$\underline{X - X}$

$\boxed{\underline{(X, d)} \longrightarrow \underline{B}}$

? $X \longrightarrow \phi(X)$

$\phi(X) - \phi(X)$

$d_H(X - X) < \infty$

$\llcorner \; B \to \mathbb{R}^k$

Mañé

Isometric embeddings of metric spaces into Banach spaces

Two results

Proposition ("Kuratowski embedding")

(X,d) compact metric space

$$x \longmapsto d(x,\cdot) \in L^\infty(X)$$

this is an isometry

Proof

(X,d) compact $\Longrightarrow \operatorname{diam}(X) < \infty$

$\mathcal{F}(x) = d(x,\cdot)$

so $\mathcal{F}(x) \in L^\infty(X)$

$$|\mathcal{F}(x)(y) - \mathcal{F}(x')(y)| = |d(x,y) - d(x',y)|$$

$$\Rightarrow \|\mathcal{F}(x) - \mathcal{F}(x')\|_{L^\infty} \leq d(x,x')$$

to show the opposite

$$|d(x,x) - d(x',x)| = d(x',x)$$

$$\leq$$

$$|\mathcal{F}(x)(x) - \mathcal{F}(x')(x)|$$

$$\|\mathcal{F}(x) - \mathcal{F}(x')\|_{L^\infty} \geq d(x',x)$$

$$\Rightarrow \|\mathcal{F}(x) - \mathcal{F}(x')\|_{L^\infty(X)} = d(x',x). \quad \square$$

[If X is not compact, choose any point $p \in X$, & set

$$\mathcal{F}(x) = d(x, \cdot) - d(p, \cdot)$$]

Here's another, where the $\underbrace{\text{target space}}_{\ell^\infty}$ is independent of X.

NB Any compact (X, d) is <u>separable</u>

- for every $n \in \mathbb{N}$ find a cover by $B(x^n_j, 1/n)$

& take $\bigcup_{j,n} x^n_j \Rightarrow$ countable dense subset

Proposition

Separable

Suppose that (X,d) has a countable dense subset $\{x_j\}_{j=0}^{\infty}$. Define

$$\mathcal{G}(x) = y \in \ell^{\infty}$$

$$y_j = d(x, x_j) - d(x_j, x_0)$$

Proof

for every $j \in \mathbb{N}$

$$|y_j^{(x)}| = |d(x, x_j) - d(x_j, x_0)|$$
$$\leq d(x, x_0)$$

So $\|\underline{y}\|_{\ell^\infty} \leq d(x, x_0)$

$$|y_j(x) - y_j(x')|$$

$$= |d(x, x_j) - d(x', x_j)| \le d(x, x')$$

$$\|\underline{y}(x) - \underline{y}(x')\|_{\ell^\infty} \le d(x, x')$$

Now given $x, x' \in X$ $\varepsilon > 0$ choose k s.t.

$$d(x', x_k) < \varepsilon$$

then

$$|y_k(x) - y_k(x')| = |d(x, x_k) - \underbrace{d(x', x_k)}_{<\varepsilon}|$$

$$\geq d(x, x') - 2\varepsilon$$

$$\Rightarrow \quad \|y(x) - y(x')\|_{\ell^\infty} \geq d(x, x'). \quad \square$$

Neither $L^\infty(X)$ nor l^∞ are separable.

Q: Can you map a compact metric space isometrically onto a subset of some separable Banach space?

Embedding subsets of Banach spaces using linear maps

We will prove

Theorem If X compact, $X \subset \overset{\text{Banach}}{B}$, $d_H(X-X) < d \in \mathbb{N}$ then a residual set in $L(B, \mathbb{R}^{d+1})$ are embeddings of X.

Lemma

Suppose that K is a compact subset of a Banach space B with $0 \notin K$.
Then $\exists \{\phi_j\}_{j=1}^{\infty}$, $\phi_j \in B^*$ with $\|\phi_j\| = 1$ s.t

$$\phi_j(x) = 0 \ \forall j \Rightarrow x \notin K.$$

Proof

Let $\{x_j\}_1^\infty$ be a countable dense subset of K & use Hahn-Banach Theorem to find $\phi_j \in B^*$ s.t.

$\|\phi_j\| = 1$ & $\phi_j(x_j) = \|x_j\|$

If $x \in K$, then choose k s.t. $\|x - x_k\| < \|x\|/3$

Now

$$\phi_k(x) = \phi_k(x_k) - \phi_k(x_k - x)$$
$$\geq \|x_k\| - \|x - x_k\|$$
$$\geq \frac{2}{3}\|x\| - \frac{1}{3}\|x\| = \frac{1}{3}\|x\| > 0$$

& the result follows. $\square$

Proof of the theorem

$$X - X := \{x - y : x, y \in X\}$$

Let $A = X - X \setminus \{0\}$

Saying that L is injective on X

$$\iff \quad LA \not\ni 0, \quad \underline{L^{-1}(0) \cap A = \emptyset}$$

$$Lx = Ly \Rightarrow x = y$$

$$\underbrace{L(x-y)}_{} = 0 \Rightarrow x - y = 0$$

$\in A \quad y \quad x \neq y$

We define

$$A_r = \left\{ a \in A : \|a\| \geq 1/r \right\}$$

$$A = \bigcup_{r=1}^{\infty} A_r$$

We now use our lemma — note that $0 \notin A_r$ & A_r is compact — to find $\{\phi_j^{(r)}\}_{j=1}^{\infty}$ s.t.

$$\phi_j^{(r)}(x) = 0 \ \forall j, \quad x \notin A_r$$

We can now write

$$A_r = \bigcup_{j,n=1}^{\infty} A_{r,j,n}$$

$$A_{r,j,n} = \left\{ a \in A_r : |\phi_j^{(r)}(a)| \geq 1/n \right\}$$

We now define

$$L_{r,j,n} = \left\{ L \in L(B, \mathbb{R}^{d+1}) : L^{-1}(0) \cap A_{r,j,n} = \emptyset \right\}$$

Note that

$$\bigcap_{r,j,n=1}^{\infty} L_{r,j,n} \quad \text{is} \quad \left\{ L : L^{-1}(0) \cap A = \emptyset \right\}$$

We will show that each $L_{r,j,n}$ is open & dense in $L(B, \mathbb{R}^{d+1})$

To show density, take

$L_0 \in L_{r,j,n}$

Since $A_{r,j,n}$ is compact $\exists \delta > 0$

s.t. $|L_0 x| \geq \delta > 0 \quad \forall x \in A_{r,j,n}$

& $\exists M > 0$ s.t. $\|x\| \leq M \quad \forall x \in A_{r,j,n}$

So if $\|L - L_0\| < \frac{\delta}{2M}$

$$|Lx| \geq -|(L - L_0)x| + |L_0 x|$$

$$> \delta - \delta/2 = \delta/2 > 0$$

$\forall x \in A_{r,j,n}$

$\Rightarrow L \in L_{r,j,n}$

We will use the following observation

Suppose that $W \subset \mathbb{R}^{d+1}$

Consider the map $\psi : \mathbb{R}^{d+1} \longrightarrow S_d$

$$\psi(x) = \begin{cases} x/|x| & x \neq 0 \\ p & x = 0 \end{cases} \qquad \{x \in \mathbb{R}^{d+1} : |x| = 1\}$$

for some $p \in S_d$

then $d_H(\psi(W)) \leq d_H(W)$

- since ψ is Lipschitz on each set $C_R = \{x \in \mathbb{R}^{d+1} : |x| \geq R\}$ for $R > 0$

So $\psi(W) = \bigcup_{\ell=1}^{\infty} \psi(C_{1/\ell} \cap W) \underbrace{\cup \{p\}}_{\text{if } 0 \in W}$

note that

$$d_H\left(\psi\left(C_{1/\ell}\cap W\right)\right)\leq d_H\left(C_{1/\ell}\cap W\right)\leq d_H(W)$$

Lipschitz on $C_{1/\ell}$

d_H monotone

+ d_H stable under countable unions

Take $L_0 \in L(B; \mathbb{R}^{d+1})$ & $\varepsilon > 0$ and show that $\exists\, L \in L_{r,j,n}$ with $\|L - L_0\| < \varepsilon$

Consider $L_0 A_{r,j,n} \subset \mathbb{R}^{d+1}$

$\{L_0 x : x \in A_{r,j,n}\}$

Now consider

$$\psi(L_0 A_{r,j,n}) \subset S_d$$

$$d_H(\psi(L_0 A_{r,j,n})) \leq d_H(L_0 A_{r,j,n})$$
$$< d_1 \text{ (by our assumption)}$$

Since $d_H(S_d) = d_1$

$\exists z \in S^d \quad z \notin \psi(L_0 A_{r,jm})$

Now consider $\|L - L_0\| \leq \varepsilon$

$$L = L_0 + \varepsilon \varphi_j(\cdot) z$$

claim that $L \in L_{r,j,n}$, i.e.

$$Lx \neq 0 \quad \forall x \in A_{r,j,n}$$

Suppose on the contrary that $\exists\, x \in A_{r,j,n}$

s.t. $\quad x \in A_{i,j,n}$

$Lx = 0 \quad$ i.e. $\quad |\varphi_j^i(x)| \geq 1/n$

$$L_0 x + \varepsilon z \varphi_j^i(x) = 0$$

$$\Rightarrow z = -\frac{1}{\varepsilon \varphi_j^i(x)} L_0 x$$

$$z = \psi(z) = \pm \psi(L_0 x)$$

$\psi(\lambda v)$
$\shortparallel$
$\psi(v)$
$\lambda > 0$

$z = \psi(l \circ y)$ $\qquad y = \pm x$

$\qquad\qquad\qquad\qquad\qquad\qquad \cap$

i.e. $z \in \psi(L \circ A_{r,j,n})$ $\qquad A_{r,j,n}$

but $z \notin$ by construction

So $L \in \mathcal{L}_{r,j,n}$ & $\mathcal{L}_{r,j,n}$ is dense

$L_{r,j,n}$ are open & dense, so

$\bigcap L_{r,j,n}$ is residual

(in particular dense). $\square$

~~There are two issues here.~~

How do we understand

$$d_H(\mathcal{F}(X) - \mathcal{F}(X)) < \infty$$

if (X, d) is a metric space & $\mathcal{F}$ is embedding into $L^\infty(X)$ or ℓ^∞ within (X, d) itself?

$\exists$ example due to Kan
appendix of Sauer, Yorke, & Casdagli
used by Ben-Artzi et al. to find
$X \subset H$ (Hilbert) $d_H(X) = 0$
$\nexists$ no linear embedding into any $\mathbb{R}^k$
$\Rightarrow d_H(X - X) = \infty$

We use two lemmas.

<u>Lemma</u>

Let H be a Hilbert space & $L : H \to \mathbb{R}^k$ s.t. $LH = \mathbb{R}^k$. Then $U = (\operatorname{Ker} L)^\perp$ has dimension k, & $L = RP$, where P is the orthogonal proj^n onto U f.s. R

& $R: U \to \mathbb{R}^k$ is an invertible linear map.

For proof see lecture notes on web.

Lemma — (Ben Aziz et al.)

Let H be a Hilbert space, P an finite rank orthogonal projection, & $\{e_j\}_{j=1}^{\infty}$ an orthonormal set in H. Then

$$\operatorname{rank} P \geq \sum_{j=1}^{\infty} \|Pe_j\|^2$$

__Proof__

$P\mathcal{H} = U$ $\quad\quad \exists$ o-n basis $\{u_1, \dots, u_k\}$

$k = \dim U = \operatorname{rank} P$

$$Px = \sum_{j=1}^{k} (x, u_j) u_j$$

$$Pe_i = \sum_{j=1}^{k} (e_i, u_j) u_j$$

$L : H \longrightarrow \mathbb{R}^k$

Can we have

$$|L^{-1}(x) - L^{-1}(y)| \leq C\omega(|x-y|)$$

Some $C > 0$?

No

Now

Bessel's inequality

$$\|Pe_i\|^2 = \sum_{j=1}^{k} |(e_i, u_j)|^2$$

$$\sum_{i=1}^{\infty} \|Pe_i\|^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{k} |(e_i, u_j)|^2$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{\infty} |(e_i, u_j)|^2 \leq \sum_{j=1}^{k} \underbrace{\|u_j\|^2}_{=1} = k \qquad \square$$

Let $f : [0,\infty) \to [0,\infty)$ $f(0) = 0$
increasing

We will show that we cannot guarantee
that $|La| \geq \varepsilon f(\|a\|)$ $\forall a \in X - X$
for any ε, whatever the
embedding dimension

$\{a_n e_n\}$

$$d_H(x, -x) \leq d$$

$$|\phi_j^r(x)| \geq \frac{1}{n}$$

$$|\phi_j^r(-x)| \geq 1/n$$

$$x \in A_{r,j,n} \Rightarrow -x \in A_{r,j,n}$$