$X \subset\subset \mathbb{R}^n \qquad d_B(X) < \infty$

i.e. $L : \mathbb{R}^n \to \mathbb{R}^k \qquad 0 < \theta < 1 - \frac{2\cancel{d}}{k}$

injective on X

$$c \| Lx - Ly \|^{\theta} \geq \| x - y \|$$

$(V_j)_{j=1}^{\infty}$ finite-dim subspaces of H

$\dim V_j = d_j$ $\qquad$ B_j unit ball in V_j

$L \in B(H, R^k)$

$\gamma > 1$

$$E = \left\{ L = (l_1, \ldots \ldots, l_k) \qquad L_j = \left(\sum_{n=1}^{\infty} n^{-\gamma} \phi_n^{(j)} \right)^* \quad \phi_n^{(j)} \in B_j \right\}$$

Key lemma

for any j

$x \in H$, $\varepsilon > 0$

$$\mu\{L \in E : |Lx| \le \varepsilon\} \le C\left(d_j^{1/2} j^{\gamma} \frac{\varepsilon}{\|P_j x\|}\right)^{k}$$

where P_j is the orthogonal projection onto V_j

Thickness exponent (Hunt & Kaloshin, 1999)

Let $d(X, \varepsilon)$ = smallest dimension of a linear subspace V s.t. $\operatorname{dist}(X, V) \leq \varepsilon$

$$\tau(X) = \limsup_{\varepsilon \to 0} \frac{\log d(X, \varepsilon)}{-\log \varepsilon}$$

Lemma

$X \subset H$

$\tau(X) \leq d_B(X)$

Proof

Cover X with $\# N(X,\varepsilon)$ ε-balls

$\{B(x_j,\varepsilon)\}$ so take $V = \operatorname{span}\{x_j\}$

then $\dim V \leq N(X,\varepsilon) \Rightarrow d(X,\varepsilon) \leq N(X,\varepsilon)$ $\square$

If $X \subset L^2(\Omega)$ $\Omega \subset \mathbb{R}^d$ then
if X is bounded in $H^s(\Omega)$ then

$$\tau(X) \leq d/s$$

Friz & R.

Can have $\tau(X) = 0$ but $d_B(X) \neq 0$.

Theorem

to be defined

$L : H \to \mathbb{R}^k$

$X \subset\subset H \qquad d_B(X) < \infty$

$k > 2 d_B(X)$ &

$$0 < \theta < \frac{k - 2d_B(X)}{k\left(1 + \frac{\tau(X)}{2}\right)}$$

then μ-a.e. $L \in E$ is $_\theta$ injective on X &

$C_L \| Lx - Ly \|^\theta \geq \|x - y\| \qquad x, y \in X$

Proof

~~a finite dim~~

Take $d > d_B(X)$ s.t.

and $t > \tau(X)$

$$0 < \alpha < \frac{k - 2d}{k(1+t/2)}$$

Now let V_j be a finite-dim subspace s.t.

$$\mathrm{dist}(X, V_j) \leq 2^{-(j+1)}$$

$$\Rightarrow \dim(V_j) = d_j \leq c\, 2^{t(j+1)} = c\, 2^{jt}$$

Now we construct E using this choice of $(V_j)_{j=1}^{\infty}$

let $Z_j = \{ z \in X - X : \|z\| \geq 2^{-j} \}$

we want to make sure that

$$\|Lz| \geq 2^{-j/\theta}$$

Let $\phi_j = \{ L \in E : \|Lz| < 2^{-j/\theta}$ for some $z \in Z_j \}$

Since $Z_j \subseteq X - X$

$d_B(Z_j) \leq d_B(X - X) \leq 2 d_B(X) < 2d$

we can cover Z_j with

$\leq M_j = 2^{2dj/\theta}$

balls of radius $2^{-j/\theta}$

Consider one of those balls $B(z_0, 2^{-j/\theta})$

& set $Y = Z_j \cap B(z_0, 2^{-j/\theta})$

Observe that every $L \in E$ satisfies

$$\|L\|_{B(H; \mathbb{R}^k)} \leq C := \sqrt{k}\, \rho(\gamma)$$

Note that if

$$|Lz_0| \geqslant (C+1)\, 2^{-j/10}$$

then $|Lz| \geqslant |Lz_0| - |L(z-z_0)|$

$$\geqslant (C+1)\, 2^{-j/10} - C\, 2^{-j/10} = 2^{-j/10}$$

So

$$\mu\left\{L \in E : \|Lz\| < 2^{-j/\theta} \text{ for some } z \in Y\right\}$$

$$\leq \mu\left\{L \in E : \|Lz_0\| < (1+c)2^{-j/\theta}\right\}$$

$$\leq C\left(j^{\gamma} d_j^{1/2} \frac{c' 2^{-j/\theta}}{\|P_j z_0\|}\right)^k$$

Now note that since

$$\operatorname{dist}(X, V_j) \leq 2^{-(j+1)}$$

$$\|P_j z_0\| \geq 2^{-(j+1)}$$

$$\|P_j z_0 - z_0 + z_0\|$$

$$\geq \|z_0\| - \|z_0 - P_j z_0\|$$

$$\geq 2^{-j} - 2^{-(j+1)}$$

$$\mu\{L \in E : \text{bad at some } z \in Y\}$$

$$\leq c'_k \left(j^\gamma \, 2^{jt/2} \, \frac{2^{-j/\theta}}{2^{-j}} \right)^k,$$

$$\mu\{L \in E : \text{bad at some } z \in Z_j\}$$

$$\leq M_{jn} \times \text{this estimate}$$

$$\mu(Q_j) \leq c_k 2^{2dj/\theta} \left(j^{\gamma} 2^{tj/2 - j/\theta + j} \right)^k$$

$$= c_k j^{2\gamma k} 2^{j\left\{ -(k-2d)/\theta + k\left(1 + t/2\right) \right\}}$$

So $\sum \mu(Q_j) < \infty$ if

$k > 2d$ & $\dfrac{k-2d}{\theta} > k\left(1 + t/2\right)$

Borel - Cantelli Lemma

$\Rightarrow$ μ-a.e. $L \in E$ lies in only finitely many (Q_j)

$\Rightarrow$ result as before. $\square$

Corollary

$X \subset\subset B$, Banach; $d_B(X) < \infty$

$\exists\, L \in \mathcal{B}(B, \mathbb{R}^k)$ s.t. inject

$k > 2 d_B(X)$

$$0 < \alpha < \frac{1}{1 + d_B(X)} \left\{ \frac{k - 2 d_B(X)}{k(1 + \tau(X)/2)} \right\}$$

$$\|x-y\| \leq c_L \|Lx - Ly\|^{\alpha} \quad \forall x, y \in X$$

L is injective on X.

Proof

We use the linear map $\Phi : B \to H$ s.t.

$$c' \|x-y\|^{\theta} \leq \|\Phi(x-y)\| \leq C \|x-y\|$$

$$\theta > 1 + d_B(X)$$

Note that since Φ is Lipschitz

$$d_B(\Phi(X)) \leq d_B(X)$$

If $\quad \mathrm{dist}(X, V) \leq \varepsilon \qquad V$ linear subspace of H

then $\mathrm{dist}(\Phi(X), \Phi(V))$

$$\leq \|\Phi\| \varepsilon$$

$$\dim \Phi(X) \leq \dim V \implies \tau(\Phi(X)) \leq \tau(X)$$

Now apply the theorem for subsets of Hilbert spaces to $\Phi(X)$ to find $L: H \to \mathbb{R}^k$

& consider

$L \circ \Phi$

$\square$

Corollary

(X, d) compact with $d_B(X) < \infty$

$\exists\, \psi : (X, d) \to \mathbb{R}^k \quad , k > d_B(X)$

ψ is Lipschitz
& ψ^{-1} is Hölder

Combine embedding of (X, d) into B with previous result
+ $\tau(X) \leq d_B(X)$

We found embeddings for $X \subset H$ with Hölder cts inverse with exponent

$$0 < \theta < \frac{k-2d}{k(1+\tau(X)/2)}$$

As $k \to \infty$

$$\text{"}\ \theta \to \frac{1}{1+\tau(X)/2}\ \text{"}$$

Is this optimal?

We return to example

$$A = \{\alpha_j e_j\}_{j=1}^{\infty} \cup \{0\}$$

$$\alpha_j \to 0 \qquad |\alpha_{j+1}| \leq |\alpha_j|$$

we saw before that

$$d_B(A) = \limsup_{n\to\infty} \frac{\log n}{-\log |a_n|} = \inf\left\{d : \sum |a_n|^d < \infty\right\}$$

We will show that $t(A) = d_B(A)$

Lemma

Take $X = \{v_1, \ldots, v_n\}$ orthogonal

Then $d(X, \varepsilon) \geq n\left(1 - \varepsilon^2/M^2\right)$

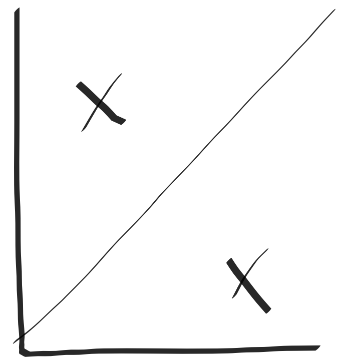
where $M = \min\{\|v_1\|, \ldots, \|v_n\|\}$

Q?

$X = \{v_1, \ldots, v_n\}$ $|v_{j+1}| \leq |v_j|$

orthogonal

$d(X, \varepsilon) = \max_j \{|v_j| > \varepsilon\}$?



We now show that $\tau(A) = \limsup_{n \to \infty} \frac{\log n}{-\log |a_n|}$

$\tau(A) \le d_B(A)$

we need $\tau(A) \ge$

take n large enough that $|a_n| < 1$

choose n' s.t.

$$|a_n| = \cdots\cdots |a_{n'}| > |a_{n'+1}|$$

Choose $\varepsilon_n^2 = \frac{1}{4}\left(|a_{n'}|^2 + |a_{n'+1}|^2\right)$

$$|a_n|^2 = |a_{n'}|^2 > 2\varepsilon_n^2$$

$$1 - \frac{\varepsilon_n^2}{|a_{n'}|^2} > \frac{1}{2}$$

$$d(A, \varepsilon_n) \geq n'\left(1 - \frac{\varepsilon_n^2}{|a_{n'}|^2}\right) > \frac{n'}{2}$$

/

So

$$\limsup_{n\to\infty} \frac{\log d(A, \varepsilon_n)}{-\log \varepsilon_n} \geq \limsup_{n\to\infty} \frac{\log n^t/2}{-\log |a_{n^t}|}$$

$$\leq \limsup_{\cancel{n\to\infty}\ \varepsilon\to 0} \frac{\log d(A, \varepsilon)}{-\log \varepsilon}$$

$\square$

We will use the result, that

$$\operatorname{rank} P \geq \sum_{j=1}^{\infty} \|Pe_j\|^2$$

Suppose that $L : H \to \mathbb{R}^k$ s.t.

$$\|Lx - Ly\| \geq c\|x-y\|^{\alpha} \quad x, y \in X$$

Since $0 \in X$,

$$\|Lx\| \geq c\|x\|^{\alpha} \quad \forall\ x \in X$$

We show before that $L : H \to \mathbb{R}^k$

$$L = T \circ P$$

invertible linear map $U \to \mathbb{R}^k$

orthogonal proj^n onto $U \simeq \mathbb{R}^k$

So

$$\|Px\| \geq c' \|x\|^{\alpha} \quad \forall\, x \in X$$

Take $X = \{\alpha_j e_j\} \cup \{0\}$

$$\|P(\alpha_j e_j)\| \geq c' \|\alpha_j e_j\|^{\alpha}$$

$$|\alpha_j| \|Pe_j\| \geq c' |\alpha_j|^{\alpha}$$

$$\|Pe_j\| \geq c' |\alpha_j|^{\alpha - 1}$$

$$\|Pe_j\|^2 \geq c'' |\alpha_j|^{2(\alpha-1)}$$

$$\operatorname{rank} P \geq \sum \|Pe_j\|^2 \geq c' \sum |\alpha_j|^{2(\alpha-1)}$$

so if $\operatorname{rank} P < \infty \Rightarrow \sum |\alpha_j|^{2(\alpha-1)} < \infty$

(R & Pinto de Moura)

So

$$2(\alpha-1) > \tau(A)$$

$$\Rightarrow \quad \alpha < \frac{1}{1+\frac{\tau(A)}{2}}$$

Hölder exponent is sharp (for the Hilbert space result) as $k \to \infty$

proof of $d(X, \varepsilon)$ lemma

If $d(X, \varepsilon) = d$

then $\exists\, v_i' \in H$ s.t. $\|v_i - v_i'\| < \varepsilon$

& $\dim \operatorname{span}(v_i') = d$

$U = \operatorname{span}(v_1, \dots, v_n)$ has dimension n

P orthogonal projection onto U

let $v_i'' = P v_i'$

then $\|v_i'' - v_i\| = \| P(v_i' - v_i) \|$

$< \varepsilon$

$\underbrace{\dim(\operatorname{span}(v_i))}_{n} \geq \underbrace{\dim(\operatorname{span}(v_i''))}_{n-r}$

We can write any element of U as

$$u = \left(\sum \alpha_j v_j''\right) + \sum_{j=1}^{r} \beta_j u_j$$

where (u_j) are a basis for the orthog. complement of $\text{Span}(v_i'')$ in U

$o-n$

$$n\varepsilon^2 \geq \sum_{i=1}^{n} \|v_i - v_i''\|^2$$

$$\geq \sum_{i=1}^{n} \sum_{j=1}^{r} |(v_i - v_i'', u_j)|^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{r} |(v_i, u_j)|^2$$

$$= \sum_{j=1}^{r} \sum_{i=1}^{n} \|v_i\|^2 \left| \left(u_j , \frac{v_i}{\|v_i\|} \right) \right|^2$$

$$\geq M^2 \sum_{j=1}^{r} \sum_{i=1}^{n} \left| \left(u_j , \frac{v_i}{\|v_i\|} \right) \right|^2$$

o.n basis for U

$$= M^2 r$$

$\|u_j\|^2 = 1.$

$$n\varepsilon^2 \geq M^2 r$$

$$r \leq \frac{n\varepsilon^2}{M^2}$$

$$d(X,\varepsilon) \geq n\left(1-\varepsilon^2/M^2\right). \quad \square.$$