Errata and additional material for
Infinite-Dimensional Dynamical Systems

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Errata in the book

p 25, l 13  \( dx \) is missing from two integrals

p 43, equation (2.3) should read \( T(0) = \text{id} \).

p 45, l 9  delete “a” before “locally”

p 46, l 5- change \( x(t) \) to \( x(s) \)

p 52, l 13  RHS should be 0

p 75, l 7- closed ball required (should be \( \overline{B}(0, 1) \))

p 77, l 8- Should be \( \inf q(x) \) and \( \sup w(x) \), not vice versa

p 80, l 3- Should be \( \bar{x}_n \to x \) and \( A\bar{x}_n \to \bar{y} \)

p 103, l 9  Should be \( \|e_j\| \)

p 105, l 5- It is not true that the Alaoglu weak-* compactness theorem is valid in any Banach space, as the following example shows (thanks to Vittorino Pata for this example): take \( X = l^\infty \), and consider the sequence of functionals \( L_n : X \to \mathbb{R} \) defined by

\[
L_n(x) = x_n \quad \text{when} \quad x = (x_1, x_2, x_3, \ldots).
\]

The \( L_n \) is clearly a bounded sequence in \( (l^\infty)^* \), but does not have a weakly-* convergent subsequence. However, Corollary 4.19 (Reflexive weak compactness) \textit{does} hold as stated, i.e. is true for \textit{any} reflexive (not necessarily separable) Banach space - this is where Yosida (1980) provides a proof.
p 109, equation (5.1) and (5.3) on page 110 all derivatives \((d/dx_j)\) should be partial derivatives \((\partial/\partial x_j)\).

p 112, Definition 5.2 - should say that the derivatives of \(\phi_n\) should converge uniformly to ‘the corresponding derivatives of \(\phi\’

p 115, l 5 should be Theorem 5.4

p 125, l 1 should be ‘a function \(u \in H^k(\mathbb{R}^m)\)’

p 132, l 8- the integral is between \(-1\) and \(1\)

p 160, l 6- should be \(u\) instead of \(u(t)\)

p 161, l 4- (5.14) (not (5.13))

p 176, l 4- The use of \(\lambda\) in Proposition 6.15 is required since we cannot prove immediately \(H^2\) regularity on the whole of \(D^+\), but only on \(\lambda D^+\) for some \(0 < \lambda < 1\).

p 190, l 8- should be \(du_h/dt \rightarrow du/dt\)

p 196, l 14 change \(\psi\) to \(\psi(t)\)

p 204, l 12 weak-* convergence of \(P_n f\) to \(f\) in \(L^2(0, T; V^*)\) follows using a much simpler argument. In fact, a straightforward application of Lebesgue’s dominated convergence theorem (theorem 1.7 (iii)) shows that \(P_n f\) converges strongly to \(f\) in \(L^2(0, T; V^*)\).

p 205, l 9- change to ‘suppose that \(x_n\) is given by the constant sequence’
p 207, l 8 should be \( t \in [0, T] \)

p 229, equation (8.30) the expression \( [1 + \|u\|_{H^1} + \|v\|_{H^1}] \) should be raised to the power of \( \gamma \) rather than \( \frac{1}{2} \): the last line of the inequality in the proof should end \( (1 + |u|_{L^{2\gamma}} + |v|_{L^{2\gamma}})^{2\gamma} \) rather than what is written.

p 231, Exercise 8.1 You need to assume that \( X \) and \( Y \) are continuously embedded in some other Banach space \( Z \). [If \( X \cap Y \) is dense in both \( X \) and \( Y \) then we have true equality \( (X \cap Y)^* = X^* + Y^* \); otherwise one should say that elements of \( (X \cap Y)^* \) can be formed by adding the restrictions to \( X \cap Y \) of elements of \( X^* \) and \( Y^* \).]

p 235, l 2- change to ‘neglecting all the nonlinear terms in (9.1) and taking \( f \) to be time-independent’

p 248 we in fact need to take \( w \in C^1(0, T; C^1(\Omega)) \) in the argument showing the convergence of \( B(u_n, u_n) \) to \( B(u, u) \), and then use the density of \( C^1(0, T; C^1(\Omega)) \) in \( L^q(0, T; V) \). In fact it is better to take \( w = \sum_{j=1}^N c_j(t)w_j \), with \( c_j(t) \) continuous: for such \( w \) it is easy to show that

\[
\int_0^T \langle P_n B(u_n, u_n), w \rangle \, dt \to \int_0^T \langle B(u, u), w \rangle \, dt,
\]

and then since such \( w \) are dense in \( L^p(0, T; V) \) we obtain convergence of \( P_n B(u_n, u_n) \) to \( B(u, u) \) in one step.

p 274, Proposition 10.12 second sentence should end ‘then \( \omega(u_0) \) is a single equilibrium point’.

p 275, Theorem 10.13. Equation (10.20) should read

\[ \mathcal{A} = \mathcal{A} \cap W^*(\mathcal{E}) = \mathcal{A} \cap \bigcup_{z \in \mathcal{E}} W^*(z). \]

p 276, l 11 change ‘double equality’ to ‘the two equalities’

p 276, l 9- change ‘\( u_0 \in \mathcal{A} \)’ to ‘\( \nu_0 \in \mathcal{A} \)’
The example is wrong (thanks to Prof. Grzegorz Łukaszewicz of Warsaw University for pointing this out). The calculations given are in fact for
\[
\begin{align*}
\frac{dx}{dt} &= -zy \\
\frac{dy}{dt} &= zx \\
\frac{dz}{dt} &= -\mu z|z|.
\end{align*}
\]
However, there are some subtleties here. In fact for this example every point has a compact \(\omega\) limit set, even though the attractor \(z \equiv 0\) is not compact, and this is why Proposition 10.14 and its corollary still apply: that problems can arise otherwise is shown in the simpler system \(\dot{z} = -z^2\) with \(\dot{x} = zx\) which has solution \(x(t) = x_0(1 + z_0 t)\) and \(z(t) = z_0/(1 + z_0 t)\): although on \(z \equiv 0\) all points are stationary, the \(x\) component of every solution has constant speed \(x_0 z_0\). An example that does away with the need for such subtleties – since it does in fact have a compact global attractor – is
\[
\begin{align*}
\frac{dx}{dt} &= (1 - (x^2 + y^2))x - yz \\
\frac{dy}{dt} &= (1 - (x^2 + y^2))y + xz \\
\frac{dz}{dt} &= -\mu z|z|,
\end{align*}
\]
or in polar coordinates
\[
\begin{align*}
\frac{dr}{dt} &= r(1 - r^2) \\
\frac{d\theta}{dt} &= z \\
\frac{dz}{dt} &= -\mu z|z|.
\end{align*}
\]
p 381, equation (14.34) the term in $k$ should be $|k|^{1/2}$.

p 390, Definition 15.2 (Strong Squeezing Property) the strictly positive constant $k$ that occurs in (15.9) depends only on the projection $P$ (i.e. only on $n$).

p 391, l 4ff the two displayed equations should read

$$|Qu - Qv| \leq e^{-kt}|Qu_t - Qv_t|$$

and

$$|Qu - Qv| \leq 2R e^{-kt},$$

the conclusion being that $Qu = Qv$. The result then follows as before.

p 391, Definition 15.4 (i), $B(0, \rho) \cap PH$ should be replaced by

$$PH \setminus [B(0, \rho) \cap PH],$$

the idea being that the portion of the ‘flat space’ $PH$ ‘outside’ the absorbing ball $B(0, \rho)$ is invariant.

p 394, l2 we need ‘$t \geq t_0(Y)$’. It would be possible to consider $B(0, |u_0|)$ instead of a general bounded set $Y$, and thus obtain more clearly

$$\text{dist}(S(t)u_0, \mathcal{M}) \leq C(|u_0|)e^{-kt}.$$  

[A more careful proof of exponential convergence can be used to show that in fact

$$\text{dist}(S(t)u_0, \mathcal{M}) \leq C \text{dist}(u_0, \mathcal{M})e^{-kt},$$

see, for example, Chow et al. (1992).]

p 418, l5 while checking that $\varphi$ satisfies the uniqueness property, the left-hand side should be $\varphi(t; \varphi(s; x))$.

**Solutions to Exercises**

Some numbering problems for Chapter 3: Solution 3.8 is for Exercise 3.7; Solution 3.9 for Exercise 3.8; and Solution 3.7 is for Exercise 3.9.
The solution for Exercise 8.1 is incorrect, since the application of the Hahn-Banach Theorem does not give a linear functional on $X$, since it extends a linear functional on $Z = X \cap Y$ which is continuous with respect to the norm of $Z$, but not that of $X$. Thanks to Vittorino Pata for a helpful email correspondence to clear this up.

We need to assume in addition that $X$ and $Y$ are continuously embedded in some other Banach space $Z$. Now take $f \in (X \cap Y)^*$. Now consider the subspace $D$ of $X \times Y$ consisting of vectors of the form $(w, w)$ with $w \in X \cap Y$.

Define a bounded linear functional $h$ on $D$ by

$$h(w, w) = f(w).$$

Now use the Hahn-Banach theorem to extend $h$ to a linear functional $g$ on $X \times Y$, and set

$$f_1(w) = g(w, 0)$$

and

$$f_2(w) = g(0, w).$$

Then $f_1 \in X^*$ and $f_2 \in Y^*$, and $g = f_1 + f_2$. Then for $w \in X \cap Y$, $f = f_1 + f_2$.

Clearly if $X \cap Y$ is dense in $X$ and $Y$ then $f_1$ and $f_2$ are uniquely defined by their restrictions to $X \cap Y$, and so the equality $(X \cap Y)^* = X^* + Y^*$ is meaningful. Otherwise, as noted above, we in fact have $f = f_1|_{X \cap Y} + f_2|_{X \cap Y}$, where $f_1 \in X^*$ and $f_2 \in Y^*$.

**Additional material**

There is a very elegant formulation of an existence result for global attractors (cf. theorem 10.5) that is due to Crauel (2001) in the case of random attractors.

**Theorem 1.** There exists a global attractor $\mathcal{A}$ iff there exists a compact attracting set $K$, and then $\mathcal{A} = \omega(K)$.
Note that the condition of a compact attracting set is much weaker than the existence of a compact absorbing set. The proof requires the following simple lemma:

**Lemma 2.** If $K$ is a compact set and $x_n$ is a sequence such that

$$\text{dist}(x_n, K) \to 0$$

then $\{x_n\}$ has a convergent subsequence whose limit lies in $K$.

As a first step to proving this new theorem first we reprove proposition 10.3 under the weaker condition here.

**Proposition 3.** If there exists a compact attracting set $K$ then the $\omega$-limit set $\omega(X)$ of any bounded set $X$ is a non-empty, invariant, closed subset of $K$. Furthermore $\omega(X)$ attracts $X$.

**Proof.** To see that $\omega(X)$ is non-empty choose some point $x \in X$. Then since $K$ is attracting

$$\text{dist}(S(n)x, K) \to 0.$$  

It follows that for some sequence $n_j \to 0$

$$S(n_j)x \to x^* \in K.$$  

As the intersection of a decreasing sequence of closed sets $\omega(X)$ is clearly closed. To show that $\omega(X) \subset K$ suppose that $t_n \to \infty$, $x_n \in X$ and

$$S(t_n)x_n \to y.$$  

Then since $K$ is attracting

$$\text{dist}(S(t_n)x_n, K) \to 0,$$

implying that a subsequence of $S(t_n)x_n$ converges to a point in $K$. Since the sequence itself converges it follows that $y \in K$. So $\omega(X)$ is compact.

Now suppose that $\omega(X)$ does not attract $X$. Then there exists a $\delta > 0$ and a sequence of $t_n$ such that

$$\text{dist}(S(t_n)X, \omega(X)) > \delta,$$
and hence $x_n \in X$ such that

$$\text{dist}(S(t_n)x_n, \omega(X)) > \delta.$$  

(1)

However, the argument above shows that a subsequence of $\{S(t_n)x_n\}$ converges to some point $z$. By (1) we should have

$$\text{dist}(z, \omega(X)) \geq \delta,$$

while by definition $z \in \omega(X)$. So $\omega(X)$ attracts $X$. 

Now observe that

$$A \subseteq B \implies \omega(A) \subseteq \omega(B),$$  

(2)

and that since $\omega(X)$ is invariant

$$\omega[\omega(X)] = \bigcap_{t \geq 0} \bigcup_{s \geq t} S(s)\omega(X) = \omega(X).$$  

(3)

Proof. (Proof of theorem 1). It follows from the previous proposition that $\omega(K)$ is non-empty, compact, invariant, and attracts $K$. So all we have to prove is that $\omega(K)$ attracts $X$. Since $\omega(X)$ attracts $X$ it suffices to show that $\omega(X) \subseteq \omega(K)$. But this follows immediately from (2) and (3).