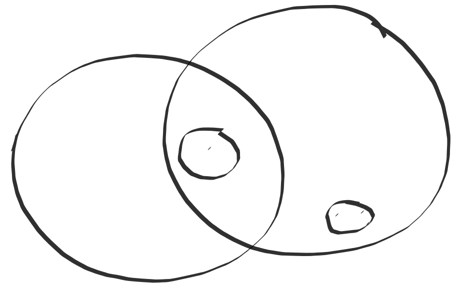
(X,d) compact

$\dim(X) \le n \iff \exists$ coverings of arb. small mesh size of order $\le n$

Proof "exercise"

$\Rightarrow$ cover X using $\varepsilon/2$ - balls

$\Leftarrow$ show that any covering has a "Lebesgue number" i.e. $\eta > 0$ s.t. any subset A of X with $|A| < \eta$ is entirely contained in some el.t of the cover $\square$



We say that $g \in C(X, \mathbb{R}^k)$ is an ε-mapping if

$$\operatorname{diam}(g^{-1}(y)) < \varepsilon \quad \forall y \in g(X)$$

i.e. if $g(x) = g(x') \Rightarrow d(x, x') < \varepsilon$

So if g is a $\frac{1}{n}$-mapping for every n

g is injective

But if $g \in C(X, \mathbb{R}^k)$

Compact

Hausdorff

is injective then it must be a homeomorphism

it's enough to show that g maps closed sets to closed sets

$K \subset X \Rightarrow K$ compact $\Rightarrow g(K)$ compact.

closed

$\Rightarrow g(K)$ closed.

Define

$$\mathcal{F}^k_\varepsilon = \{\text{set of all } \varepsilon\text{-mappings in } C(X, \mathbb{R}^k)\}$$

We will show that $\mathcal{F}^{2n+1}_{1/m}$ is open & dense for every m.

[We use sup metric on $C(X, \mathbb{R}^k)$

$$d(f,g) = \|f-g\|_\infty = \sup_{x \in X} |f(x) - g(x)|$$]

Lemma

$\mathcal{F}_\varepsilon$ is open $\forall\ \varepsilon > 0$

proof Take $g \in \mathcal{F}_\varepsilon$

$X \times X$ is compact

$Z = \{(x,x') \in X \times X : d(x,x') \geq \varepsilon\}$ is compact

$h : Z \to \mathbb{R}$ $\quad h(x,x') = |g(x) - g(x')|$

$h > 0$ on $Z \Rightarrow h(x,x') \geq \delta > 0$

if $f \in C(X; \mathbb{R}^k)$

$\|f - g\|_\infty < \delta/2$

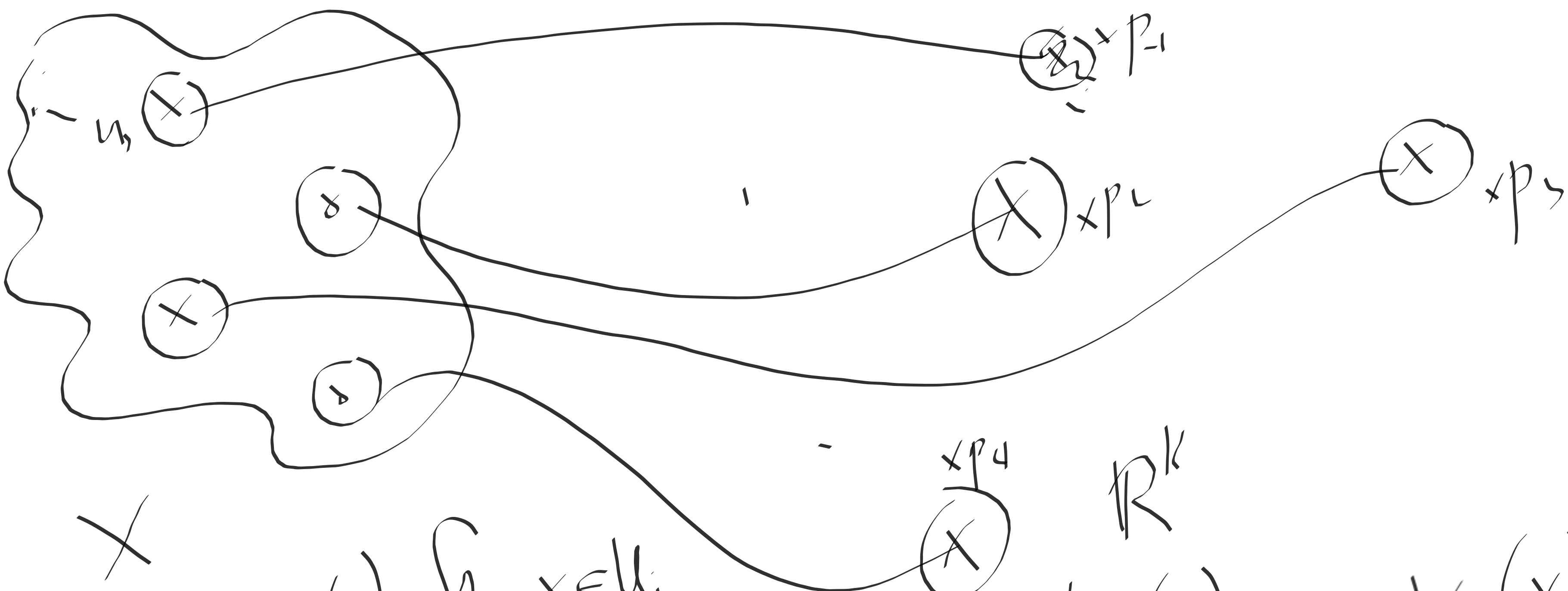
if $f(x) = f(x')$

then $\|g(x) - g(x')\| < \delta$

$\Rightarrow d(x, x') < \varepsilon$

$\Rightarrow f$ is an ε-mapping $\square$

To show density we need more ideas



$$w_j(x) = \begin{cases} 1 & x \in U_i \\ 0 & x \notin U_i \end{cases}$$

$$\phi_i(x) = \frac{w_i(x)}{\sum w_j(x)}$$

try

$$g(x) = \sum_{i=1}^{r} \phi_i(x) p_i \qquad \left[\|g - f\|_\infty < \text{small} \right]$$

we want

$$g(x) = g(x') \qquad x \simeq x'$$

$$\sum_{i=1}^{r} \phi_i(x) p_i = \sum_{i=1}^{r} \phi_i(x') p_i$$

$$\sum_{i=1}^{r} \underbrace{\left[\phi_i(x) - \phi_i(x') \right]}_{c_i} p_j = 0 \qquad \sum c_i = 0$$

Def

A set of points $\{x_0, \ldots, x_N\}$ in $\mathbb{R}^k$ is

geometrically independent if

$$\sum_{i=0}^{N} \alpha_i x_i = 0 \quad \& \quad \sum \alpha_i = 0$$

$$\Longrightarrow \alpha_i = 0 \quad \forall\, i = 0, 1, \ldots, N$$

if $\sum_{i=1}^{N} \alpha_i (x_i - x_0) = 0 \Longrightarrow \alpha_i = 0 \ \forall\, i = 1, \ldots, N$

vectors $(x_i - x_0)_{i=1}^{N}$ are l.i.

Any set of points generates a hyperplane P

$$\underline{x} = \sum_{i=0}^{N} t_i \underline{x}_i \qquad \text{where} \quad \sum_{i=0}^{N} t_i = 1$$

$$\Longleftrightarrow \quad \underline{x} = \underline{x}_0 + \sum_{i=1}^{N} a_i (x_i - x_0)$$

- if $N < k$ then P has empty interior
- if $\underline{y} \notin P$ then $(\underline{x}_0, \ldots, x_N, \underline{y})$ is geometrically indep

Proof

By induction. Take $y_1 = x_1$.

Suppose $(y_1, \ldots, y_m)$ in GP.

Consider all hyperplanes determined by $\leq k$ elements

Their union has empty interior / measure zero

$\exists\, y_{m+1} \in B(x_{m+1}, \delta)$ not in any of these planes.

these are ~~geometrical~~ in general pos:

take $k+1$

- if one of them is y_{m+1} then geom indep since $y_{m+1} \in P(m)$
- if not then geom indep by induction. $\square$

__Lemma__ in $\mathbb{R}^k$

$(x_1, \ldots, x_N)$ $\delta > 0$

$\exists (y_1, \ldots, y_N)$ in gen. pos.

$$|x_j - y_j| < \delta$$

A set of points in $\mathbb{R}^k$ is *in general position* if any collection of $\leq k+1$ points is geometrically independent

Lemma

Given $(x_1, \ldots\ldots, x_n) \in \mathbb{R}^k$ & $\delta > 0$ then $\exists (y_1, \ldots, y_n) \in \mathbb{R}^k$ in general position & $|y_j - x_j| < \delta$

Prop

If (X,d) compact $\dim(X) \leq n$

$\mathcal{F}_{\varepsilon}^{2n+1}$ is dense in $C(X;\mathbb{R}^{2n+1})$

Proof take $f \in C(X;\mathbb{R}^{2n+1})$ & $\eta > 0$

. X compact $\Rightarrow$ f uniformly cts

$\exists\, \delta < \varepsilon$ s.t. $d(x,x') < \delta \Rightarrow |f(x) - f(x')| < \eta/2$

$\exists$ covering of X $(U_j)_{j=1}^r$ s.t. $|U_j| < \delta$ $\forall j$

& order $\leq n$

[any point in X is in at least one U_j & at most $n+1$ U_j's]

$\operatorname{diam}(f(U_j)) < \eta/2$

Find points (p_j) in general position in $\mathbb{R}^{2n+1}$

s.t. $\operatorname{dist}(p_j, f(U_j)) < \eta/2$

Define

$$w_i(x) = \mathrm{dist}(x, X \setminus U_i)$$

$$w_i(x) > 0 \iff x \in U_i$$

$$w_i(x) = 0 \iff x \notin U_i$$

$$\phi_i(x) = \frac{w_i(x)}{\sum_i w_i(x)}$$

$$\sum \phi_j(x) = 1 \quad \forall x \in X$$

Set $g(x) = \sum_{i=1}^{r} \phi_i(x) p_i$

NB $g \in C(X, \mathbb{R}^{2n+1})$

& $|f(x) - g(x)| = \left| \sum_{i=1}^{r} \phi_i(x)(p_i - f(x)) \right|$

$\leq \sum_{i=1}^{r} \phi_i(x) \underbrace{|p_i - f(x)|}_{< \eta \text{ if } x \in U_i}$

$\leq \left\{ \sum_{j=1} \phi_j(x) \right\} \eta = \eta$

i.e. $\|f-g\|_\infty < \eta$

Now show that g is an ε-mapping

- if $g(x) = g(x')$

$$\sum_{i=1}^{r} \phi_i(x) p_i = \sum_{i=1}^{r} \phi_i(x') p_i$$

$$\sum_{i=1}^{r} \underbrace{\left[\phi_i(x) - \phi_i(x')\right]}_{c_i} p_i = 0$$

$$\sum c_i = 0$$

at most $(n+1)$ values of i make

$$\phi_i(x) \neq 0$$

similarly for $\phi_i(x')$

So at most $2n+2$ values of $c_i \neq 0$.

$$\sum c_i p_i = 0 \quad + \quad \sum c_i = 0$$

$\Rightarrow c_i = 0 \ \forall i$ as (p_j) in gen pos in $\mathbb{R}^{2n+1}$

$g(x) = g(x') \Rightarrow \phi_i(x) = \phi_i(x')$
$\forall i$

So if $g(x)=g(x')$

$x \in U_i \Rightarrow x' \in U_i$

but $|U_i| < \delta < \varepsilon$

$\Rightarrow d(x,x') < \varepsilon$

$\Rightarrow g$ is an ε-mapping $\square$

$\mathcal{J}_\varepsilon^{2n+1}$ is open & dense in $C(X, \mathbb{R}^{2n+1})$

So

$$\boxed{\bigcap_{m\geq 1}^{\infty} \mathcal{F}_{1/m}^{2n+1}}$$ is (dense) residual in $C(X, \mathbb{R}^{2n+1})$

by Baire Category Theorem

homeomorphisms of (X, d) onto

Subsets of $\mathbb{R}^{2n+1}$