$X \subset\subset B \qquad d_B(X) < \infty$

$\sigma > 2 d_B(X)$

$\phi_n \in L(B; \mathbb{R}^{M_n}) \qquad M_n \leq C 2^{n\sigma}$

$\|\phi_n\| \leq \sqrt{M_n}$

$z \in X - X \qquad \|z\| \geqslant 2^{-n} \Rightarrow \|\phi_n(z)\| \geqslant 2^{-(n+1)}$

Theorem

If $X \subset\subset B$ $\quad d_B(X) < \infty$. For any $\theta > 1 + d_B(X)$ $\quad \exists\, L \in \mathcal{B}(B; H)$, where H is a separable Hilbert space s.t.

$$c_\theta^{-1} \|x-y\|^\theta \le \|Lx - Ly\| \le c_\theta \|x-y\|$$

$x, y \in X$

<u>Proof</u> Let $\{e_\alpha\}$ be an o-n basis for a Hilbert space H; set $s = \Theta - 1$

Choose σ s.t. $2\alpha_B(X) < \sigma < 2s$ & apply the lemma to find $\{\phi_n\}_{n=1}^{\infty}$

Set $$Lx = \sum_{n=1}^{\infty} 2^{-ns} \hat{\phi}_n(x)$$

Where $\hat{\phi}_1 \in H$ with components ϕ_1 in directions $(e_1, \ldots, e_{M_1})$

$\hat{\phi}_2 \in H$ —— ϕ_2 —— $(e_{M_1+1}, \ldots, e_{M_1+M_2})$

etc.

1/ L is bounded

$$\|Lx\|^2 \leq c \sum 2^{-2ns} 2^{n\sigma} < \infty$$

by choice of σ

So $L \in L(B; H)$

2/ lower bound

if $x \in X - X$ with

$$2^{-k} \leq \|x\| < 2^{-(k-1)}$$

$$\|Lx\| \geq 2^{-ks} |\phi_k(x)|$$

$$\geq 2^{-ks} 2^{-(k+1)} \geq \frac{1}{2} 2^{-k(s+1)}$$

$$\|Lx\| \geq c\,\|x\|^{1+s} = c\|x\|^{\theta} \qquad \square$$

NB $\theta > 1$

$$\|Lx - Ly\| \geq c_\theta \|x - y\|^{\theta} \quad x, y \in X$$

$$\Rightarrow \|L^{-1}a - L^{-1}b\| \leq c_\theta' \, \|a-b\|^{1/\theta}$$

Hölder cts inverse $\qquad a, b \in LX$

Embedding $X \subset \mathbb{R}^N$ into $\mathbb{R}^k$

$(N \gg k)$ in terms of $d_B(X)$

"Most" linear maps $\mathbb{R}^N \to \mathbb{R}^k$ are embeddings with Hölder inverses if $k > 2d_B(X) + 1$

A linear map $L : \mathbb{R}^N \to \mathbb{R}^k$

$L = (\ell_1, \ldots, \ell_k)$ $\quad \ell_j : \mathbb{R}^N \to \mathbb{R}$

each $\ell_j = (l_j, \cdot)$ for some $l_j \in \mathbb{R}^N$

$=: l_j^*$

We consider the set

$$E = \{ L = (\ell_1^*, \ldots\ldots, \ell_k^*) : \ell_j \in B_N \}$$

unit ball in $\mathbb{R}^N$

NB that

$$\|L\| \leq \sqrt{k}$$

We put a probability measure μ on E, by selecting each ℓ_j from a unif. dist^n μ_j on B_N.

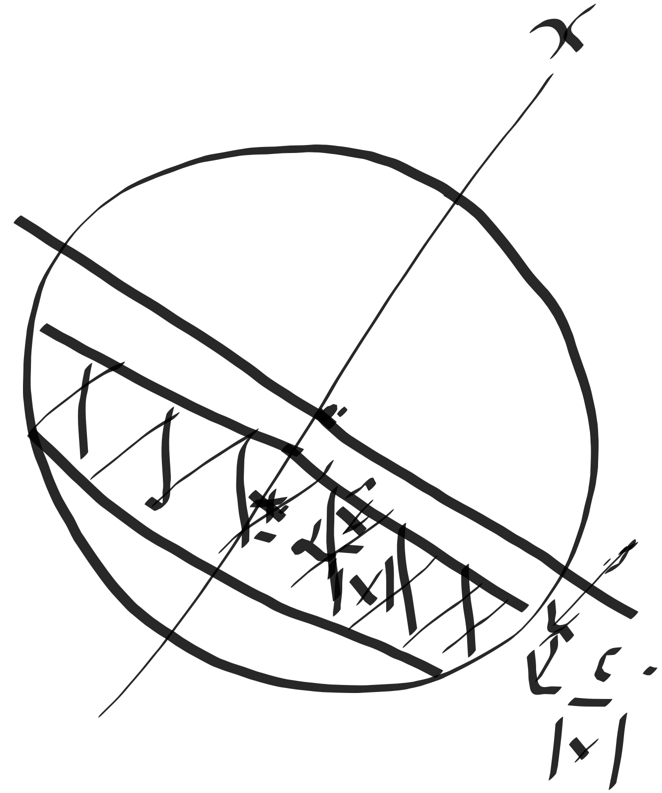
note that $\mu_j = \dfrac{\text{Lebesgue measure}}{\ell_N}$ volume of B_r

We will use two key lemmas

Lemma

For any $\alpha \in \mathbb{R}^k$ & any $x \in \mathbb{R}^N$

$$\mu\{L \in E : |\alpha + Lx| \leq \varepsilon\} \leq C_k N^{k/2} \left(\frac{\varepsilon}{|x|}\right)^k$$



Proof Let $\alpha = (\alpha_1, \dots, \alpha_k)$

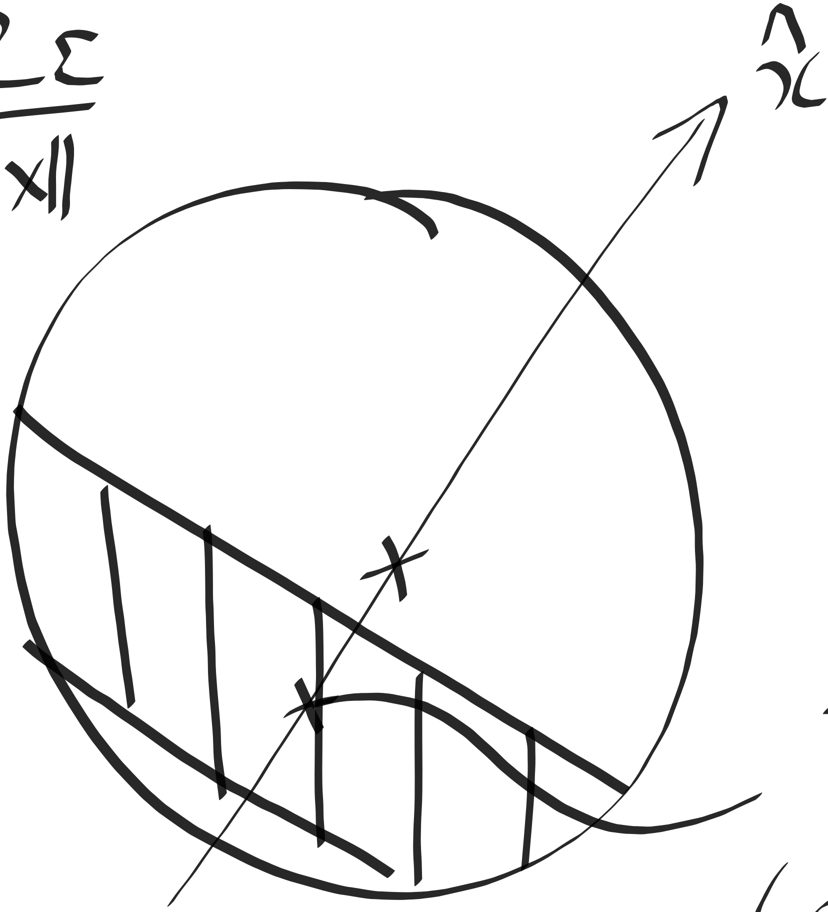
$$\mu\{L \in E : |\alpha + Lx| \leq \varepsilon\}$$

$$\leq \prod_{j=1}^{k} \mu\{L \in E : |\alpha_j + \ell_j^* x| \leq \varepsilon\}$$

but $\mu\{\ell_j \in B_N : |\alpha_j + (x, \ell_j)| \leq \varepsilon\}$

$\leq \dots$

$$\mathrm{vol} \le \Omega_{N-1} \frac{2\varepsilon}{\|x\|}$$



$$\mathrm{prob} \le \frac{2\Omega_{N-1}\,\varepsilon}{\Omega_N \|x\|}$$

So our required bound is

$$\left(2\,\frac{\Omega_{N-1}}{\Omega_N}\,\frac{\varepsilon}{\|x\|}\right)^k \qquad \le \sqrt[c]{N}$$

$$\Omega_n = \pi^{n/2} \Big/ \Gamma\left(\frac{n}{2}+1\right) \Rightarrow \frac{\Omega_{N-1}}{\Omega_N} = \frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(\frac{N}{2}+\frac{1}{2}\right)}\,\frac{1}{\sqrt{\pi}}$$

Borel-Cantelli Lemma

Let μ be a probability measure on E & $\Phi_j \subset E$ s.t.

$$\sum_{j=1}^{\infty} \mu(\Phi_j) < \infty$$

Then μ-a.e. $x \in E$ belongs to only finitely many $\{\Phi_j\}$.

Proof

Note that
$$\mathcal{L} = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} \varphi_j = \text{elements of } E \text{ in } \infty\text{ly many of the } \varphi_j\text{'s}$$

$$\mu(\mathcal{L}) \leq \mu\left(\bigcup_{j=n}^{\infty} \varphi_j\right) \leq \sum_{j=n}^{\infty} \mu(\varphi_j)$$

Since $\sum_{j=1}^{\infty} \mu(Q_j) < \infty$

$\forall \varepsilon > 0 \ \exists n_0$ s.t

$$\sum_{j=n_0}^{\infty} \mu(Q_j) \leq \varepsilon$$

$\mu(\mathcal{L}) \leq \varepsilon \quad \forall \varepsilon > 0 \Rightarrow \mu(\mathcal{L}) = 0.$ $\square$

Theorem

$X \subset\subset \mathbb{R}^N$. If $k > 2d_B(X)$ then for any α with

$$0 < \alpha < 1 - \frac{2d_B(X)}{k}$$

μ-a.e. $L \in E$ satisfies

$$|x-y| \le c_L |Lx - Ly|^\alpha \quad \forall x, y \in X.$$

Proof

We want to make sure that

$$z \in X - X \implies \|Lz\| \geq |z|^{1/\alpha}$$

Split Z up into

$$Z_n = \{ z \in X - X : |z| \geq 2^{-n} \}$$

and consider

the maps that are 'bad' for some $z \in Z_n$

$$\Phi_n = \{ L \in E : \|Lz\| < 2^{-n/\alpha} \text{ for some } z \in Z_n \}$$

We cover $Z_n \subset X - X$ by balls of radius $2^{-n/\alpha}$

for any $d > d_B(\mathcal{Z}_n)$ $\leq 2 d_B(X)$

we need $\leq N_n = 2^{2nd/\alpha}$ such balls

$B(z_i, 2^{-n/\alpha})$

Note that if $|Lz_i| \geq 2^{-n/\alpha} + \sqrt{k}\, 2^{-n/\alpha}$

then $|Lz| \geq |Lz_i| - |L(z - z_i)| \geq 2^{-n/\alpha}$ $\forall z \in$

So

$$\mu\{L : \|Lz\| < 2^{-n/\alpha} \text{ some } z \in B(z_j, 2^{-n/\alpha})\}$$

$$\leq \mu\{L : \|Lz_j\| < (1+\sqrt{k})\, 2^{-n/\alpha}\}$$

$$\underset{\text{lemma}}{\leq} c_k N^{k/2} \left\{ \frac{(1+\sqrt{k})\, 2^{-n/\alpha}}{2^{-n}} \right\}^k,$$

$$\leq c_{N,k}\, 2^{nk(1-1/\alpha)}$$

$$\mu(\Phi_n) \leq N_n\, c_{N,k}\, 2^{nk(1-1/\alpha)}$$

$$= 2^{2nd/\alpha}\, c_{N,k}\, 2^{nk(1-1/\alpha)}$$

$$= c_{N,k}\, 2^{n\left[k - \frac{1}{\alpha}(k-2d)\right]}$$

$$\mu(\Phi_n) \le c_{N,k}\, 2^{n\left[k - \frac{1}{\alpha}(k-2d)\right]}$$

We want

$$\sum \mu(\Phi_n) < \infty$$

- $k - 2d > 0 \Rightarrow k > 2d_B(X)$
- $k - \frac{1}{\alpha}(k-2d) < 0 \Rightarrow \alpha < 1 - \frac{2d}{k}$

Borel–Cantelli

$\Rightarrow$

a.e. $L \in$ only finitely many $\{Q_j\}$

given such an L

$\exists\, j_0$ s.t. $L \notin Q_j \ \forall j \geq j_0$

for $z \in X - X$ $2^{-(j+1)} \leq |z| \leq 2^{-j}$ $j \geq j_0$

$$|Lz| \geq 2^{-(j+1)/\alpha}$$

$$\geq \left(\frac{|z|}{2}\right)^{1/\alpha} = 2^{-1/\alpha} |z|^{1/\alpha}$$

for $z \in X - X$ with $|z| \geq 2^{j_0}$

if $X \subset B(0, R) \Rightarrow X - X \subset B(0, 2R)$

$$|Lz| \geq 2^{j_0/\alpha} \geq 2^{j_0/\alpha} \frac{|z|^\alpha}{(2R)^\alpha}$$

$$|Lz| \geq \min\left(2^{-1/\alpha}, \frac{2^{j\cdot\alpha}}{(2R)^{\alpha}}\right) |z|^{\alpha}$$

$\square$

Q: How to set up something similar in a Hilbert space? $\mathcal{H}$.

Take a sequence $\{V_j\}_{j=1}^{\infty}$ of finite-dim linear subspaces of $\mathcal{H}$.

Write B_j for the unit ball in V_j

$\cong B_{\mathbb{R}^{d_j}}$ $\quad d_j = \dim(V_j)$

$$E = \left\{ L = (l_1, \ldots, l_k) : l_j = \underbrace{\left(\sum_{n=1}^{\infty} n^{-\gamma} \phi_n^j \right)^*}_{} \right\}$$

$$\cap$$

$$L(H, \mathbb{R}^k)$$

if $\gamma > 1$ $\quad l \in H$

and if $u \in H$, $u^* \in H^*$ $\quad u^*(v) := (v, u)$

We define a probability measure on E by choosing all of the ϕ_n^j at random from a unif. distribution on

$$B_j \simeq \mathbb{R}^{d_j}$$

Call this distn λ_j

Lemma

$\alpha \in \mathbb{R}^k$

$$\lambda_j\left(\phi \in S_j : |\alpha + (\phi, x)| < \varepsilon\right\} \leq C\, d_j^{1/2} \left(\frac{\varepsilon}{\|P_j x\|}\right)$$

where P_j = orthogonal projection onto V_j

since $(\phi, X) = (\phi, P_j X)$

Lemma

$x \in \underline{H}$

$$\mu\{L \in E : |Lx| \le \varepsilon\} \le C\left(j^\gamma d_j^{1/2} \frac{\varepsilon}{\|P_j x\|}\right)^k$$

for any j

__Proof__

$$\leq \mu\left\{L : |L_j x| \leq \varepsilon \quad \forall\ j = 1, \dots, k\right\}$$

$$= \prod_{j=1}^{k} \mu\left\{|L_1 x| \leq \varepsilon\right\}$$

$$\mu\{|L_{\varphi}\lambda| \leq \varepsilon\}$$

$$\leq \bigotimes \lambda_i \left\{ \left| \sum_{n=1}^{\infty} n^{-\gamma} (\phi_n, x) \right| \leq \varepsilon \right\}$$

$$= \bigotimes \lambda_i \left\{ \left| \sum_{n \neq j}^{\infty} n^{-\gamma} \phi_n(x) + j^{-\gamma} (\phi_j, x) \right| \leq \varepsilon \right\}$$

$$\leq c\, j^{\gamma}\, \alpha_j^{1/2} \sum \|P_j x\|$$

$\Rightarrow$ result.

$\square$