7 Assouad dimension

(X, d) When is there a bi-Lipschitz embedding into some $\mathbb{R}^k$?

$\phi : (X, d) \to \mathbb{R}^k$ s.t.

$$\frac{1}{L} d(x,y) \leq |\phi(x) - \phi(y)| \leq L\, d(x,y)$$

A metric space (X, ℓ) is (M, s)-homogeneous if

$$N\big(X \cap B(x, r), \rho\big) \leq M\left(\frac{r}{\rho}\right)^s$$

$\forall x \in X, \quad 0 < \rho < r$

Lemma

Any subset A of $\mathbb{R}^n$ is $(2^n n^{n/2}, n)$-homogeneous

Proof

$$A \cap B(x,r) \subseteq B(x,r) \subseteq Q(x,r)$$

cube

Cover $Q(x,r)$ by

$$\left(\frac{r}{\rho}+1\right)^n$$

cubes of side ρ

any cube of side ρ $\subset$ ball of radius $\sqrt{n}\,\rho$

$$N(B(x,r),\rho) \leq \left(\frac{\sqrt{n}\,r}{\rho}+1\right)^n \leq \underline{2^n n^{n/2}} \left(\frac{r}{\rho}\right)^n \quad \square$$

Lemma

If (X, d_X) is (M, s)-homogeneous & $\phi : (X, d_X) \to (Y, d_Y)$ that is L-bi-Lipschitz then $\phi(X)$ is

$$(ML^{2s}, s)\text{-homogeneous}$$

Proof

take $y \in \phi(X)$ $\quad y = \phi(\xi)$

& consider

$$\phi^{-1}[B(y,r) \cap \phi(X)] \subseteq B(\xi, Lr) \cap X$$

cover $B(\xi, Lr) \cap X$ with $\leq M \left(\frac{Lr}{(\rho/L)}\right)^s$

balls of radius ρ/L

$\Rightarrow$ cover by $\leq M L^{2s} \left(\frac{r}{\rho}\right)^s$ ρ-balls in Y. $\square$

Any (X,d) bi-Lip embeddable into $\mathbb{R}^k$ must be (M,k)-homogeneous for some $M>0$.

The Assouad dimension $d_A(X)$

$$d_A(X) = \inf\{s : X \text{ is } (M,s)\text{-homogeneous for some } M>0\}.$$

Assouad dim. is invariant under bi-Lipschitz maps.

<u>**Lemma**</u> (Properties of d_A)

(i) If $A \subseteq B$ $\quad d_A(A) \le d_A(B)$

(ii) $d_A(A \cup B) \le \max\left(d_A(A), d_A(B)\right)$

(iii) $d_A(X) = n$ if X is an open subset of $\mathbb{R}^n$

(iv) If X is compact then

$$d_B(X) \leq d_A(X)$$

to see (iv), $X \subset B(x_0, R)$ some $R > 0$, $x_0 \in X$

$$N\left(X \cap B(x,R), \rho\right) \leq M\left(\frac{R}{\rho}\right)^s = MR^s \rho^{-s}$$

$s > d_A(X) \implies d_B(X) \leq s$

So

$$\dim(X) \leq d_H(X) \leq d_B(X) \leq d_A(X)$$

[X compact]

All these inequalities can be strict

Consider $A = \{\frac{1}{n}\} \cup \{0\} \subset [0,1]$

$d_H(A) = 0$; $d_B(A) = 1/2$; $d_A(A) = 1$.

$N\left(A \cap \left(\frac{1}{n}, \frac{1}{n}\right), 1/n^2\right) \approx n$

$n \quad \left(\frac{1}{m} - \frac{1}{m+1}\right) \sim \frac{n}{m^2} \lesssim \frac{1}{n}$

$m \geq n \Rightarrow d_{A,1}(A) = 1$

e.g. $A = \{n^{-\alpha} e_n\} \cup \{0\}$

in a Hilbert space

$d_A(A) = \infty$

Assouad dim can be much bigger than the box dimension.

Assouad's embedding theorem

(X, d) homogeneous metric space.

Then $\forall \alpha \in (0,1)$ $\exists$ bi-Lipschitz embedding of (X, d^α) into $\mathbb{R}^{N(\alpha)}$

Proof

An ε-net A for X is a collection of points s.t. $\forall\, x \in X$

$$d(x,a) < \varepsilon \text{ for some } a \in X$$

i.e. $\bigcup_{a \in A} B(a, \varepsilon) \supseteq X$

it is <u>maximal</u> if

$$d(x,y) \geq \varepsilon \quad \forall\ x,y \in A \quad x \neq y$$

Suppose that A_1 is a maximal 1-net in $X \leftarrow X$ is (M,s)-homog.

$$|A_1 \cap B(x,12)| \leq M' \quad \forall x \in X$$

$$|A_1 \cap B(x,12)| \le N\left(B(x,12), 1/2\right)$$

$$\le M 24^s = M'$$

A (K,δ)-<u>colouring</u> of a set A is a map $K : A \longrightarrow \{1, \dots, K\}$ s.t.

$$K(a) \neq K(b) \text{ if } d(a,b) \le \delta$$

Claim: there is an $(M', 12)$ colouring of A_1

Let $Y = \{y_1, y_2, y_3 \cdots\}$ be a denumeration of A_1 & suppose we have defined $K_i : \{y_1, \ldots, y_i\} \to \{1, \ldots, M'\}$

We want to define $K_{i+1}(y_{i+1})$ appropriately

$$|\{y_1, \ldots, y_i\} \cap B(y_{i+1}, 12)| \leq M' - 1$$

$$\subset A'_1 \cap B(y_i, 12)$$

$$|\sim\sim\sim\sim\sim| \leq M'$$

So we can define $K_{i+1}(\tilde{y}_{i+1})$ to ensure "the colouring property" let $\{e_j\}$ be a basis for $\mathbb{R}^{n'}$, &

define $\hat{K} : A_1 \longrightarrow \mathbb{R}^{n'}$

$$\hat{K}(a) = e_{K(a)}$$

Define $$\phi_1(x) = \sum_{a \in A_1} \max(2 - d(x,a), 0)\, \hat{K}(a)$$

$$\phi_1(x) = \sum_{a \in A} \max(2 - d(x,a), 0)\, \hat{k}(a),$$

$$\Delta_a(x) := \max(2 - d(x,a), 0)$$

(i) $0 \le \Delta_a(x) \le 2$;

(ii) if $d(x,y) > 4$ then $\Delta_a(x) \neq 0$
$\Rightarrow \Delta_a(y) = 0$

(iii) $|a : \Delta_a(x) \neq 0| \leq M'$

(iv) $D_a := |\Delta_a(x) - \Delta_a(y)|$

$$\leq \min(d(x,y), 2)$$

[case-by-case]

Now, suppose that

$$\frac{1}{2} \cdot 8 < d(x,y) \leq 8 \quad (*)$$

(i) $|\phi_1(x)| \leq \sqrt{M}\, 2 \quad \forall x \in X$

(ii) $|\phi_1(x) - \phi_2(y)| \leq \sqrt{2M} \min\big(d(x,y), 2\big) \quad \forall x, y \in X$.

(iii) - If ⊛ holds for

$$|\phi_1(x) - \phi_2(y)| \geq 1$$

- $d(x,a) < 1$ for at least one $a \in A$

So $\Delta_a(x) > 1$

since $d(x,y) > 4$ $\quad d(y,a) > 2$

$\Delta_a(y) = 0$

if $d(y,a') < 2$ for some $a' \in A$

then $d(a', a) \leq d(a, x)$

$+ d(x, y)$

$+ d(y, a') < 12$

every $a \in A_1$ s.t.

$\Delta_a(x) \neq 0 \quad \Delta_a(y) \neq 0$

has a different label

$$\Rightarrow \quad |\phi_1(x) - \phi_2(y)| \geq 1$$

$$|\phi_1(x) - \phi_2(y)| \leq 2 \min\left(\overline{d(x,y), 1}\right)$$

$y \qquad 4 < d(x,y) \leq 8$

$$|\phi_2(x) - \phi_2(y)| \geq 1$$

By applying the same construction to $(X, 2^{+j} d)$ or considering a 2^{-j}-net we obtain

$$\phi_j : X \to \mathbb{R}^{M'} \text{ s.t}$$

$$|\phi_j(x) - \phi_j(y)| \leq 2M' \min(2^j d(x,y), 1)$$

$$\& \quad 2^{-(j+1)} 8 < d(x,y) \leq 2^{-j} 8 \Rightarrow |\phi_j(x) - \phi_j(y)| \geq 1$$

Let $(\eta_j)_{j=0}^{2N-1}$ be a basis for $\mathbb{R}^{2N}$

Set $\hat{\eta}_j = \eta_{j \bmod 2N}$ $\quad j \in \mathbb{Z}$

$$\left[\phi(x) = \sum_{j \in \mathbb{Z}} \phi_j(x) \otimes \hat{\eta}_j \, 2^{-\alpha j} \right]$$

pick $x_0 \in X$ change $\phi_j \mapsto \phi_j(x) - \phi_j(x_0)$, We now show that ϕ is bi-Lip from (X, d^α) into $\mathbb{R}^{2NM'}$ for N large enough

Suppose that
$$2^{-(k+1)}\delta < d(x,y) \le 2^{-k}\delta$$

$$|\phi(x) - \phi(y)| \le \sum_{j \in \mathbb{Z}} |\phi_j(x) - \phi_j(y)| 2^{-\alpha j}$$
$$\le \sum_{j \le k} |\phi_j(x) - \phi_j(y)| 2^{-\alpha j} + \sum_{j > k} \frac{|\phi_j(x) - \phi_j(y)|}{2^{-\alpha j}}$$

$$\leq c' \sum_{j \leq k} 2^{-\alpha j} 2^{j} d(x,y)$$

$$+ c' \sum_{j > k} 2^{-\alpha j}$$

$$\leq c' 2^{-(\alpha-1)k} d(x,y) + c' 2^{-\alpha k}$$

$$\leq C d(x,y)^{\alpha}$$

$$2^{-(k+1)}\delta < d(x,y) \leq 2^{-k}\delta$$

$|\phi(x)-\phi(y)|$

$$\geq \Big|\sum_{k-N<j\leq k+N} 2^{-\alpha j}\left(\phi_j(x)-\phi_j(y)\right)\otimes \hat{e}_j\Big|$$

$$-\sum_{j\leq k-N} 2^{-\alpha j}|\phi_j(x)-\phi_j(y)|$$

$$-\sum_{j>k+N} 2^{-\alpha j}|\phi_j(x)-\phi_j(y)|$$

I'll

$$\geq 2^{-\alpha k} - \underbrace{\sum_{j<k-N} 2^{-\alpha j}\, 2^{j}\, d(x,y)} - \sum_{j \geq k+N} 2^{-\alpha j}$$

$$\sum_{j \geq k+N} 2^{-\alpha j} \leq 2^{-\alpha(k+N)} = 2^{-\alpha N}\, 2^{-\alpha k} < 2^{-\alpha k}/4$$

$$d(x,y)\, \underbrace{2^{-k}\, 2^{-k(\alpha-1)}}_{2^{-\alpha k}}\, 2^{(\alpha-1)N} < 2^{-\alpha k}/4$$

$$\geqslant \frac{1}{2} 2^{-\alpha k} \geqslant c\, d(x,y)^{\alpha}$$

$$c\, d(x,y)^{\alpha} \leq |\phi(x) - \phi(y)| \leqslant c'\, d(x,y)^{\alpha}$$

$\square$

a variant of this proof gives the following

$d_A(X) < \infty$ & $\gamma > 1/2$

$\exists\ \phi : X \to H$ s.t

$$c \frac{d(x,y)}{|\log_2 d(x,y)|^\gamma} \leq \|\phi(x) - \phi(y)\| \leq C d(x,y)$$