Hausdorff dimension

To define the Hausdorff measure of $A \subset (X,d)$ start with

$$H^s_\delta(A) = \inf \left\{ \sum |U_j|^s , A \subset \bigcup_j U_j , |U_j| \le \delta \right\}$$

$$H^s(A) = \lim_{\delta \to 0} H^s_\delta(A)$$

s-dimensional Hausdorff measure

In $\mathbb{R}^n$, $H^n \propto L^n$ Lebesgue measure

– Falconer (1985)

Note that if $H^s(A) < \infty$ then $H^{s'}(A) = 0$ for $s' > s$.

- For any $\delta > 0$ $\exists$ cover of A by $\{U_j\}$ s.t. $|U_j| \leq \delta$

$$\sum |U_j|^s \leq H^s(A) + 1$$

$$\sum |U_j|^{s'} \leq \delta^{s'-s}[H^s(A) + 1],$$

We can now define

$$d_H(A) = \inf\{s : H^s(A) = 0\}$$

This is real valued

Lemma

$H^s(A) = 0 \iff \forall\, \varepsilon > 0\ \exists$ cover of A by $\{U_j\}$ s.t.

$$\sum |U_j|^s < \varepsilon$$

Proof (exercise)

Proposition

(i) $B \subset A$, $d_H(B) \leq d_H(A)$

(ii) stable under countable unions. If $(A_k)_{k=1}^{\infty} \subset X$ then

$$d_H\left(\bigcup_{k=1}^{\infty} A_k\right) = \sup_k d_H(A_k)$$

(iii) $f : (X, d) \longrightarrow (Y, \varrho)$

θ-Hölder cts, i.e.

$$\varrho(f(x), f(y)) \leq C\, d(x, y)^{\theta} \quad \theta \in (0, 1]$$

$$d_H(f(A)) \leq \frac{d_H(A)}{\theta}$$

d_H non-increasing under Lipschitz maps ($\theta = 1$).

(iv) $A \subset \mathbb{R}^n$ $\quad d_H(A) \leq n$

$A \subset \mathbb{R}^n$ contains an open set then

$d_H(A) = n$

Proof

(i) immediate from definition

(ii) if $\sup_k d_H(A_k) = \infty$ nothing to do

$\sup_k d_H(A_k) = \sigma < \infty$

for any $s > \sigma$ $\exists$ $U_j^{(k)}$ s.t.

$$\sum |U_j^{(k)}|^s < \varepsilon 2^{-k}$$

then $\{U_j^{(k)}\}_{j,k}$ covers union

$$\sum_{j,k} |U_j^{(k)}|^s < \varepsilon$$

$\xRightarrow{\text{by } \cancel{\text{lemma}}}$ $d_H(\text{union}) \leq \sigma$

Since $\forall \delta > 0$ $\exists k$ s.t

$d_H(\text{union}) \geq d_H(A_k) > \sigma - \varepsilon$ (part (i))

$\Rightarrow d_H(\text{union}) \geq \sigma$

$\Rightarrow d_H(U) = \sigma$

(iii) take $s > d_H(A)$ for any $\varepsilon > 0$

$\exists \{U_j\}$ s.t $\sum |U_j|^s < \varepsilon$

cover of A

$\Rightarrow \{f(U_j)\}$ covers $f(A)$

We have

$$|f(u_j)| \leq C|u_j|^\theta$$

So

$$\sum |f(u_j)|^{s/\theta}$$

$$\leq C^{s/\theta} \sum |u_j|^s < C^{s/\theta} \varepsilon$$

$$\Rightarrow d_H(f(A)) \leq s/\theta$$

$[0,1]^n$

(iv) take $A \subset \mathbb{R}^n$

then $A \subset \bigcup_{j=1}^{\infty}$ unit cubes

So it's enough to show that $d_H(Q) \leq n$

Take $s > n$

Cover Q by m^n cubes of side $\frac{1}{m}$.

then $\sum_{i=1}^{m^n} \left(\sqrt{n}\,\frac{1}{m}\right)^s \leq n^{s/2} m^n m^{-s}$

$< \varepsilon$ when m large enough

now suppose that A is open

so $A \supset$ open ball

if $d_H(A) < n$, $d_H(B) < n$ open ball

take $s = n$ $\mathcal{H}^s(B) = 0$

$B \subset (U_j)$ s.t. $\sum |U_j|^n < \varepsilon$

$U_j \subset B(x_j, |U_j|)$ volume of $B \leq \omega_n \sum (2|U_j|)^n$

$\square$

Note that if

$$f : (X, d) \longrightarrow (Y, \rho)$$

is bi-Lipschitz, i.e. f & f^{-1} are Lipschitz

then $d_H(f(A)) = d_H(A)$

bi-Lipschitz maps preserve the d_H.

We now show that $\dim(A)^{n} \leq d_H(A)$

<u>Lemma</u> (X, d) compact

Suppose that for every open cover $\{U_1, \dots, U_{n+2}\}$ of X we can find a cover $\{F_1, \dots, F_{n+2}\}$ closed with $F_j \subseteq U_j$

& $\bigcap_{j=1}^{n+2} F_j = \emptyset$

Then $\dim(X) \leq n$

Proof First we show that the same thing is true, replacing the closed sets (F_j) by open sets (V_j)

$$\underbrace{F_1}_{\text{closed}} \subseteq \underbrace{U_1 \cap \left\{ X \setminus \bigcap_{j=2}^{n+2} F_j \right\}}_{\text{open}}$$

So $\exists$ open V_1 s.t.

$$F_1 \subseteq V_1 \subseteq \overline{V_1} \subseteq \text{RHS}$$

So $V_1 \subseteq \overline{V_1} \subseteq U_1$

& $\{V_1, U_2, \ldots\ldots, U_{n+2}\}$ is a cover

& $\overline{V_1} \cap \bigcap_{j=2}^{n+2} F_j = \emptyset$

Now

$$F_2 \subseteq U_2 \cap \left\{ X \setminus \left(\overline{V_1} \cap \bigcap_{i=3}^{n+2} F_i \right) \right\}$$

So $\exists$ open V_2 $\quad F_2 \subseteq V_2 \subseteq \overline{V_2} \subseteq$ RHS

$\{V_1, V_2, U_3, \ldots, U_{n+2}\}$ is a cover &

$$\bar{V}_1 \cap \bar{V}_2 \cap \bigcap_{i=3}^{n+2} F_i = \phi$$

"and so on" $\Longrightarrow$ open sets $(V_j)_{j=1}^{n+2}$ as required.

Now we show that if $\{U_1, \ldots, U_k\}$ an open cover, there is a refinement of order $\leq n$

We perform the following construction on every collection of $n+2$ elements from $(U_j)_{j=1}^{k}$

Set $W_j = \begin{cases} U_j & j = 1, \ldots, n+1 \\ \bigcup_{j=n+2}^{k} U_j & j = n+2 \end{cases}$

(W_i) open cover of X,

We can find $V_j \subseteq W_j$ V_j open cover

s.t. $\bigcap_{j=1}^{n+2} V_j = \phi$

Now set

$$U_j' = \begin{cases} V_j & j = 1, \ldots, n+1 \\ U_j \cap V_{n+2} & j = n+2, \ldots, k \end{cases}$$

$U_j' \subseteq U_j$ & $\bigcap_{j=1}^{n+2} U_j' = \phi$

this yields a refinement of order $\leq n$

$\Rightarrow \dim(X) \leq n$. $\square$

We use this to show —

Theorem (X, d) compact (Edgar, 2008)

$\dim(X) \leq d_H(X)$

Proof

Suppose that $\dim(X) = n$

In particular $\dim(X) \not\leq n-1$

Lemma implies that $\exists$ $(U_j)_{j=1}^{n+1}$ open

cover of X s.t. if $F_j \subseteq U_j$ closed &

$(F_j)_{j=1}^{n+1}$ covers X, then $\bigcap_{j=1}^{n+1} F_j \neq \emptyset$

Given $(U_j)_{j=1}^{n+1}$ define

$$w_j(x) = \operatorname{dist}(x, X \setminus U_j)$$

- $w_j(x) > 0 \iff x \in U_j$
- $|w_j(x) - w_j(x')| \leq d(x, x')$

Lipschitz

$$\sum_{j=1}^{n+1} w_j(x) > 0 \quad \forall x \in X \qquad \text{also Lipschitz}$$

Since $x \longmapsto \sum w_j(x)$ is cts & > 0 on X

X compact $\Rightarrow$ $\exists\, \gamma > 0$ s.t. $\sum w_j(x) \geq \gamma$ $\forall x \in X$

the map

$$x \longmapsto \phi_i(x) = \frac{w_i(x)}{\sum w_i(x)}$$

is also Lipschitz

NB $\sum \phi_i(x) = 1$

the map $\Phi : X \to \mathbb{R}^{n+1}$

defined by

$$\Phi(x) = (\varphi_1(x), \ldots, \phi_{n+1}(x))$$

is again Lipschitz

Now

$$d_H(\Phi(X)) \leq d_H(X)$$

We will show that $\Phi(X)$ contains

$$T = \left\{ (t_1, \ldots, t_{n+1}) : t_j > 0, \ \sum_{j=1}^{n+1} t_j = 1 \right\}$$

& $d_H(T) = n$

T is the image of

$$\left\{(t_1,\dots,t_n) : 0 < \sum^n t_j < 1\right\}$$

open subset of $\mathbb{R}^n$

under the map

$$(t_1,\dots,t_n) \longmapsto \left(t_1,\dots,t_n, 1-\sum_{j=1}^n t_j\right)$$

this is a bi-Lipschitz map

so $d_H(T)=n$

take some $\underline{t} \in T$

$t_j > 0 \quad \sum t_j = 1$

show that $\exists x \in X$ s.t. $\underline{\Phi}(x) = \underline{t}$

Consider

$$F_i = \{ x \in X : \phi_i(x) \geq t_i \}$$

- F_i are closed
- $F_i \subseteq U_i$

(F_i) form a cover

if not $\exists\, x \in X$ s.t.

$x \notin F_i \quad \forall i$

i.e. $\phi_i(x) < t_i \quad \forall i$

$$1 = \sum \phi_i(x) < \sum t_i = 1 \quad \times$$

It follows that $\bigcap_{i=1}^{n+1} F_i \neq \emptyset$

i.e. $\exists x \in X$ s.t. $x \in F_i \quad \forall i$

$$\phi_i(x) \geq t_i \quad \forall i$$

$$\sum \phi_i(x) = 1$$

$$\sum t_i = 1$$

$$\Rightarrow \phi_i(x) = t_i \quad \forall i$$

i.e. $\Phi(x) = \underline{t}$

So

$$T \subseteq \Phi(X)$$

$$n = d_H(T) \leq d_H(\Phi(X)) \leq d_H(X)$$

$$\| \quad \dim(X)$$

$\square_{L1}$

So using our embedding result from before,

<u>Corollary</u>

If $d_H(X) \leq n$ residual collection in $C(X, \mathbb{R}^{2n+1})$ are homeomorphisms between X and its image

Remarks

There are many examples in which this inequality is strict: any set with $d_H(X) \notin \mathbb{Z}$

middle-third Cantor set $\quad d_H = \frac{\log 2}{\log 3}$

(Falconer, 1990)

Lemma (X,d) compact (H & W)

If $\dim(X) = n$ $\exists$ homeomorphism $f \in C(X, \mathbb{R}^{2n+1})$ s.t.

$d_H(f(X)) = n$

"No topological fractals" $\dim(X) < d_H(X)$

not topologically invariant

Mañé

$X \subset$ Banach space

$d_H(X - X) < \infty$

$\{x - y : x, y \in X\}$

$\exists$ many $L : B \to \mathbb{R}^k$

$$(X, d) \downarrow \ell^\infty(X)$$