Chapter 1

Let \( \{x_j\} \) be a countable dense subset of \( X \), and let \( \{y_k\} \) be a countable dense subset of \( Y \). Then the countable collection \( \{(x_j, y_k)\} \) is dense in \( X \times Y \), since for any \((x, y) \in X \times Y \) and any \( \epsilon > 0 \) there exist \( x_j \) and \( y_k \) such that

\[
\|x - x_j\|_X < \frac{\epsilon}{2} \quad \text{and} \quad \|y - y_k\|_Y < \frac{\epsilon}{2},
\]

and so

\[
\|(x_j, y_k) - (x, y)\|_{X \times Y} \leq \epsilon.
\]

It follows that \( X \times Y \) is separable, and by induction it follows that any finite product of separable spaces is separable.

If \( M \) is a linear subspace of \( X \) then let \( \{x_j\} \) be a countable subset of \( X \) such that for each \( x \in X \) there is an \( x_j \) such that \( |x - x_j| < \epsilon \). Now discard any element \( x_j \) of this collection for which \( B(x_j, \epsilon) \) does not intersect \( M \). For each remaining \( x_j \), it follows that there exists an element \( m_j \in M \) such that \( B(m_j, 2\epsilon) \supset B(x_j, \epsilon) \). Thus this collection \( \{m_j\} \) has the property that for each element \( m \in M \) there exists an \( m_j \) such that \( |m - m_j| < 2\epsilon \). Applying this construction for the sequence \( \epsilon_n = 2^{-n} \) gives a countable dense subset of \( M \), as required.

1.2 Cover \( X \) with the collection of open balls

\[
\bigcup_{x \in X} B(x, \epsilon).
\]

Since \( X \) is compact it follows that there exists a finite covering by such
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balls:

\[ X \subset \bigcup_{j=1}^{N} B(x_j, \epsilon). \]

It follows that for each \( x \in X \) there exists an \( x_j \) with \( |x - x_j| < \epsilon \) as required.

1.3 We first consider the case of \( \Omega \) bounded. If \( u \in C^0_\infty(\Omega) \) then clearly \( u = 0 \) on \( \partial \Omega \); it follows that if \( u_n \in C^0_\infty(\Omega) \) converges to \( u \) uniformly on \( \Omega \) then \( u = 0 \) on \( \partial \Omega \) too. We now show that any function in

\[ C^0_\infty(\Omega) = \{ u \in C^0(\Omega) : u = 0 \text{ on } \partial \Omega \} \]

can be arrived at in this way and hence that this space is the completion of \( C^0_\infty(\Omega) \) in the sup norm. Let \( \theta \) be the continuous function

\[ \theta(x) = \begin{cases} 
  x, & x \geq 1, \\
  2x - 1, & 1 > x > \frac{1}{2}, \\
  0, & x \leq \frac{1}{2},
\end{cases} \]

and define

\[ u_\epsilon(x) = \theta(|u(x)|/\epsilon)u(x). \]

Clearly \( u_\epsilon \) is continuous on \( \Omega \), and since \( u \) is uniformly continuous on \( \Omega \) there exists a \( \delta \) such that

\[ \text{dist}(x, \partial \Omega) < \delta \quad \Rightarrow \quad |u(x)| < \epsilon/2, \]

that is, such that \( u_\epsilon(x) = 0 \) when \( \text{dist}(x, \partial \Omega) < \delta \). It follows that \( u_\epsilon \in C^0_\infty(\Omega) \), and since,

\[ |u(x) - u_\epsilon(x)| \leq \epsilon, \]

\( u_\epsilon \) converges uniformly to \( u \) on \( \Omega \).

It follows that \( C^0_0(\Omega) \neq C^0_\infty(\Omega) \) is the completion of \( C^0_\infty(\Omega) \) in the sup norm, and \( C^0_\infty(\Omega) \) is therefore not complete.

When \( \Omega = \mathbb{R}^m \) the limit of any convergent sequence of functions in \( C^0_0(\mathbb{R}^m) \) must tend to zero as \( |x| \to \infty \). This is clear, since given \( \epsilon > 0 \) there exists an \( N \) such that \( |u_n - u| \leq \epsilon \) for all \( n \geq N \). In particular, \( u_N \) is zero for all \( x > R_N \), say, and so \( |u| \leq \epsilon \) for all \( x > R_N \). The space of all such \( u \),

\[ C^0_0(\mathbb{R}^m) = \{ u \in C^0(\mathbb{R}^m) : u(x) \to 0 \text{ as } |x| \to \infty \}, \]

is the appropriate completion of \( C^0_\infty(\mathbb{R}^m) \). For any \( u \in C^0_0(\mathbb{R}^m) \), we
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can use the argument above to find an approximating sequence of $u_\epsilon \in C^0_\epsilon(\mathbb{R}^m)$.

1.4 If $\{f_j\}$ is Cauchy in the $\|\cdot\|_\epsilon$ norm then it is Cauchy in each $C^n(\overline{\Omega})$ norm. Since each $C^n(\overline{\Omega})$ is complete, $f_j \to f$ in each of these spaces, so that $f \in C^n(\overline{\Omega})$ for every $n$ and thus $f \in C^\infty(\overline{\Omega})$. It remains to show that in fact

$$\|f\|_\epsilon < \infty$$

and that

$$\|f_j - f\|_\epsilon \to 0$$
as $j \to \infty$. Since $\{f_j\}$ is Cauchy it certainly follows that for $j, k \geq N$ we have

$$\sum_{n=1}^l c_n \|f_j - f_k\|_{C^n(\overline{\Omega})} < \epsilon$$

for each $l < \infty$, and taking the limit as $k \to \infty$ gives

$$\sum_{n=1}^l c_n \|f_j - f\|_{C^n(\overline{\Omega})} < \epsilon. \quad (S1.1)$$

Using the triangle inequality in each $C^n(\overline{\Omega})$, $0 \leq n \leq l$, shows that

$$\sum_{n=1}^l c_n \|f\|_{C^n(\overline{\Omega})} < \epsilon + \sum_{n=1}^l c_n \|f_j\|_{C^n(\overline{\Omega})},$$

and so

$$\|f\|_\epsilon \leq \epsilon + \|f_j\|_\epsilon.$$ Since (S1.1) holds for all $l$, we can let $l \to \infty$ to show that

$$\sum_{n=1}^\infty c_n \|f_j - f\|_{C^n(\overline{\Omega})} < \epsilon,$$

and so $f_j \to f$ in the $\|\cdot\|_\epsilon$ norm.

1.5 We show that $C^{0,\gamma}(\overline{\Omega})$ is a Banach space; the case $C^{r,\gamma}$ then follows easily. If the sequence $\{f_j\}$ is Cauchy in $C^{0,\gamma}(\overline{\Omega})$ then given $\epsilon > 0$ there
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exists an $N$ such that for $j, k \geq N$ we have

$$\|f_j - f_k\|_\infty + \sup_{x, y \in \Omega} \left| \frac{[f_j(x) - f_k(x)] - [f_j(y) - f_k(y)]}{|x - y|^\gamma} \right| \leq \epsilon.$$

Because $C^0(\overline{\Omega})$ is complete we know that $f_j$ converges to some $f \in C^0(\overline{\Omega})$. We just need to show that $f$ is Hölder. However, since $f_k \to f$ uniformly we have

$$|[f_j(x) - f(x)] - [f_j(y) - f(y)]| \leq \epsilon |x - y|^\gamma$$

and so

$$|f(x) - f(y)| \leq |f_j(x) - f_j(y)| + |[f_j(x) - f(x)] + [f_j(y) - f(y)]| \leq C_j |x - y|^\gamma + \epsilon |x - y|^\gamma,$$

which shows that $f \in C^0,\gamma(\overline{\Omega})$.

1.6 If $f \in C^1(\overline{\Omega})$ then $|Df(x)|$ is uniformly bounded on $\overline{\Omega}$, by $L$, say. Since $\Omega$ is convex, given any two points $x, y \in \Omega$ the line segment joining $x$ and $y$ lies entirely in $\Omega$. It follows that

$$|f(x) - f(y)| = \left| \int_0^1 Df(y + \xi(x - y)) \cdot (x - y) d\xi \right| \leq L |x - y|,$$

so $f$ is Lipschitz.

1.7 We have

$$|u_h(x) - u_h(y)| = \left| h^{-m} \int_{\Omega} \left[ \rho \left( \frac{x - z}{h} \right) - \rho \left( \frac{y - z}{h} \right) \right] u(z) dz \right| \leq h^{-m} \int_{\Omega} \rho \left( \frac{x - z}{h} \right) |u(z) - u(z + y - x)| dz \leq C |y - x|^\gamma$$

by using (1.7) so that $u_h$ is also Hölder.

1.8 We prove the result by induction, supposing that it is true for $n = k$. Then for $n = k + 1$ we take $p$ such that

$$\left( \sum_{j=1}^{k-1} \frac{1}{p_j} \right) + \frac{1}{p} = 1,$$
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to obtain
\[ \int_{\Omega} | f_1(x) \cdots f_{k+1}(x) | dx \leq \| f_1 \|_{L^{n_1}} \cdots \| f_{k-1} \|_{L^{n-k+1}} \| f_k f_{k+1} \|_{L^p}. \]

(S1.2)

Now, we use the standard Hölder inequality, noting that
\[ 1 = \frac{p}{p_k} + \frac{p}{p_{k+1}}; \]

thus
\[ \int_{\Omega} (f_k f_{k+1})^p dx \leq \left( \int_{\Omega} f_k^{p_k} dx \right)^{p/p_k} \left( \int_{\Omega} f_{k+1}^{p_{k+1}} dx \right)^{p/p_{k+1}}, \]

and so
\[ \| f_k f_{k+1} \|_{L^p} \leq \| f_k \|_{L^{n_k}} \| f_{k+1} \|_{L^{n_{k+1}}}, \]

which combined with (S1.2) gives (1.31) for \( n = k + 1 \). Since the standard Hölder inequality is (1.31) for \( n = 2 \) the result follows.

1.9 Write
\[ \int_{\Omega} |u(x)|^p dx = \int_{\Omega} |u(x)|^{q(r-p)/(r-q)} |u(x)|^{r(p-q)/(r-q)} dx. \]

Now note that
\[ \frac{r-p}{r-q} + \frac{p-q}{r-q} = 1, \]

and so using Hölder’s inequality we have
\[ \int_{\Omega} |u(x)|^p dx \leq \left( \int_{\Omega} |u(x)|^q dx \right)^{(r-p)/(r-q)} \left( \int_{\Omega} |u(x)|^r dx \right)^{(p-q)/(r-q)}, \]

which becomes
\[ \| u \|_{L^p} \leq \| u \|_{L^q}^{(r-p)/p(r-q)} \| u \|_{L^r}^{(p-q)/p(r-q)}, \]

as required.

1.10 If \( s \in S(\Omega) \) then it is of the form of (1.10),
\[ s(x) = \sum_{j=1}^{n} c_j \chi_{I_j}(x), \]
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where the $I_j$ are $m$-dimensional cuboids, each of the form

$$I = \prod_{k=1}^{m} [a_k, b_k].$$

It clearly suffices to approximate $\chi[I]$ to within $\epsilon$ in the $L^p$ norm using an element of $C^0_c(\Omega)$. To do this, consider the function

$$\chi_\eta = \prod_{k=1}^{m} \phi_\eta(x_k; a_k, b_k),$$

where

$$\phi_\eta(x; b, a) = \begin{cases} 
\frac{(x - a)}{\eta}, & a \leq x \leq a + \eta, \\
1, & a + \eta < x < b - \eta, \\
\frac{(b - x)}{\eta}, & b - \eta \leq x \leq b.
\end{cases}$$

Clearly $\chi_\eta \in C^0_c(\Omega)$ and converges to $\chi[I]$ in $L^p(\Omega)$ as $\eta \to 0$.

1.11 Since $|g(x)| \leq \|g\|_\infty$ almost everywhere, it follows that

$$|f(x)g(x)| \leq |f(x)||g|_\infty$$

almost everywhere, and so

$$\int_\Omega |f(x)g(x)| \, dx \leq \int_\Omega |f(x)||g|_\infty \, dx \leq \|f\|_{L^1} \|g\|_\infty,$$

as claimed.

1.12 Since $\{x^{(n)}\}$ is Cauchy, given $\epsilon > 0$ there exists an $N$ such that

$$\|x^{(n)} - x^{(m)}\|_\infty \leq \epsilon \quad \text{for all} \quad n, m \geq N.$$

This implies that

$$|x^{(n)}_j - x^{(m)}_j| \leq \epsilon \quad \text{for all} \quad n, m \geq N. \quad (S1.3)$$

In particular, we have $x^{(n)}_j$ is Cauchy for each $j$. So $x^{(n)}_j \to x_j$ as $n \to \infty$. It is then clear that $x = \{x_j\} \in l^\infty$, and taking the limit $m \to \infty$ in (S1.3) shows that

$$|x^{(n)}_j - x_j| \leq \epsilon \quad \text{for all} \quad n \geq N, \quad \text{for all} \quad j.$$

It follows that $x^{(n)} \to x$ in $l^\infty$, and so $l^\infty$ is complete.
Chapter 2

1.13 We know that the norm is positive definite, and so
\[ \|x + \lambda y\|^2 = (x + \lambda y, x + \lambda y) = \|x\|^2 + 2\lambda (x, y) + \lambda^2\|y\|^2 \geq 0. \]
In particular, the quadratic equation for \(\lambda\),
\[ \lambda^2\|y\|^2 + 2\lambda (x, y) + \|x\|^2 = 0, \]
can have only one distinct real root. Therefore the discriminant \(b^2 - 4ac\) cannot be positive (which would give two real roots). In other words,
\[ 4(x, y)^2 - 4\|y\|^2\|x\|^2 \leq 0 \]
or
\[ |(x, y)| \leq \|x\||\|y\|, \]
which is the Cauchy–Schwarz inequality. We can now write
\[ \|x + y\|^2 = \|x\|^2 + 2(x, y) + \|y\|^2 \leq \|x\|^2 + 2\|x\||\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2, \]
giving the triangle inequality.

1.14 We simply expand the left-hand side,
\[ \|u + v\|^2 + \|u - v\|^2 = \|u\|^2 + 2(u, v) + \|v\|^2 + \|u\|^2 - 2(u, v) + \|v\|^2 = 2\|u\|^2 + 2\|v\|^2, \]
as required.

1.15 If \(\{u_j\}\) is a dense subset of \(L^2(\Gamma)\) then for each element \(\gamma \in \Gamma\) there must exist a \(u_j\) that is within \(\epsilon\) of 1 at \(\gamma\) and within \(\epsilon\) of 0 for all other elements of \(\Gamma\). Each such \(u_j\) is distinct. It follows that if \(\Gamma\) is uncountable then so are the \(\{u_j\}\), and so \(L^2(\Gamma)\) cannot be separable.

Chapter 2

2.1 We can apply the contraction mapping theorem to \(h^n\) to deduce that \(h^n\) has a unique fixed point \(x^*_n\),
\[ h^n(x^*_n) = x^*_n. \]
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If we apply $h$ to both sides then

$$h(h^n(x^*)) = h^{n+1}(x^*) = h^n(h(x^*)) = h(x^*),$$

showing that $h(x^*)$ is also a fixed point of $h^n$. Since the contraction mapping theorem guarantees that the fixed point of $h^n$ is unique, we must have $h(x^*) = h^*$, and so $h^*$ is also a fixed point of $h$.

2.2 The interval $[1, \infty)$ is closed but not compact, and the map $h : [1, \infty) \to [1, \infty)$ given by $x \mapsto x + 1/x$ satisfies

$$|h(x) - h(y)| = |x - y|(1 - (xy)^{-1}) < |x - y|$$

but clearly has no fixed point.

However, if $X$ is compact and $h : X \to X$ satisfies

$$\|h(x) - h(y)\| < \|x - y\|, \quad (S2.1)$$

suppose that $h$ has no fixed point. Then

$$\|h(x) - x\| > 0 \quad \text{for all} \quad x \in X,$$

and since $\|h(x) - x\|$ is continuous from $X$ into $\mathbb{R}$ it obtains its lower bound, so that

$$\|h(x) - x\| \geq \epsilon \quad \text{for all} \quad x \in X,$$

and there exists some $y \in X$ such that $\|h(y) - y\| = \epsilon$. However, if we take $z = h(y)$ then from (S2.1) we have

$$\|h(z) - z\| < \epsilon,$$

a contradiction. So $h$ has at least one fixed point. Uniqueness follows as in the proof of the standard contraction mapping theorem.

2.3 Take $\epsilon_n = 2^{-n}$ and apply the result of Exercise 1.2 so that there exists finite set $\{x_j^{(k)}\}$, $1 \leq j \leq M_k$, such that $|x - x_j^{(k)}| \leq 2^{-k}$. Set $N_k = \sum_{j=1}^{k} M_j$, and let $\{x_j\}$ be the sequence

$$x_1^{(1)}, \ldots, x_{M_1}, x_1^{(2)}, \ldots, x_{M_2}, x_1^{(3)}, \ldots$$

2.4 Suppose that there are solutions $x_n(t)$ of

$$dx/dt = f(x) \quad \text{with} \quad x(0) = x_0 \quad (S2.2)$$
such that \( x_n(\tau) \to x^* \). We need to show that there is a solution of (S2.2) with \( x(\tau) = x^* \). Now, if \( f \) is bounded then the sequence \( x_n(t) \) satisfies

\[
\sup_{t \in [0, \tau]} |x_n(t)| \leq |x_0| + \tau \| f \|_{\infty} \quad \text{and} \quad |x_n(t) - x_n(s)| \leq \| f \|_{\infty} |t - s|,
\]

the conditions of the Arzelà–Ascoli theorem (Theorem 2.5). It follows that there is a subsequence that converges uniformly on \([0, \tau]\), and as in the proof of Theorem 2.6 the limit \( x(t) \) satisfies (S2.2). Since \( x_n \to x \) uniformly on \([0, \tau]\), in particular we have \( x(\tau) = x^* \) as required.

2.5 When \(|x| \neq 0\) then it follows that

\[
\frac{d}{dt} |x|^2 = 2|x| \frac{d}{dt} |x|,
\]

and (2.27) follows immediately. When \(|x(t_0)| = 0\), since \( C(t) \) is continuous, for any \( \epsilon > 0 \) we have

\[
\frac{1}{2} \frac{d}{dt} |x|^2 \leq [C(t_0) + \epsilon] |x|
\]

for \( t - t_0 \) small enough, and so it follows from Lemma 2.7 that

\[
|x(t)|^2 \leq \left( [C(t_0) + \epsilon] (t - t_0) \right)^2.
\]

Therefore

\[
|x(t + h)| \leq [C(t_0) + \epsilon] t,
\]

and so

\[
\frac{d}{dt^+} |x| \leq C(t_0) + \epsilon.
\]

Since this holds for any \( \epsilon > 0 \) we have (2.27).

2.6 If

\[
y(t) = \int_0^t b(s)x(s) \, ds
\]

then

\[
\frac{dy}{dt} = b(t)x(t) \leq a(t)b(t) + b(t)y(t),
\]

and so

\[
\left( \frac{dy}{dt} - b(t)y(t) \right) \exp \left( - \int_0^t b(s) \, ds \right) \leq a(t)b(t) \exp \left( - \int_0^t b(s) \, ds \right).
\]
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If \( a(t) \) is increasing then we can replace \( a(t) \) on \([0, T]\) with \( a(T)\), and so

\[
\frac{d}{dt} \left[ y(t) \exp \left( - \int_0^t b(s) \, ds \right) \right] \leq a(T) b(t) \exp \left( - \int_0^t b(s) \, ds \right).
\]

Integrating both sides between 0 and \( T \) gives us

\[
y(T) \exp \left( - \int_0^T b(s) \, ds \right) \leq a(T) \int_0^T b(t) \exp \left( - \int_0^t b(s) \, ds \right) \, dt
\]

and so

\[
y(T) \leq a(T) \int_0^T b(t) \exp \left( \int_0^t b(s) \, ds \right) \, dt.
\]

We can integrate the right-hand side to obtain

\[
y(T) \leq a(T) \left[ \exp \left( \int_0^T b(s) \, ds \right) - 1 \right],
\]

and so, using (2.28), we have

\[
x(T) \leq a(T) \exp \left( \int_0^T b(s) \, ds \right)
\]

as claimed.

2.7 As in the proof of Proposition 2.10 we consider the difference of two solutions, \( z(t) = x(t) - y(t) \), which satisfies

\[
\frac{dz}{dt} = f(x) - g(y) = f(x) - f(y) + f(y) - g(y).
\]

We now use Lemma 2.9 to deduce that

\[
\frac{d}{dt} |z| \leq |f(x) - f(y)| + |f(y) - g(y)| \leq L |z| + \| f - g \|_{\infty}.
\]

An application of Gronwall’s inequality [(2.21) in Lemma 2.8] now yields (2.29).
Chapter 3

3.1 We denote
\[ \|A\|_1 = \{ \text{smallest } M \text{ such that } \|Ax\|_Y \leq M\|x\|_X \text{ for all } x \in X \} \]
and
\[ \|A\|_2 = \sup_{\|x\|_X = 1} \|Ax\|_Y. \]
First we take \( x \neq 0 \) and put \( y = x/\|x\|_X \); then we have
\[ \|Ay\|_Y \leq \|A\|_2 \Rightarrow \|Ax\|_Y \leq \|A\|_2 \|x\|_X \]
for all \( x \in X \), and so \( \|A\|_1 \leq \|A\|_2 \). Furthermore, it is clear that, for any \( M \),
\[ \|Ax\|_Y \leq M\|x\|_X \text{ for all } x \in X \Rightarrow \|A\|_2 \leq M, \]
and so \( \|A\|_2 \leq \|A\|_1 \). Thus \( \|A\|_1 = \|A\|_2 \).

3.2 \( I \) is clearly bounded from \( C^0([O, L]) \) into itself, since
\[ \|I(f)\|_\infty \leq L\|f\|_\infty. \]
For the \( L^2 \) bound, first observe, by using the Cauchy–Schwarz inequality, that \( I(f)(x) \) is defined for all \( x \) if \( f \in L^2 \). Then
\[
|I(f)|^2 = \int_0^L |I(f)(x)|^2 \, dx
\]
\[
= \int_0^L \left( \int_0^x f(s) \, ds \right)^2 \, dx
\]
\[
= \int_0^L \left( \int_0^x ds \right) \left( \int_0^x |f(s)|^2 \, ds \right) \, dx
\]
\[
\leq L^2 \int_0^L |f(s)|^2 \, ds
\]
\[
\leq L^2 |f|^2. \]
Thus \( I \) is a bounded operator on both spaces.

3.3 Suppose that \( A^{-1}y_1 = x_1 \) and that \( A^{-1}y_2 = x_2 \). Then it is clear that
\[ A(x_1 + x_2) = y_1 + y_2. \]
Since the inverse is unique it follows that
\[ A^{-1}(y_1 + y_2) = A^{-1}y_1 + A^{-1}y_2. \]
Solutions to Exercises

3.4 For each \(x \in X\), \(P_n x\) converges to \(x\), and so it follows that the sequence \(\{P_n x\}_{n=1}^{\infty}\) is bounded:

\[
\sup_{n \in \mathbb{Z}^+} \|P_n x\|_X < \infty
\]

for each \(x \in X\). From the principle of uniform boundedness (Theorem 3.7) we immediately obtain

\[
\sup_{n \in \mathbb{Z}^+} \|P_n\|_{op} < \infty,
\]

as claimed.

3.5 It is clear that \(\phi_i(x)\phi_j(y)\) is an element of \(L^2(\Omega \times \Omega)\) and that

\[
\int_{\Omega \times \Omega} [\phi_i(x)\phi_j(y)][\phi_k(x)\phi_l(y)] \, dx \, dy = \delta_{ik}\delta_{jl},
\]

and so they certainly form an orthonormal set. If \(k \in L^2(\Omega \times \Omega)\) then \(k(\cdot, y) \in L^2(\Omega)\), and we can write

\[
k(x, y) = \sum_{i=1}^{\infty} u_i(y)\phi_i(x),
\]

where

\[
u_i(y) = \int_{\Omega} k(x, y)\phi_i(x) \, dx.
\]

Since

\[
\int_{\Omega} |u_i(y)|^2 \, dy = \int_{\Omega} \left| \int_{\Omega} k(x, y)\phi_i(x) \, dx \right|^2 \, dy
\]

\[
\leq \int_{\Omega} \left( \int_{\Omega} |k(x, y)|^2 \, dx \int_{\Omega} |\phi_i(x)|^2 \, dx \right) \, dy
\]

\[
\leq \int_{\Omega \times \Omega} |k(x, y)|^2 \, dx \, dy,
\]

we have \(u_i \in L^2(\Omega)\). So we can write

\[
u_i(y) = \sum_{j=1}^{\infty} \left( \int_{\Omega} u_i(y)\phi_j(y) \, dy \right)\phi_j(y),
\]

which yields the expression

\[
k(x, y) = \sum_{i,j=1}^{\infty} \left( \int_{\Omega \times \Omega} k(x, y)\phi_i(x)\phi_j(y) \, dx \, dy \right)\phi_i(x)\phi_j(y),
\]

as claimed.
Chapter 3

3.6 We consider the approximations to $A$ given by the truncated sums,

$$A_n u = \sum_{j=1}^{n} \lambda_j(u, w_j)w_j.$$ 

Using Lemma 3.12 we see that each operator $A_n$ is compact. We now want to show that

$$\|A - A_n\|_{\text{op}} \to 0,$$

and it then follows from Theorem 3.10 that $A$ is compact. However, this convergence is clear, since

$$\|(A - A_n)u\| = \left\| \sum_{j=n+1}^{\infty} \lambda_j(u, w_j)w_j \right\|$$

$$\leq \lambda_{n+1} \left\| \sum_{j=n+1}^{\infty} (u, w_j)w_j \right\|$$

$$\leq \lambda_{n+1} \left( \sum_{j=n+1}^{\infty} |(u, w_j)|^2 \right)^{1/2}$$

$$\leq \lambda_{n+1} \|u\|,$$

and $\lambda_{n+1} \to 0$ as $n \to \infty$. Thus $A$ is compact. That $A$ is symmetric follows by taking the inner product of $Au$ with $v$ to give

$$(Au, v) = \sum_{j=1}^{\infty} \lambda_j(u, w_j)(v, w_j) = (u, Av).$$

3.7 We know from Lemma 3.4 that $A^{-1}$ exists iff $\text{Ker}(A) = 0$. So we show that if $Ax = 0$ then $x = 0$. Because $A$ is bounded below we have

$$0 = \|Ax\|_Y \geq k\|x\|_X,$$

so that $\|x\|_X = 0$. For $y \in R(A)$ we can use the lower bound on $A$ to deduce that

$$\|A^{-1}y\|_X \leq \frac{1}{k} \|AA^{-1}y\|_Y = \frac{1}{k} \|y\|_Y,$$

so that $A^{-1}$ is bounded.
Solutions to Exercises

3.8 Since $G$ is a solution of the homogeneous equation on both sides of $x = y$, we must have

$$G(x, y) = \begin{cases} \ C_1(y)u_1(x), & a \leq x < y, \\ C_2(y)u_2(x), & y \leq x \leq b \end{cases}$$

The conditions at $y$ require

$$C_1(y)u_1(y) = C_2(y)u_2(y),$$

$$C_1(y)u'_1(y) + p(y)^{-1} = C_2(y)u'_2(y).$$

Solving these simultaneous equations for $C_1$ and $C_2$ gives

$$C_1(y) = u_2(y)/W_p(y) \quad \text{and} \quad C_2(y) = u_1(y)/W_p(y),$$

where

$$W_p(y) = p(y)[u_1(y)u'_2(y) - u_2(y)u'_1(y)].$$

Differentiating $W_p$ with respect to $y$ and cancelling the $pu'_1u'_2$ terms gives

$$W'_p = p'(u_1u'_2 - u_2u'_1) + p[u_1u''_2 - u_2u''_1].$$

If we use the differential equation $L[u_1] = L[u_2] = 0$ to substitute for the terms $pu''_1$ and $pu''_2$ we see that in fact $W'_p = 0$, so that $W_p$ is a constant. We therefore obtain (3.28), and $G(x, y)$ is symmetric.

3.9 Proposition 3.13 and Lemma 3.16 show that the integral operator $K$ defined by

$$[Ku](x) = \int_{\Omega} k(x, y)u(y) \, dy$$

is a compact symmetric mapping from $L^2(\Omega)$ into $L^2(\Omega)$. It follows from Theorem 3.18 that $K$ has a set of eigenfunctions $u_n(x)$ with corresponding eigenvalues $\lambda_n$, so that $Ku_n = \lambda_n u_n$:

$$\int_{\Omega} k(x, y)u_n(y) \, dy = \lambda_n u_n(x).$$

Since $\lambda_j \neq 0$ for all $j$ there is no nonzero $u$ such that $Ku = 0$. In this case Ker$K = \{0\}$, and so we can expand any $f \in L^2(\Omega)$ in terms of the
Chapter 3

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eigenfunctions of $K$,

$$f = \sum_{j=1}^{\infty} (f, u_j) u_j.$$  

It is now easy to see that the solution of (3.29) is given by

$$u(x) = \sum_{j=1}^{\infty} \frac{(f, u_j)}{\lambda_j} u_j(x),$$

as claimed.

3.10 We have

$$A^{-\alpha} w_j = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\lambda_j t} dt w_j.$$  \hspace{1cm} (S3.1)

Now,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$$

and so, substituting $u = \lambda_j t$ in (S3.1), we have

$$\int_0^\infty \lambda_j^{1-\alpha} t^{\alpha-1} e^{-u} \frac{du}{\lambda_j} = \lambda_j^{-\alpha} \Gamma(\alpha),$$

which gives

$$A^{-\alpha} w_j = \lambda_j^{-\alpha} w_j$$

as required. Since $A^{-\alpha}$ is characterised by its action on the eigenfunctions the two expressions are equivalent.

3.11 We have

$$\| A^\alpha \|^2 = \sum_{j=1}^{\infty} \lambda_j^{2\alpha} |c_j|^2$$

$$= \sum_{j=1}^{\infty} \lambda_j^{2(s-\alpha)/\psi} |c_j|^{2\psi} \lambda_j^{2\alpha/(1-\psi)} |c_j|^{2(1-\psi)}$$

$$\leq \left( \sum_{j=1}^{\infty} \lambda_j^{2(s-\alpha)/\psi} |c_j|^{2\psi} \lambda_j^{2\alpha/(1-\psi)} |c_j|^{2(1-\psi)} \right)^{1-\psi}$$

$$\leq \| A^{(s-\alpha)/\psi} u \|^{2\psi} \| A^{\alpha/(1-\psi)} u \|^{2(1-\psi)},$$

which gives the result on setting $\varphi = (k - s)/(k - l)$ and $\alpha = k(s - l)/(k - l)$. 
Solutions to Exercises

3.12 Take \( x = \sum_{j=1}^{\infty} x_j w_j \) and consider the series expansion

\[
\frac{(e^{-Ah} - I)}{h} x + Ax = \sum_{j=1}^{\infty} \left[ \frac{(e^{-\lambda_j h} - 1)}{h} + \lambda_j \right] x_j w_j, \tag{S3.2}
\]

Observe that

\[
\sum_{j=n+1}^{\infty} \left[ \frac{(e^{-\lambda_j h} - 1)}{h} + \lambda_j \right] x_j w_j = \sum_{j=n+1}^{\infty} \left[ \frac{(e^{-\lambda_j h} - 1)}{\lambda_j h} + 1 \right] \lambda_j x_j w_j.
\]

The mean-value theorem tells us that \( \frac{(e^{-z} - 1)}{z} \leq 1 \), and so

\[
\left| \sum_{j=n+1}^{\infty} \left[ \frac{(e^{-\lambda_j h} - 1)}{h} + \lambda_j \right] x_j w_j \right|^2 \leq 4 \sum_{j=n+1}^{\infty} \lambda_j^2 |x_j|^2, \tag{S3.3}
\]

which tends to zero as \( n \to \infty \).

It follows that given \( \epsilon > 0 \) we can choose an \( n \) such that the infinite sum in (S3.3) is bounded above by \( \epsilon/2 \). It is then clear that the finite sum

\[
\sum_{j=1}^{n} \left[ \frac{(e^{-\lambda_j h} - 1)}{h} + \lambda_j \right] x_j w_j
\]

converges to zero as \( h \to 0 \), and so for small enough \( h \) the whole expression in (S3.2) is bounded by \( \epsilon \), as required.

Chapter 4

4.1 Let \( P = \{\text{orthonormal subsets of } H\} \), and define an order on \( P \) such that \( a \leq b \) if \( a \subseteq b \). If \( \{C_i\} \) is a chain \( (i \in I) \) then \( C = \cup_i C_i \) is an upper bound. Zorn’s lemma implies that there is a maximal orthonormal set \( \{e_i\}_{i \in I} \). The argument of the second part of Proposition 1.23 now shows that the \( \{e_i\} \) form a basis.

4.2 Take \( z \notin Y \). Then if \( w \) is contained in the linear span of \( z \) and \( Y \) it has a unique decomposition of the form

\[
w = y + \alpha z \quad \text{with} \quad y \in Y,
\]

as in the proof of the Hahn–Banach theorem. We can therefore define a nonzero linear functional on the linear span of \( z \) and \( Y \) via

\[
f(y + \alpha z) = \alpha.
\]

The functional \( f \) is zero on \( Y \), and we can extend it to a nonzero linear functional on \( X \) by using the Hahn–Banach theorem.
Chapter 4

4.3 It is immediate from Hölder’s inequality that

$$|L_f(g)| \leq \|f\|_{L^\infty} \|g\|_{L^1}$$

and so

$$\|L_f\|_{(L^1)^*} \leq \|f\|_{L^\infty}. \quad (S4.1)$$

To show equality consider the sequence of functions

$$g_p(x) = |f(x)|^{p-2} f(x).$$

Since $f \in L^\infty(\Omega)$ and $\Omega$ is bounded we have $g_p(x) \in L^1(\Omega)$ for every $p$, with

$$\|g_p\|_{L^1} = \|f\|_{L^p}^{p-1}.$$ It follows from

$$|L_f(g_p)| = \|f\|_{L^p}^p$$

that

$$\|L_f\|_{(L^1)^*} \geq \frac{\|f\|_{L^p}^p}{\|f\|_{L^{p-1}}^{p-1}}.$$ Since $f \in L^\infty$ we can use the result of Proposition 1.16,

$$\|f\|_{L^\infty} = \lim_{p \to \infty} \|f\|_{L^p},$$

to deduce that

$$\|L_f\|_{(L^1)^*} \geq \|f\|_{L^\infty},$$

which combined with (S4.1) gives the required equality.

4.4 Since $M$ is a linear subspace of $H$ it is also a Hilbert space. The Riesz theorem then shows that given a linear functional $f$ on $M$ there exists an $m \in M$ such that

$$f(x) = (m, x) \quad \text{for all} \quad x \in M.$$ Now define $F$ on $H$ by

$$F(u) = (m, u);$$

it is clear that $F$ is an extension of $F$ and that $\|F\| = \|f\|.$

4.5 If $x \not\in M$ then the argument of Solution 4.2 shows that there exists an element $f \in X^*$ with $f|_M = 0$ but $f(x) \neq 0$. So if $f(x) = 0$ for all
Solutions to Exercises

such \( f \) then we must have \( x \in M \). Now if \( x_n \to x \) then for each \( f \in X^* \) with \( f|_M = 0 \) we have

\[
f(x) = \lim_{n \to \infty} f(x_n) = 0,
\]

and so it follows that \( x \in M \).

The linear span of the \( \{x_n\} \) forms a linear subspace \( M \) of \( X \), and clearly \( x_n \in M \) for each \( n \). It follows that \( x \) is contained in the linear span of the \( \{x_n\} \) and so can be written in the form

\[
x = \sum_{j=1}^{\infty} c_j x_j.
\]

(In fact \( x \) can be written as a convex combination of the \( \{x_j\} \), that is, (S4.2) with \( c_j \geq 0 \) and \( \sum_j c_j = 1 \); see Yosida (1980, p. 120).]

4.6 For any \( t \in [a, b] \),

\[
\delta_t : x \mapsto x(t)
\]

is a bounded linear functional on \( C^0([a, b]) \). Since \( x_n \to x \), we have

\[
\delta_t(x_n) \to \delta_t(x),
\]

and so \( x_n(t) \to x(t) \) for each \( t \in [a, b] \).

4.7 Write

\[
\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2(x, x_n),
\]

and then take limits on the right-hand side, using norm convergence on \( \|x_n\|^2 \) and weak convergence on \( (x, x_n) \), to show that

\[
\lim_{n \to \infty} \|x_n - x\|^2 = 0,
\]

which is \( x_n \to x \).

Chapter 5

5.1 Simply write

\[
\langle D^\alpha u, \phi_n \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \phi_n \rangle,
\]

and then using the definition of convergence in \( D(\Omega) \) (Definition 5.2)
we have $D^\alpha \phi_n \to D^\alpha \phi$ in $\mathcal{D}(\Omega)$, and so

$$
\langle D^\alpha u, \phi_n \rangle \to (-1)^{|\alpha|} \langle u, D^\alpha \phi \rangle
= \langle D^\alpha u, \phi \rangle,
$$

so that $D^\alpha u$ is indeed a distribution.

5.2 If $\phi_n \in \mathcal{D}(\Omega)$ with $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$ then $\psi \phi_n \to \psi \phi$ in $\mathcal{D}(\Omega)$. It follows that

$$
\langle \psi u, \phi_n \rangle = \langle u, \psi \phi_n \rangle \to \langle u, \psi \phi \rangle = \langle \psi u, \phi \rangle,
$$

and so $\psi u \in \mathcal{D}'(\Omega)$.

Given $\phi \in \mathcal{D}(\Omega)$ we have

$$
\langle D(\psi u), \phi \rangle = -\langle \psi u, \phi' \rangle
= -\langle u, \psi \phi' \rangle
= -\langle u, \psi \phi' + \phi \psi' \rangle + \langle u, \phi \psi' \rangle
= \langle Du, \psi \phi \rangle + \langle u D\psi, \phi \rangle
= \langle \psi Du + u D\psi, \phi \rangle,
$$

as claimed.

5.3 Assume that $|f_n| \leq M$ for every $n$. For every $\phi \in C_c^\infty(\Omega)$ we know that

$$
\int_{\Omega} f_n \phi \, dx \tag{S5.1}
$$

is a Cauchy sequence. Since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$ (Corollary 1.14), for each $u \in L^2(\Omega)$ we can find a sequence of $\phi_n \in C_c^\infty(\Omega)$ with $\phi_n \to u$ in $L^2(\Omega)$. Then, given $\epsilon > 0$, choose $K$ such that

$$
|\phi_k - u| \leq \epsilon / 4M
$$

for all $k \geq K$

and then choose $N$ such that

$$
\left| \int_{\Omega} (f_n - f_m) \phi_k \, dx \right| \leq \epsilon / 2
$$

for all $n, m \geq N$.

It follows that for all $n, m \geq N$

$$
\left| \int_{\Omega} (f_n - f_m) u \, dx \right| \leq |f_n - f_m| |u - \phi_k| + \epsilon / 2
\leq (2M)(\epsilon / 4M) + \epsilon / 2 = \epsilon,
$$

and so (S5.1) is a Cauchy sequence for every $u \in L^2(\Omega)$, showing that $f_n \to f$ in $L^2(\Omega)$.
5.4 Suppose that the result is true for \( k = n \). We show that it holds for \( k = n + 1 \), which then gives a proof by induction since the statement of Proposition 5.8 gives (5.45) for \( k = 1 \). We know that
\[
\|u\|_{H^{n+1}}^2 = \|u\|_{H^n}^2 \sum_{|\alpha| = n+1} |D^\alpha u|^2,
\]
which along with the induction hypothesis becomes
\[
\|u\|_{H^{n+1}}^2 \leq C(n) \sum_{|\alpha| = n} |D^\alpha u|^2 + \sum_{|\alpha| = n+1} |D^\alpha u|^2. \tag{S5.2}
\]
We therefore consider \( |D^\alpha u| \) for \( |\alpha| = n \). Since \( u \in H_0^{n+1}(\Omega) \) we must have \( D^\alpha u \in H_0^1(\Omega) \), and so
\[
|D^\alpha u| \leq C|D_1 D^\alpha u| = C|D^\beta u|
\]
with \( |\beta| = n + 1 \), by using (5.11) from the proof of Proposition 5.8. It follows from (S5.2) that
\[
\|u\|_{H^{n+1}}^2 \leq C(n + 1) \sum_{|\alpha| = n+1} |D^\alpha u|^2,
\]
which is the result for \( k = n + 1 \).

5.5 Consider a sequence of \( u_n \in C^\infty(\Omega) \) that approximates \( u \) in \( H^k(\Omega) \). Then the derivatives of \( \psi u_n \) are given by the Leibniz formula (1.6)
\[
D^\alpha (\psi u_n) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \psi D^{\alpha-\beta} u_n,
\]
and so
\[
|D^\alpha (\psi u_n)| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D^\beta \psi| |D^{\alpha-\beta} u_n|
\]
\[
\leq \left( \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D^\beta \psi| \right) \|u_n\|_{H^k}.
\]
In this way the derivatives up to and including order \( k \) are bounded in \( L^2 \) by a constant (depending on \( \psi \)) times the \( H^k \) norm of \( u_n \), and so
\[
\|\psi u_n\|_{H^k} \leq C(\psi) \|u_n\|_{H^k}.
\]
It follows that \( \psi u_n \) is Cauchy in \( H^k(\Omega) \), and so in the limit as \( n \to \infty \) we have \( \psi u \in H^k(\Omega) \) with
\[
\|\psi u\|_{H^k} \leq C(\psi) \|u\|_{H^k}
\]
as required.
Chapter 5

5.6 First we show that \( u \in L^2(B(0, 1)) \):

\[
\int_{B(0,1)} \left[ \log \log \left(1 + \frac{1}{|x|}\right) \right]^2 \, dx \, dy = \int_0^{2\pi} \int_0^1 r \log \log (1 + 1/r) \, dr \, d\theta,
\]

which is finite since the integrand is bounded. Now, since

\[
\frac{\partial u}{\partial x} = \frac{1}{\log(1 + 1/|x|)} \frac{x}{|x|^2(1 + |x|)}
\]

we have

\[
\int_{B(0,1)} |\nabla u(x)|^2 \, dx \, dy
\]

\[
= \int_{B(0,1)} \frac{1}{\log(1 + 1/|x|)^2} \frac{1}{|x|^2(1 + |x|)^2} \, dx \, dy
\]

\[
= \int_0^{2\pi} \int_0^1 \frac{1}{\log(1 + 1/r)^2} \frac{1}{r(1 + r)^2} \, dr.
\]

(S5.3)

If we make the substitution \( u = 1/r \) this becomes

\[
\int_1^\infty \frac{1}{u + (1/u) \log(1 + u)^2} \, du.
\]

This integral is bounded by

\[
\int_1^\infty \frac{1}{u(\log u)^2} \, du,
\]

and since the integrand is the derivative of \(-1/\log u\) it follows that the integral in (S5.3) is finite. Therefore \( u \in H^1(B(0, 1)) \), even though it is unbounded.

5.7 First integrate (5.46) with respect to \( x_1 \), so that

\[
\int_{-\infty}^{\infty} |u(x)|^3 \, dx_1 \leq 6 \left( \int_{-\infty}^{\infty} uD_1u \, dy_1 \right)^{1/2} \left( \int_{-\infty}^{\infty} uD_2u \, dy_2 \right)^{1/2}
\]

\[
\times \left( \int_{-\infty}^{\infty} uD_3u \, dy_3 \right)^{1/2} \, dx_1
\]

\[
\leq 6 \left( \int_{-\infty}^{\infty} uD_1u \, dy_1 \right)^{1/2} \left( \int_{-\infty}^{\infty} uD_2u \, dx_1 \, dy_2 \right)^{1/2}
\]

\[
\times \left( \int_{-\infty}^{\infty} uD_3u \, dx_1 \, dy_3 \right)^{1/2}.
\]
Now integrate with respect to $x$ to obtain
\[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x)|^3 \, dx \, dx_2 \leq 6 \left( \int_{-\infty}^{\infty} uD_2 u \, dx_1 \, dy_2 \right)^{1/2} \times \left( \int_{-\infty}^{\infty} uD_1 u \, dx_1 \, dx_2 \right)^{1/2} \left( \int_{-\infty}^{\infty} uD_3 u \, dx_1 \, dx_2 \, dy_3 \right)^{1/2}.\]

Finally, integrating with respect to $x_3$ gives
\[\int_{\Omega} |u(x)|^3 \, dx \leq 6 \prod_{j=1}^{3} \left( \int_{\Omega} uD_j u \, dx \right)^{1/2},\]
and so
\[\|u\|_{L^3}^3 \leq C|u|^{3/2}|Du|^{3/2},\]
which gives
\[\|u\|_{L^1} \leq C|u|^{1/2}\|u\|_{H^1}^{1/2},\]
as required.

5.8 We simply apply the argument of Theorem 5.29 to the functions $v = D^\alpha u$ for each $\alpha$ with $|\alpha| \leq j$. It follows that $v \in H^{k-j}(\Omega)$, and since $k-j > m/2$ we can use Theorem 5.29 to deduce that $v \in C^0(\overline{\Omega})$ with
\[\|v\|_{C^0(\overline{\Omega})} \leq C\|u\|_{H^{k-j}(\Omega)} \leq C\|u\|_{H^k}.\]
Combining the estimates for each $|\alpha| \leq j$ shows that $u \in C^j(\overline{\Omega})$ with
\[\|u\|_{C^j(\overline{\Omega})} \leq C\|u\|_{H^k(\Omega)}\]
as claimed.

5.9 Suppose that the inequality does not hold. Then for each $k \in \mathbb{Z}^+$ there must exist $u_k \in V$ such that
\[|u_k| \geq k|\nabla u_k|. \quad (S5.4)\]
If we set $v_k = u_k/|u_k|$ so that $|v_k| = 1$, (S5.4) becomes
\[|\nabla v_k| \geq k^{-1}. \quad (S5.5)\]
It follows that $v_k$ is a bounded sequence in $H^1(\Omega)$, and so using Theorem 5.32 it has a subsequence that converges in $L^2(\Omega)$ to some
Chapter 5

\[ v \in V \text{ with } \int_{\Omega} v(x) \, dx = 0 \quad \text{and} \quad |v| = 1. \]  
(S5.6)

However, we can also use (S5.5) along with the \( L^2 \) convergence of \( v_k \) to \( v \) to show that for any \( \phi \in D(\Omega) \) and any \( j \)

\[ \int_{\Omega} v D^j \phi \, dx = \lim_{k \to \infty} \int_{\Omega} v_k D_j \phi \, dx = -\lim_{k \to \infty} \int_{\Omega} D_j v_k \phi \, dx = 0. \]

It follows that \( Dv = 0 \), and so, using the hint, \( v \) is constant almost everywhere. This contradicts (S5.6), and so we have the inequality (5.47).

5.10 Suppose that \( \{u_n\} \) is a bounded sequence in \( L^2(\Omega) \). Then, since \( L^2 \) is reflexive, there is a subsequence that converges weakly in \( L^2(\Omega) \), i.e. for every \( \phi \in L^2(\Omega) \) we have

\[ (u_n, \phi) \to (u, \phi) \]

for some \( u \in L^2(\Omega) \). Now, suppose that \( u_n \) does not converge to \( u \) in \( H^{-1}(\Omega) \), so that there exists an \( \epsilon > 0 \) such that, for some subsequence \( \{u_n\} \),

\[ \sup_{\|\phi\|_{H^1_0(\Omega)}} |(u_n - u, \phi)| \geq \epsilon. \]

Then there exist \( \phi_n \) with \( \|\phi_n\|_{H^1_0(\Omega)} = 1 \) such that

\[ |(u_n - u, \phi_n)| \geq \epsilon/2. \]

Since \( \{\phi_n\} \) is a bounded sequence in \( H^1_0(\Omega) \) and \( H^1_0(\Omega) \) is compactly embedded in \( L^2(\Omega) \), there exists a subsequence that is convergent in \( L^2(\Omega) \) to some \( \phi \). It follows (on relabelling) that

\[ |(u_n - u, \phi)| \geq \epsilon/4 \]

for \( n \) large enough. But this contradicts the weak convergence of \( u_n \) to \( u \) in \( L^2 \), and so we must have \( u_n \to u \) in \( H^{-1}(\Omega) \).

5.11 Since

\[ u = \sum_{k \in \mathbb{Z}^m} c_k e^{2\pi ik \cdot x/L} \]
Solutions to Exercises

we have

\[
Du = \sum_{k \in \mathbb{Z}} \frac{2\pi i k}{L} c_k e^{2\pi i k x/L}.
\]

It follows that

\[
|u|^2 = L^n \sum_{k \in \mathbb{Z}^n} |c_k|^2 \quad \text{and} \quad |Du|^2 = L^n \sum_{k \in \mathbb{Z}^n} (4\pi/L)^2 |k|^2 |c_k|^2,
\]

and so

\[
|u| \leq \left( \frac{L}{2\pi} \right) |Du|
\]

as claimed.

Chapter 6

6.1 Start with

\[
\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f(x)v(x) \, dx,
\]

and integrate the left-hand side by parts to give

\[
\int_{\Omega} (\Delta u - f)v \, dx = 0.
\]

Since \( u \in C^2(\Omega) \) and \( f \in C^0(\Omega) \), we have

\[
\varphi \equiv \Delta u - f \in C^0(\Omega).
\]

It therefore suffices to show that if

\[
\int_{\Omega} \varphi v \, dx = 0 \quad \text{for all} \quad v \in C^1_c(\Omega)
\]

then \( \varphi = 0 \). Suppose that \( \varphi(x) \neq 0 \) for some \( x \in \Omega \). Then since \( \varphi \) is continuous there is a neighbourhood \( N \) of \( x \) on which \( \varphi(x) \) is of constant sign. Taking a function \( v \) that is positive and has compact support within \( N \) implies that

\[
\int_{\Omega} \varphi v \, dx = \int_{N} \varphi v \, dx \neq 0,
\]

a contradiction. That \( u \) satisfies \( u|_{\partial \Omega} = 0 \) follows from \( u \in H^1_0(\Omega) \cap C^0(\overline{\Omega}) \), using Theorems 5.35 and 5.36.
Chapter 6

6.2 Take the inner product of $Lu = f$ with a $v \in C^1_c(\Omega)$,

\[- \int_{\Omega} \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) v(x) + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} v(x) + c(x)u(x)v(x) \, dx\]

\[= \int_{\Omega} f(x)v(x) \, dx,\]

and integrate the first term by parts,

\[\int_{\Omega} \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} v(x) + c(x)u(x)v(x) \, dx\]

\[= \int_{\Omega} f(x)v(x) \, dx.\]

We can now introduce a bilinear form

\[a(u, v) = \int_{\Omega} \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} v(x) + c(x)u(x)v(x) \, dx,\]

and write the equation as

\[a(u, v) = \langle f, v \rangle \quad \text{for all} \quad v \in C^1_c(\Omega).\]

As before we use the density of $C^1_c(\Omega)$ in $H^1_0(\Omega)$ to generalise to $f \in H^{-1}(\Omega)$ and the weak form of the problem is thus to find $u \in H^1_0(\Omega)$ such that

\[a(u, v) = \langle f, v \rangle \quad \text{for all} \quad v \in H^1_0(\Omega).\]

6.3 By definition

\[a(u, u) = \sum_{i,j=1}^{m} \int_{\Omega} a_{ij}(x) D_j u D_i u \, dx + \sum_{i=1}^{m} \int_{\Omega} b_i(x) D_i u u \, dx\]

\[+ \int_{\Omega} c(x) u^2 \, dx\]

\[\geq \theta \int_{\Omega} |\nabla u|^2 \, dx - \max_{i} \|b_i\|_{L^\infty} \int_{\Omega} |\nabla u| |u| \, dx\]

\[- \|c\|_{L^\infty} \int_{\Omega} |u|^2 \, dx.\]
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Now we use Young’s inequality with $\epsilon$,

$$ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon},$$

to split the second term,

$$a(u, u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \left( \theta \max_i \|b_i\|_{L^\infty} \right)^{-1} \int_{\Omega} |u|^2 \, dx$$

and so

$$a(u, u) \geq C \|u\|_{H^1}^2 - \lambda |u|^2,$$

as required.

6.4 Consider the bilinear form $b(u, v)$ corresponding to the operator $L + \alpha$. Then

$$b(u, v) = a(u, v) + \alpha (u, v)$$

is a continuous bilinear form on $H^1_0$; clearly $(u, v)$ is, and $a(u, v)$ is since

$$|a(u, v)| \leq \sum_{i,j=1}^m \int_\Omega |a_{ij}| |D_j u| |D_i v| \, dx + \sum_{i=1}^m \int_\Omega |b_i| |D_i u| |v| \, dx$$

$$+ \int_\Omega |c||u||v| \, dx$$

$$\leq C \|u\|_{H^1} \|v\|_{H^1}.$$ 

Furthermore, $b$ satisfies the coercivity condition, since

$$b(u, u) = a(u, u) + \alpha (u, u)$$

$$\geq C \|u\|_{H^1}^2 - \lambda |u|^2 + \alpha |u|^2$$

$$\geq C \|u\|_{H^1}^2.$$ 

We can now apply the Lax–Milgram lemma to obtain the conclusion.

6.5 First, it is easy to see that if (6.31) holds for all $v \in H^1(\Omega)$ then choosing $v = 1$ we have

$$\int_\Omega f(x) \, dx = 0.$$
Chapter 6

We cannot immediately apply the Lax–Milgram lemma to the equation

\[ a(u, v) = (f, v), \]

since

\[ |a(u, u)| = |\nabla u|^2 = ||u||^2_{H^1} - |u|^2, \]

and so \( a(u, v) \) is not coercive. To deal with the \( L^2 \) part we need a Poincaré-type inequality. Note that if \( \int_{\Omega} f(x) \, dx = 0 \), then

\[ (f, v) = (f, v - \int_{\Omega} v(x) \, dx), \]

since subtracting the constant from \( v \) does not make any difference, and similarly

\[ a(u, v) = a(u, v - \int_{\Omega} v(x) \, dx). \]

The weak form of the equation in this case \( \int_{\Omega} f(x) \, dx = 0 \) is therefore equivalent to

\[ a(u, v) = (f, v) \quad \text{for all} \quad v \in V, \]

where

\[ V = \left\{ u \in H^1(\Omega) : \int_{\Omega} u(x) \, dx = 0 \right\}. \]

It is was shown in Exercise 5.9 that in this space

\[ |u| \leq C|\nabla u|, \]

and so we have

\[ |a(u, u)| = |\nabla u|^2 \geq \frac{1}{2C} |u|^2 + \frac{1}{2} |\nabla u|^2 \geq k||u||^2_{H^1}. \]

We can therefore apply the Lax–Milgram lemma to deduce the existence of a weak solution of the Neumann problem.

6.6 Without the imposition of the condition \( \int_{Q} u(x) \, dx = 0 \) Laplace’s equation on \( Q \) with periodic boundary conditions does not have a unique
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A weak solution. In terms of the Lax–Milgram lemma this translates into the weak problem

\[
a(u, v) = \int_Q \nabla u \cdot \nabla v \, dx = \langle f, v \rangle \quad \text{with} \quad f \in H^{-1}(Q),
\]

where we seek \( u \in L^2(Q) \). But then \( a \) is not coercive on \( L^2(Q) \), since \( a(c, c) = 0 \) for any constant \( c \).

6.7 First,

\[
D_h^i(uv)(x) = \frac{u(x + he_i)v(x + he_i) - u(x)v(x)}{h}
\]

\[
= u(x)\left(\frac{v(x + he_i) - v(x)}{h}\right) + v(x + he_i)\left(\frac{u(x + he_i) - u(x)}{h}\right)
\]

\[
= u(x)D_h^i v(x) + v(x + he_i)D_h^i u(x).
\]

Next, we write

\[
\int_{\Omega} \frac{u(x + he_i) - u(x)}{h} v(x) \, dx = \int_{\Omega} \frac{u(x + he_i)}{h} v(x) \, dx - \int_{\Omega} \frac{u(x)}{h} v(x) \, dx
\]

and change variables in the first integral, putting \( y = x + he_i \), to obtain

\[
\int_{\Omega} \frac{u(y)}{h} v(y - he_i) \, dy - \int_{\Omega} \frac{u(x)}{h} v(x) \, dx
\]

\[
= -\int_{\Omega} \frac{u(x)}{h} \frac{v(x - he_i) - v(x)}{-h} \, dx
\]

\[
= -\int_{\Omega} u(x)D_{-h}^i v(x) \, dx.
\]

Finally, both expressions are equal to

\[
\frac{D_i u(x + he_i) - D_i u(x)}{h}.
\]

6.8 The inverse of \( \Phi \) is just the map \( y \mapsto x \), given by

\[
x_i = \begin{cases} y_i + z_i, & i = 1, \ldots, m - 1, \\ y_m + \psi(y_1 + z_1, \ldots, y_{m-1} + z_{m-1}), & i = m. \end{cases}
\]
Chapter 7

Therefore

\[ \nabla \psi = \begin{pmatrix} 1 & 0 & 0 & \ldots & D_1 \psi \\ 0 & 1 & 0 & \ldots & D_2 \psi \\ 0 & 0 & 1 & \ldots & D_3 \psi \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \]

It follows immediately that \( \text{det} \nabla \psi = 1 \), as required.

6.9 The result of Lemma 3.26 shows that, for a general positive symmetric linear operator whose inverse is compact,

\[ |A^j u| \leq C |A^{j/2} u|^{(k-s)/(k-l)} |A^{j/2} u|^{(s-l)/(k-l)} \]

for \( 0 \leq l < s < k \). \( A = -\Delta \) on \( \Omega' \) with Dirichlet boundary conditions certainly satisfies these conditions.

Taking \( u \in H_0^k(\Omega') \), Proposition 6.19 shows that \( u \in D(A^{j/2}) \) for all \( j = 0, 1, \ldots, k \), and so

\[ \| u \|_{H^j(\Omega')} \leq |A^{j/2} u| \leq C_j \| u \|_{H^j(\Omega')} \]

Therefore we have

\[ \| u \|_{H^j(\Omega')} \leq C \| u \|_{H^j(\Omega')}^{(k-s)/(k-l)} \| u \|_{H^j(\Omega')}^{(s-l)/(k-l)} \] \hspace{1cm} (S6.1)

for all such \( u \).

Now take \( u \in H^k(\Omega) \), and use Theorem 5.20 to extend \( u \) to a function \( Eu \in H_0^k(\Omega') \) for some \( \Omega' \supset \Omega \). Then (S6.1) holds for \( Eu \), and since \( E \) is bounded from \( H^j(\Omega) \) into \( H_0^j(\Omega') \) for each \( 0 \leq j \leq k \), we have

\[ \| u \|_{H^j(\Omega)} \leq \| Eu \|_{H^j(\Omega')} \leq C \| Eu \|_{H^j(\Omega')}^{(k-s)/(k-l)} \| Eu \|_{H^j(\Omega')}^{(s-l)/(k-l)} \]

\[ \leq C \| u \|_{H^j(\Omega)}^{(k-s)/(k-l)} \| u \|_{H^j(\Omega)}^{(s-l)/(k-l)} , \]

which is (6.32) for \( u \in H^k(\Omega) \), as required.

Chapter 7

7.1 Define an element \( l \in X^{**} \) by

\[ \langle l, L \rangle = \int_0^T \langle L, f(t) \rangle \, dt \quad \text{for all} \quad L \in X^* . \] \hspace{1cm} (S7.1)
This map \( I \) is clearly linear, and it is bounded since
\[
|\langle I, L \rangle| \leq \int_0^T \| L \|_X \cdot \| f(t) \|_X \, dt
\leq \left( \int_0^T \| f(t) \|_X \, dt \right) \| L \|_{X^*},
\]
and
\[
\int_0^T \| f(t) \|_X \, dt < \infty
\]
from (7.31). Since \( X \) is reflexive, it follows that there exists an element \( y \in X \) such that
\[
\langle I, L \rangle = \langle L, y \rangle \quad \text{for all} \quad L \in X^*.
\]
Therefore, using (S7.1), we have (7.29).

That the integral is well defined follows from Lemma 4.4, which shows that if
\[
\langle L, y_1 \rangle = \langle L, y_2 \rangle \quad \text{for all} \quad L \in X^*
\]
then \( y_1 = y_2 \).

7.2 Corollary 4.5 shows that there exists an element \( L \in X^* \) such that \( \| L \|_{op} = 1 \) and \( Ly = \| y \|_X \). Then, using (7.29), we have
\[
\left\| \int_0^T f(t) \, dt \right\|_X \leq \int_0^T |\langle L, f(t) \rangle| \, dt
\leq \int_0^T \| f(t) \|_X \, dt,
\]
as required.

7.3 An element \( v \) of \( L^p(0, T; V) \) is the limit in the \( L^p \) norm of a sequence of functions \( v_n \) in \( C^0([0, T]; V) \). Since such functions are uniformly continuous on \([0, T]\), given \( \epsilon > 0 \) we can find an integer \( N \) such that \( \delta = T/N \) satisfies
\[
|t - s| < \delta \quad \Rightarrow \quad \| v_n(t) - v_n(s) \|_V \leq \epsilon / T^{1/p}.
\]
We can approximate \( v_n \) to within \( \epsilon \) in \( L^p(0, T; V) \) by
\[
\sum_{j=1}^N v_n(j\delta) \chi((j\delta, (j + 1)\delta]),
\]
an expression of the form (7.32). It follows that such elements are dense in \( L^p(0, T; V) \). Since \( C^1([0, T]) \) is dense in \( L^p(0, T) \) we could also use elements of the form of (7.32) with \( \alpha \in C^1([0, T]) \); similarly, \( C_\infty^\infty(\Omega) \) is dense in \( L^p(\Omega) \), so we could take \( v_j \in C_\infty^\infty(\Omega) \).

7.4 Taking the inner product of (7.33) with \( A^k u_n \) yields

\[
\frac{1}{2} \frac{d}{dt} |A^{k/2} u_n|^2 + |A^{k/2} f|^2 \leq |A^{k/2} f||A^{k/2} u_n|,
\]

and so, using Young’s inequality, we obtain

\[
\frac{d}{dt} |A^{k/2} u_n|^2 + |A^{k/2} u_n|^2 \leq |A^{k/2} f|^2,
\]

which shows that

\[
|A^{k/2} u_n(t)|^2 + \int_0^t |A^{k/2} u_n(s)|^2 \, ds \leq |A^{k/2} u(0)|^2 + |A^{k/2} f|^2,
\]

which yields (7.34), and then (7.35) follows from (7.33). Therefore, using Proposition 6.18, we get

\[
u_n \in L^\infty(0, T; H^k) \cap L^2(0, T; H^{k+1})
\]

and

\[
du_n/dt \in L^2(0, T; H^{k-1}).
\]

Extracting a subsequence shows that the solution \( u \) satisfies

\[
u \in L^2(0, T; H^{k+1}) \quad \text{and} \quad du/dt \in L^2(0, T; H^{k-1}).
\]

It follows from Corollary 7.3 that \( u \in C^0([0, T]; H^k) \).

7.5 Since the \( \{w_j\} \) are orthogonal in \( H_0^1 \) and orthonormal in \( L^2 \) the equation for \( u_n \) becomes

\[
\lambda_j u_{nj} = f_j,
\]

where \( \lambda_j \equiv \|w_j\|^2 \) and \( f_j = (f, w_j) \). It follows that

\[
u_{nj} = f_j/\lambda_j,
\]

independent of \( n \). In particular we have

\[
u_n = \sum_{j=1}^n \frac{f_j}{\lambda_j} w_j.
\]
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and so, for $m > n$,

$$\|u_m - u_n\|^2 \leq \sum_{j=n+1}^{m} \frac{f_j^2}{\lambda_j}.$$ 

Since we have the Poincaré inequality we must have $\lambda_j \geq \lambda$, for some $\lambda$, and so

$$\|u_m - u_n\|^2 \leq \frac{1}{\lambda} \sum_{j=n+1}^{m} f_j^2.$$ 

Since $f \in L^2(\Omega)$ it follows that $u_n$ converges in $H^1_0(\Omega)$ to $u = \sum_{j=1}^{\infty} f_j w_j / \lambda_j$.

Now, we know that

$$((u_n, v)) = (P_n f, v) \quad \text{for all} \quad v \in P_n H^1_0(\Omega).$$

Since

$$((u_n, v)) = ((u_n, P_n v)) \quad \text{and} \quad (P_n f, v) = (P_n f, P_n v)$$

for all $v \in H^1_0(\Omega)$, we in fact have

$$((u_n, v)) = (P_n f, v) \quad \text{for all} \quad v \in H^1_0(\Omega).$$

Since $u_n \to u$ in $H^1_0(\Omega)$ we know that

$$((u_n, v)) \to ((u, v)),$$

and since $P_n f \to f$ in $L^2(\Omega)$ we must have

$$((u, v)) = (f, v) \quad \text{for all} \quad v \in H^1_0(\Omega),$$

and $u$ is a weak solution of (7.36) as required.

Chapter 8

8.1 We show in general that if $Z = X \cap Y$, with norm

$$\|u\|_Z = \|u\|_X + \|u\|_Y,$$
then $Z^* = X^* + Y^*$. First, it is clear that if $f = f_1 + f_2$, with $f_1 \in X^*$ and $f_2 \in Y^*$, then for $u \in X \cap Y$

$$|(f_1 + f_2, u)| \leq |(f_1, u)| + |(f_2, u)|$$

$$\leq \|f_1\|_{X^*} \|u\|_X + \|f_2\|_{Y^*} \|u\|_Y$$

$$\leq (\|f_1\|_{X^*} + \|f_2\|_{Y^*}) \|u\|_{X \cap Y}.$$ 

Thus $X^* + Y^* \subset (X \cap Y)^*$. Now, if $f \in (X \cap Y)^*$ then, since it is a linear functional on a linear subspace of $X$, application of the Hahn–Banach theorem (Theorem 4.3) tells us it has an extension $f_1$ that is a linear functional on the whole of $X$ (we could use $Y$ rather than $X$ here if we wished). Thus $(X \cap Y)^* \subset X^* \subset X^* + Y^*$, and so we have the required equality.

8.2 Follow the argument of Theorem 7.2, except approximate $u$ by a sequence $u_n \in C^1([0, T]; H^1)$ such that

$$u_n \to \text{ in } L^2(0, T; H^1) \cap L^p(\Omega_T)$$

and

$$du_n/dt \to du/dt \text{ in } L^2(0, T; H^{-1}) + L^q(\Omega_T).$$

We will denote by $X(t_1, t_2)$ the space

$$L^2(t_1, t_2; H^1) \cap L^p(\Omega \times (t_1, t_2)),$$

and by $X^*(t_1, t_2)$ the space

$$L^2(t_1, t_2; H^{-1}) + L^q(\Omega \times (t_1, t_2)).$$

We now estimate

$$\int_\Omega |u_n(t)|^2 \, dx = \frac{1}{T} \int_\Omega \int_0^T |u_n(t)|^2 \, dt \, dx + 2 \int_\Omega \int_0^T \dot{u}_n(s) u_n(s) \, ds$$

$$\leq \frac{1}{T} \int_\Omega \int_0^T |u_n(t)|^2 \, dt \, dx + 2 \|\dot{u}_n\|_{H^1(t_1, T)} \|u_n\|_{H^1(t_1, T)}$$

$$\leq \frac{1}{T} \int_\Omega \int_0^T |u_n(t)|^2 \, dt \, dx + 2 \|\dot{u}_n\|_{H^{-1}(0, T)} \|u_n\|_{H^0(0, T)},$$

showing once again that $u_n$ is also a Cauchy sequence in $C^0([0, T]; L^2)$ and hence that $u \in C^0([0, T]; L^2)$ as claimed.
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8.3 Integrating by parts gives

\[- \int_{\Omega} \sum_{j} f(u_n) \frac{\partial^2 u_n}{\partial x_j^2} \, dx \]

\[= \int_{\Omega} \sum_{j} f'(u_n) \left| \frac{\partial u_n}{\partial x_j} \right|^2 \, dx + \int_{\partial\Omega} f(u_n) \nabla u_n \cdot n \, dS.\]

We can estimate the extra term by

\[\int_{\partial\Omega} f(u_n) \nabla u_n \cdot n \, dS \leq |f(0)| \int_{\partial\Omega} |\nabla u_n| \, dS \]

\[\leq |f(0)||\partial\Omega|^{1/2} \|\nabla u_n\|_{L^2(\partial\Omega)} \]

\[\leq C \|u_n\|_{H^1(\Omega)},\]

using the trace theorem (Theorem 5.35). Since we have

\[\|u\|_{H^2(\Omega)} \leq C|Au|\]

from Theorem 6.16, we can write

\[\frac{1}{2} \frac{d}{dt} \|u_n\|^2 + |Au_n|^2 \leq \frac{l}{2} \|u_n\|^2 + C|Au_n|.\]

Using Young’s inequality on the last term and rearranging finally gives

\[\frac{d}{dt} \|u_n\|^2 + |Au_n|^2 \leq 2l \|u_n\|^2 + C,\]

which integrates to give the bound

\[\|u_n(T)\|^2 + \int_0^T |Au_n(s)|^2 \, ds \leq 2l \int_0^T \|u_n(t)\|^2 \, dt + \|u_0\|^2 + CT.\]

Thus \(u_n\) is uniformly bounded in \(L^2(0, T; D(A))\) [and \(L^\infty(0, T; V)\)], where (8.19) is used as before to guarantee that \(u_n \in L^2(0, T; V)\).

8.4 In this case we can follow the proof of Proposition 8.6 until the line

\[|F(u) - F(v)|^2 \leq C|u - v|^2_{L^p} (1 + |u|^2_{L^{2p}} + |v|^2_{L^{2p}}).\]

We now have to be more careful with our use of the Sobolev embedding theorem, since the highest we can go is \(H^1 \subset L^6\). We therefore need

\[2p \leq 6 \quad \text{and} \quad 2q \gamma \leq 6, \quad \text{where} \ (p, q) \ \text{are conjugate}.\]

The first conditions forces us to take \(p \leq 3\), and hence we must have
Chapter 8

$q \geq 3/2$, which shows that the largest possible value for $\gamma$ is 2, as claimed. Provided that $\gamma \leq 2$ we can write

$$|F(u) - F(v)|^2 \leq C\|u - v\|_{H^1}^2 (1 + \|u\|_{H^1} + \|v\|_{H^1}),$$

as in Proposition 8.6.

8.5 Since $u, v \in L^2(0, T; D(A))$ we can use Corollary 7.3 to take the inner product of

$$dw/dt + Aw = F(u) - F(v)$$

with $Aw$ to obtain, using (8.31),

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 + |Aw|^2 = (F(u) - F(v), Aw) \leq C(1 + |Au| + |Av|)^{3/2}\|w\|^{1/2}|Aw|^{3/2}.$$

We now use Young’s inequality to split the right-hand side,

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 + |Aw|^2 \leq \frac{3}{4}|Aw|^2 + C(1 + |Au| + |Av|)^2\|w\|^2,$$

and so

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 \leq C(1 + |Au| + |Av|)^2\|w\|^2.$$

This yields

$$\|w(t)\|^2 \leq \|w(0)\|^2 \exp \left( \int_0^t C(1 + |Au(s)|^2 + |Av(s)|^2) \, ds \right),$$

which gives continuous dependence on initial conditions since we know that both $u$ and $v$ are elements of $L^2(0, T; D(A))$.

8.6 Setting $g(s) = e^{-A(t-s)}u(s)$, we have

$$\frac{dg}{ds} = Ae^{-A(t-s)}u(s) + e^{-A(t-s)}\frac{d}{ds}u(s)$$

$$= Ae^{-A(t-s)}u(s) + e^{-A(t-s)}[-Au + f(u(s))]$$

$$= e^{-A(t-s)}f(u(s)),$$

so that integrating with respect to $s$ between 0 and $t$ gives

$$g(t) - g(0) = \int_0^t e^{-A(t-s)}f(u(s)) \, ds.$$
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Chapter 9

9.1 Taking the divergence of the governing equation yields

\[ \Delta u = \nabla \cdot f, \]

since all the other terms are divergence free. A solution of this equation in the periodic case when \( f \in \dot{L}^2(Q) \) has been obtained as Equation (9.10). Note that if \( f \in H \) then this implies that \( p = 0 \) (or, equivalently, a constant).

9.2 We have

\[
|u|_{L^4}^4 = \int_Q |u|^4 \, dx \leq \left( \int_Q |u|^6 \right)^{1/2} \left( \int_Q |u|^2 \right)^{1/2} \\
= \|u\|_{L^6}^3 |u| \leq k \|u\|_V |u|,
\]

since \( H^1(Q) \subset L^6(Q) \) (see Theorem 5.31).

9.3 Applying the Cauchy–Schwarz inequality first in the variable \( j \) and then in the variable \( i \), we get

\[
\left| \sum_{i,j=1}^m a_i b_{i,j} c_j \right| \leq \left( \sum_{i,j=1}^m |a_i b_{i,j}|^2 \right)^{1/2} \left( \sum_{j=1}^m |c_j|^2 \right)^{1/2} \\
= \left( \sum_{i=1}^m a_i \left( \sum_{j=1}^m b_{i,j} \right) \right) \left( \sum_{j=1}^m |c_j|^2 \right)^{1/2} \\
\leq \left( \sum_{i=1}^m |a_i|^2 \right)^{1/2} \left( \sum_{i,j=1}^m |b_{i,j}|^2 \right)^{1/2} \left( \sum_{j=1}^m |c_j|^2 \right)^{1/2},
\]

as claimed.

9.4 If \( m = 2 \), we have

\[
|b(u, v, w)| \leq k |u|^{1/2} \|u\|^{1/2} |v|^{1/2} \|v\|^{1/2} \|w\| \|w\|
\]

[using \( b(u, v, w) = -b(u, w, v) \)], so that

\[
\langle B(u, u), w \rangle \leq k \|u\| \|u\| \|w\|,
\]

and therefore

\[
\|B(u, u)\|_{V^*} \leq k |u| \|u\|.
\]
Chapter 9

If \( m = 3 \), we have

\[
|b(u, v, w)| \leq k |u|^{1/4} |u|^{3/4} |v|^{1/4} |v|^{3/4} |w|^{1/4} |w|^{3/4}
\]

[using \( b(u, v, w) = -b(u, w, v) \) again], and so

\[
\langle B(u, u), w \rangle \leq k |u|^{1/2} |u|^{3/2} |w|^{3/2}.
\]

giving

\[
\|B(u, u)\|_{V^*} \leq k |u|^{1/2} |u|^{3/2}.
\]

9.5 Take \((p, q) = (2, 2)\) if \( m = 2 \) and \((p, q) = (4/3, 4)\) if \( m = 3 \). We know that \( B_n \rightharpoonup B \) in \( L^p(0, T; V^*) \), where \( B_n = B(u_n, u_n) \) and \( B = B(u, u) \). We need to show that \( P_n B_n \rightharpoonup B \) in the same sense. For \( \psi \in L^q(0, T; V) \) we have

\[
\int_0^T \langle P_n B_n(t) - B, \psi \rangle dt = \int_0^T \langle P_n B_n - B_n, \psi \rangle dt + \int_0^T \langle B_n - B, \psi \rangle dt.
\]

The second term converges since \( B_n \rightharpoonup B \), so we have to treat only the first term. We rewrite this as

\[
\int_0^T \langle B_n(t), Q_n \psi \rangle dt.
\]

Since functions of the form

\[
\psi = \sum_{j=1}^k \psi_j \alpha_j(t), \quad \psi_j \in V, \ \alpha_j \in C^1([0, T], \mathbb{R}) \quad (S9.1)
\]

are dense in \( L^q(0, T; V) \) (see Exercise 7.3) we can consider

\[
\int_0^T \left\langle B_n, \sum_{j=1}^k Q_n \psi_j \right\rangle \alpha_j(t) dt.
\]

Since \( B_n \) is uniformly bounded in \( L^p(0, T; V^*) \) when \( m = 2 \), we can use the fact that \( Q_n \psi_j \rightharpoonup \psi_j \) in \( V \) to show the required convergence for all \( \psi \) of the form \((S9.1)\). The density of such \( \psi \) in \( L^q(0, T; V) \) then gives the full result.
9.6 If \( u \in L^4(0, T; V) \) then we can estimate \( b(w, u, w) \) in (9.41) differently, writing
\[
\frac{1}{2} \frac{d}{dt} |w|^2 + v\|w\|^2 \leq k|w|^{1/2}\|w\|^{3/2}\|u\|
\leq \frac{\nu}{2} \|w\|^2 + \frac{c}{\nu^3} |w|^2 \|u\|^4,
\]
which becomes, dropping the terms in \( \|w\|^2 \),
\[
\frac{d}{dt} |w|^2 \leq C|w|^2 \|u\|^4.
\]
Integrating gives
\[
|w(t)|^2 \leq |w(0)|^2 \exp \left( \int_0^t \|u(s)\|^4 ds \right),
\]
which implies uniqueness provided that \( u \in L^4(0, T; V) \).

10.1 If not, then there exist an \( \epsilon > 0 \) and sequences \( \delta_n \to 0 \), \( x_n \in K \), \( y_n \in H \), such that
\[
|x_n - y_n| \leq \delta_n \quad \text{and} \quad |f(x_n) - f(y_n)| > \epsilon.
\]
Since \( K \) is compact there is a subsequence of the \( \{x_n\} \) (relabel this \( x_n \)) such that \( x_n \to x^* \in X \). Now,
\[
|x^* - y_n| \leq |x^* - x_n| + |x_n - y_n| \to 0 \quad \text{as} \quad n \to \infty, \quad (S10.1)
\]
and
\[
|f(x^*) - f(y_n)| \geq |f(x_n) - f(y_n)| - |f(x_n) - f(x^*)| \geq \epsilon/2 \quad (S10.2)
\]
if \( n \) is sufficiently large, since \( f \) is continuous at \( x^* \). But then (S10.1) and (S10.2) say precisely that \( f \) is not continuous at \( x^* \), which is a contradiction.

10.2 The set in (10.23) is bounded since
\[
\bigcup_{t \geq \theta(B)} S(t)B, \quad (S10.3)
\]
Chapter 10

with $t_0(B)$ from Definition 10.2, is a subset of $B$, and

$$\bigcup_{0 \leq t \leq t_0(B)} S(t)B$$

(S10.4)

is bounded since $B$ is bounded and $S(t)$ is continuous. Similarly, if $B$ is compact then (S10.3) is a closed subset of $B$, and (S10.4) is the continuous image of the compact set $B \times [0, t_0(B)]$: both parts are compact, and therefore so is (10.23). That (10.23) is positively invariant is clear by definition.

10.3 In this example $\omega(0) = 0$ and $\omega(x) = \{|x| = 1\}$ if $x \neq 0$. So

$$\Lambda(B) = \{(0, 0)\} \cup \{|x| = 1\}$$

(which is clearly not connected). Since $\omega(x) = (1, 0)$ for all $x$ with $|x| = 1$,

$$\Lambda[\Lambda(B)] = \{(0, 0), (1, 0)\},$$

so that $\Lambda[\Lambda(B)] \neq \Lambda(B)$ as claimed.

10.4 We show that, for a bounded set $X$,

$$\omega_1(X) = \{y : S(t_n)x_n \to y\},$$

where $t_n \to \infty$ and $x_n \in X$, is equal to

$$\omega_2(X) = \bigcap_{t \geq 0} \bigcup_{s \geq t} S(s)X.$$

If $y \in \omega_1(X)$ then clearly

$$y \in \bigcup_{s \geq t} S(s)X$$

for all $t \geq 0$ and hence in $y \in \omega_2(X)$. So $\omega_1(X) \subset \omega_2(X)$.

Conversely, if $y \in \omega_2(X)$ then for any $t \geq 0$

$$y \in \bigcup_{s \geq t} S(s)X,$$

and so there are sequences $\{r_m^{(1)}\}$, with $r_m^{(1)} \geq t$, and $\{x_m^{(1)}\} \in X$ with $S(r_m^{(1)})x_m^{(1)} \to y$. Now consider $t = 1, 2, \ldots$ and pick $t_n$ from $r_n^{(1)}$ and
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Let \( x_n \) from \( x_m^{(a)} \) such that

\[
|S(t_n^{(a)})x_m^{(a)} - y| \leq 1/n.
\]

Then \( S(t_n)x_n \rightarrow y \) with \( t_n \rightarrow \infty \), since \( t_n \geq n \), showing that \( y \in \omega_1(X) \). This gives \( \omega_2(X) \subseteq \omega_1(X) \), and so \( \omega_1(X) = \omega_2(X) \).

10.5 If \( y \in S(t)B \) for all \( t \geq 0 \) then for any \( t_n \) there is an \( x_n \in B \) with \( y = S(t_n)x_n \), so clearly \( y \in \omega_2(B) \) (as defined in the previous solution).

Conversely, if \( y \in \omega_2(B) \) then we must have \( y \in \bigcup_{s \geq t} S(s)B \).

Now, if \( \tau \geq t_0(B) \), then

\[ S(t)B \supset S(t + \tau)B, \]

and so then

\[ S(t)B \supset \bigcup_{\tau \geq t_0(B)} S(t + \tau)B. \]

Since \( S(t)B \) is closed

\[ S(t)B \supset \bigcup_{\tau \geq t_0(B)} S(t + \tau)B \ni y, \]

that is, \( y \in S(t)B \) for all \( t \geq 0 \).

Clearly we have

\[ \bigcap_{t \geq 0} S(t)B \subset \bigcap_{n \in \mathbb{Z}^+} S(nT)B. \]

If \( u \in S(nT)B \) for all \( n \in \mathbb{Z}^+ \) then in particular \( u \in S(n_0T)B \), provided that \( n_0 \) is large enough that \( n_0T \geq t_0(B) \), where

\[ S(t)B \subset B \quad \text{for all} \quad t \geq t_0(B). \]

Since \( u \in S(nT)B \) we have \( u = S(nT)y \) with \( y \in B \), and it follows that for all \( t \geq 0 \)

\[ S(t)u = S(t + nT)y = S(\tau)y \in B, \]

since \( \tau \geq t_0(B) \). Therefore \( u \in S(t)B \) for all \( t \geq n_0T \). It follows that \( u \in \omega_2(B) \), giving the required equality.
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10.6 \( y \in \omega(Y) \) if \( S(t_n) y_n \to y \) with \( t_n \to \infty \) and \( y_n \in Y \). Then \( y_n \in X \) also and so \( \omega(X) \supset \omega(Y) \). If \( Y \) absorbs \( X \) in a time \( t_0 \) (assuming \( X \) to be bounded) and if \( S(t_n)x_n \to x \), then

\[
S(t_n - t_0)[S(t_0)x_n] \to x,
\]

and since \( t_n - t_0 \to \infty \) and \( S(t_0)x_n \in Y \), \( \omega(X) \subset \omega(Y) \), so then \( \omega(X) = \omega(Y) \).

10.7 First, the set

\[
\bigcup_{i=j}^{\infty} K_i \quad \text{(S10.5)}
\]

is clearly closed, and since all sets \( K_i \) lie within \( 1/j \) of \( K_j \) if \( i \geq j \) it is also bounded, and hence compact. It follows that \( K_\infty \), the intersection of a decreasing sequence of compact sets, is itself compact.

Now, it is clear by a similar argument that

\[
dist(K_\infty, K_j) \leq j^{-1}.
\]

Conversely, if \( u \in K_j \) then \( dist(u, K_i) \leq j^{-1} \) for all \( i \geq j \). So certainly

\[
dist \left( u, \bigcup_{i=j}^{\infty} K_i \right) \leq j^{-1}.
\]

In particular, there exist points \( u_i \in K_i, i \geq j \), such that

\[
|u_i - u| \leq j^{-1}.
\]

Since each \( u_i \) is contained in the compact set \( \text{(S10.5)} \) (with \( j = 1 \)) then there exists a subsequence of the \( u_i \) that converges to some \( u^* \). It follows that \( u^* \in K_\infty \), and by construction \( |u - u^*| \leq j^{-1} \). Therefore

\[
dist(K_j, K_\infty) \leq j^{-1},
\]

and so

\[
dist_2(K_j, K_\infty) \leq j^{-1}.
\]

\( K_j \) converges to \( K_\infty \) in the Hausdorff metric.

10.8 To show that the inverse is continuous, suppose not. Then there exist an \( \epsilon > 0 \) and a sequence \( \{x_n\} \in f(X) \) with \( x_n \to y \in f(X) \) but

\[
|f^{-1}(x_n) - f^{-1}(y)| \geq \epsilon. \quad \text{However,} \quad f^{-1}(x_n) \in X, \quad \text{and since} \quad X \] is
Solutions to Exercises

compact there exists a subsequence $x_{n_j}$ such that $f^{-1}(x_{n_j}) \to z$. Since $f$ is continuous, it follows that $x_{n_j} \to f(z)$. Since $f$ is injective, it follows from $f(z) = y$ that $z = f^{-1}(y)$, which is a contradiction. So $f^{-1}$ is continuous on $f(X)$.

10.9 Proposition 10.14 says that, given $\epsilon_1$ and $T > 0$, there exists a time $\tau_1$ such that, for all $t \geq \tau_1$, 
\[
\text{dist}(u(t), A) \leq \delta(\epsilon_1, T).
\]
So we can track the trajectory $u(t)$ within a distance $\epsilon_1$ for a time $T$ starting at any time $t \geq \tau_1$.

We can replace $T$ with $2T$ and apply the same argument for $\epsilon_2 = \epsilon_1/2$, that is, there exists a time $\tau_2$ such that, for all $t \geq \tau_2$, 
\[
\text{dist}(u(t), A) \leq \delta(\epsilon_2, 2T),
\]
and then the trajectory $u(t)$ can be tracked for a time $2T$ starting at any time $t \geq \tau_2$.

Thus $u(t)$ can be followed from $\tau_1$ to $\tau_2$ by a distance $\epsilon_1$ with a finite number of trajectories on $A$ of time length $T$, and when we reach $\tau_2$, we can start to track $u(t)$ within a distance $\epsilon_2$ with trajectories on $A$ of time length $2T$, until we reach a $\tau_3$ after which we can track within a distance $\epsilon_3$ for a time length $3T$, etc.

The “jumps” are bounded by $\epsilon_k + \epsilon_{k+1}$, since 
\[
|v_{k+1} - S(t_{k+1} - t_k) v_k| \\
\leq |v_{k+1} - u(t_{k+1})| + |u(t_k + (t_{k+1} - t_k)) - S(t_{k+1} - t_k) v_k| \\
\leq \epsilon_{k+1} + \epsilon_k.
\]

10.10 Take $\epsilon > 0$. Then there is a $T > 0$ such that 
\[
\text{dist}(S(t)B_1, A) + \text{dist}(S(t)B_2, A) < \epsilon \quad \text{for all} \quad t \geq T.
\]
Also, by the uniform continuity of the semigroup, there is a $\delta > 0$ such that 
\[
\text{dist}(S(t)B_1, S(t)B_2) \leq \epsilon \quad \text{for all} \quad t \in [0, T]
\]
provided that $\text{dist}(B_1, B_2) \leq \delta$. The argument is symmetric, which gives the result.
Chapter 11

11.1 Using Young’s inequality on (11.30) we can deduce that

\[ |u|^2 \leq \frac{p}{2} \int_{\Omega} |u|^p \, dx + \frac{p}{p-2} |\Omega|. \]

So we can write (11.6) as

\[ \frac{1}{2} \frac{d}{dt} |u|^2 + \frac{2\alpha^2}{p} |u|^2 \leq \left( \frac{2}{p-2} + k \right) |\Omega|. \]

Neglecting the \( \|u\|^2 \) term we can write

\[ \frac{1}{2} \frac{d}{dt} |u|^2 + \frac{2\alpha^2}{p} |u|^2 \leq \left( \frac{2}{p-2} + k \right) |\Omega|. \]

We can now apply the Gronwall inequality to deduce an asymptotic bound on \( |u(t)| \), as in Proposition 11.1. (The expression for the bound will be a more complicated expression than before.)

11.2 Proceeding as advised, we obtain

\[ \frac{d}{ds} \left( y(s) \exp \left( -\int_s^t g(\tau) \, d\tau \right) \right) \leq h(s) \exp \left( -\int_s^t g(\tau) \, d\tau \right) \leq h(s), \]

and integrating both sides between \( s \) and \( t + r \) gives

\[ y(t + r) \leq y(s) \exp \left( \int_s^{t+r} g(\tau) \, d\tau \right) + \left( \int_s^{t+r} h(\tau) \, d\tau \right) \exp \left( \int_s^{t+r} g(\tau) \, d\tau \right) \leq (y(s) + a_2) \exp(a_1). \]

Integrating both sides for \( t \leq s \leq t + r \) gives the result as stated.

11.3 Taking the inner product of

\[ \frac{du_n}{dt} + Au_n = P_n f(u_n) \]

with \( t^2 Au_n \) we obtain

\[ \left( \frac{du_n}{dt}, t^2 Au_n \right) + t^2 |Au_n|^2 = (P_n f(u_n), t^2 Au_n), \]
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which, using the methods leading to (8.27) for the right-hand side, becomes

\[ \frac{1}{2} \frac{d}{dt} \| tu_n \|^2 - 2t \| u_n \|^2 + t^2 |A u_n|^2 \leq lt^2 \| u_n \|^2. \]

Integrating from 0 to \( T \) gives

\[ \| T u_n \|^2 + \int_0^T t^2 |A u_n|^2 dt \leq \int_0^T (2t + lt^2) \| u_n \|^2 dt. \]

Since we already know that \( u_n \in L^2(0, T; V) \), it follows that \( u_n \in L^2(t, T; D(A)) \) for any \( t > 0 \). Since \( H^2(\Omega) \subset C^0(\overline{\Omega}) \) if \( m \leq 3 \) we also have \( P_n f(u_n) \in L^2(t, T; L^2) \), and so it follows that \( u_n / dt \in L^2(t, T; H) \). Taking limits shows that the solution \( u \) satisfies

\[ u \in L^2(t, T; D(A)) \quad \text{and} \quad du/dt \in L^2(t, T; L^2). \]

Application of Corollary 7.3 then makes the “formal” calculations at the beginning of Section 11.1.2 rigorous.

11.4 Observe that for \( s < 0 \) we have

\[ f(s) |s| \geq \alpha_2 |s|^p - k, \]

and so in particular

\[ f(s) \geq 0 \quad \text{for all} \quad s < (k/\alpha_2)^{1/p}. \]  

(S11.1)

Now set \( M = (k/\alpha_2)^{1/p} \), multiply Equation (11.1) by \( (u(x) + M)_- \), and integrate to obtain

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} (u(x) + M)_-^2 + \int_{\Omega} |\nabla (u + M)_-|^2 = \int_{\Omega} f(u)(u + M)_- dx \leq 0, \]

using (S11.1). It follows, using the Poincaré inequality, that

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} (u(x) + M)_-^2 dx \leq -C \int_{\Omega} (u(x) + M)_-^2 dx, \]

and so as in the last part of the argument given in Theorem 11.6, we must have

\[ \int_{\Omega} (u(x) + M)_-^2 dx = 0 \]

for all \( u \in A \).
Chapter 12

12.1 If \( u \) is smooth then \( Au = -\Delta u \), and we have

\[
b(u, u, A^2 u) = \sum_{i,j,k,l=1}^2 \int_{\Omega} u_i(D_j u_j) D_k^2 D_l^2 u_j \, dx
\]

\[
= \sum_{i,j,k,l=1}^2 \int_{\Omega} D_k^2(u_i(D_j u_j)) (D_l^2 u_j) \, dx
\]

\[
= \sum_{i,j,k,l} \int_{\Omega} \left[ (D_k^2 u_i)(D_l^2 u_j) + 2(D_k u_i)(D_l D_j u_j)

+ u_i(D_k D_l^2 u_j) \right] (D_l^2 u_j)
\]

\[
= b(Au, u) + 2 \sum_{i,j,k,l} (D_k u_i)(D_l D_j u_j) (D_l^2 u_j) \, dx

+ b(u, Au)
\]

\[
= b(Au, u) + 2 \sum_{k=1}^2 b(D_k u, D_k u, Au),
\]

as claimed. The result follows for general \( u \) by taking limits.

To obtain inequality (12.23), use (9.26) to give

\[
|b(u, u, A^2 u)| \leq k |A^{3/2} u| ||Au|| ||u|| + 2 \sum_{j=1}^2 |b(D_j u, Au, D_j u)|
\]

\[
\leq k |A^{3/2} u| ||Au|| ||u|| + 2k \sum_{j=1}^2 |D_j u||\|D_j u\|\||Au||
\]

\[
\leq k |A^{3/2} u| ||Au|| ||u|| + 2k \left( \sum_{j=1}^2 |D_j u|^2 \right)^{1/2} \left( \sum_{j=1}^2 \|D_j u\|^2 \right)^{1/2} \|Au\|.
\]

Since

\[
\|u\|^2 = a(u, u) = \langle Au, u \rangle = (A^{1/2} u, A^{1/2} u) = |A^{1/2} u|^2,
\]

this becomes

\[
|b(u, u, A^2 u)| \leq 3k |A^{3/2} u| ||Au|| ||u||
\]

\[
\leq \frac{3}{4} |A^{3/2} u|^2 + \frac{9k^2}{4} ||u||^2 |Au|^2,
\]

as required.
Solutions to Exercises

12.2 Take the inner product of 
\[ \frac{d}{dt} \frac{1}{2} |Au|^2 + \nu |A^{3/2}u|^2 = -b(u, u, A^2 u) + (f, A^2 u), \]
with \( A^2 u \) to obtain
\[ \frac{1}{2} \frac{d}{dt} |Au|^2 + \nu |A^{3/2}u|^2 \leq -b(u, u, A^2 u) + (f, A^2 u), \]
and use the estimate (12.23) from the previous exercise to write
\[ \frac{1}{2} \frac{d}{dt} |Au|^2 + \nu |A^{3/2}u|^2 \leq \frac{v}{4} |A^{3/2}u|^2 + \frac{C}{v} \|u\|^2 |Au|^2 + \frac{v}{4} |A^{3/2}u|^2, \]
so that
\[ \frac{d}{dt} |Au|^2 + \nu |A^{3/2}u|^2 \leq 2 \|f\|^2 + \frac{C}{v} \|u\|^2 |Au|^2. \]
Using a similar trick as we did for the absorbing set in \( V \), we integrate this equation between \( s \) and \( t \), with \( t < s < t + 1 \), so that
\[ |Au(t + 1)|^2 \leq |Au(s)|^2 + \frac{2M}{v} + \frac{C}{v} \int_s^{t+1} \|u(s)\|^2 |Au(s)|^2 ds, \]
where we have used (12.24). Integrating again with respect to \( s \) between \( t \) and \( t + 1 \) gives
\[ |Au(t + 1)|^2 \leq \int_t^{t+1} |Au(s)|^2 ds + \frac{2M}{v} + \frac{C}{v} \int_t^{t+1} \|u(s)\|^2 |Au(s)|^2 ds. \]

(S12.1)

Now, if \( t \geq t_1 (\|u_0\|) \) then we know that
\[ \|u(s)\| \leq \rho v \quad \text{and} \quad \int_t^{t+1} |Au(s)|^2 ds \leq I_A, \]
and so if it follows that then
\[ |Au(t + 1)|^2 \leq \rho_A \equiv I_A + \frac{2M}{v} + \frac{C}{v} \rho v^2 I_A, \]
an absorbing set in \( D(A) \).

12.3 Suppose that \( u_n \in V \) with \( \|u_n\| \leq k \) and that \( u_n \to u \) in \( H \). Then there exists a subsequence \( u_{n_j} \) such that \( u_{n_j} \to v \) in \( V \), so that \( \|v\| \leq k \). Since \( V \subset H \), it follows that \( u_{n_j} \to v \) in \( V \), and so in particular we must have \( u = v \), which implies that \( \|u\| \leq k \).
Chapter 12

12.4 If \( u \in D(A) \) with
\[
    u = \sum_{k \in \mathbb{Z}^2} u_k e^{2\pi ik \cdot x / L}
\]
then we can estimate \( \|u\|_\infty \) by
\[
    \|u\|_\infty \leq \sum_{k \in \mathbb{Z}^2} |u_k|.
\]
Split the sum into two parts,
\[
    \|u\|_\infty \leq \sum_{|k| \leq \kappa} |u_k| + \sum_{|k| > \kappa} |u_k|.
\]
We now use the Cauchy–Schwarz inequality on each piece,
\[
    \|u\|_\infty \leq \left( \sum_{|k| \leq \kappa} |u_k|^2 \right)^{1/2} \left( \sum_{|k| \leq \kappa} 1 \right)^{1/2}
    + \left( \sum_{|k| > \kappa} |u_k|^2 |k|^4 \right)^{1/2} \left( \sum_{|k| > \kappa} |k|^{-4} \right)^{1/2}.
\]
Since
\[
    \sum_{|k| \leq \kappa} 1 \leq C \kappa^2 \quad \text{and} \quad \sum_{|k| > \kappa} |k|^4 \leq C \kappa^{-2},
\]
this becomes
\[
    \|u\|_\infty \leq C (\kappa |u| + \kappa^{-1} |Au|).
\]
To make both terms on the right-hand side the same, we choose \( \kappa = |Au|^{1/2} |u|^{-1/2} \), obtaining
\[
    \|u\|_\infty \leq C |u|^{1/2} |Au|^{1/2}.
\]

12.5 We have already derived in (12.20) the inequality
\[
    \frac{d}{dt} \|u\|^2 + \nu |Au|^2 \leq \frac{2|f|^2}{\nu} + C \|u\|^6,
\]
and since we have a uniform bound on \( \|u\| \) for \( t \) large enough, we obtain a uniform bound on the integral of \( |Au(s)|^2 \),
\[
    \int_{b_0}^{b_{n+1}} |Au(s)|^2 \, ds \leq C_1. \quad (S12.2)
\]
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Following the analysis in Proposition 12.4, we estimate

\[ |u_t| \leq v|Au| + |B(u, u)| + |f|, \]

and using (12.25) this becomes

\[ |u_t| \leq v|Au| + k\|u\|^3/2|Au|^{1/2} + |f|. \]

An application of Young’s inequality yields

\[ |u_t| \leq c|Au| + C\|u\|^2 + |f|, \]

and so for \( t \) large enough,

\[ |u_t| \leq c|Au| + C\rho^2 + |f|. \]

The bound in (S12.2) therefore implies a bound on \( \int |u_t|^2 \),

\[ \int_{t_0}^{t_0+1} |u_t(s)|^2 \, ds \leq C_3. \quad \text{(S12.3)} \]

Now differentiate

\[ u_t + vAu + B(u, u) = f \]

with respect to \( t \) to obtain

\[ u_{tt} + vAu_t + B(u_t, u) + B(u, u_t) = 0 \]

and take the inner product with \( u_t \) so that

\[ \frac{1}{2} \frac{d}{dt} |u_t|^2 + v\|u_t\|^2 \leq |b(u_t, u, u_t)| \]

\[ \leq k\|u\|^1/2|u_t|^{3/2} \]

\[ \leq \frac{3v}{4}\|u_t\|^2 + \frac{k^4\|u\|^4|u_t|^2}{4v^5}. \]

Using once again the asymptotic bound on \( \|u\| \), we have for \( t \geq t_0 \) that

\[ \frac{d}{dt} |u_t|^2 \leq C_3|u_t|^2. \]

We use the usual trick, integrating between \( s \) and \( t + 1 \), with \( t < s < t + 1 \),

\[ |u_t(t+1)|^2 \leq |u_t(s)|^2 + C_4 \int_t^{t+1} |u_t(s)|^2 \, ds, \]
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and then between \( t \) and \( t + 1 \) (with respect to \( s \)) so that

\[
|u_t(t + 1)|^2 \leq (1 + C_4) \int_t^{t+1} |u_s|^2 \, ds
\]

\[
\leq (1 + C_4) C_3,
\]

(S12.4)

by (S12.3).

To end, we show that \( |u_t| \) bounds \( |Au| \). From the equation we have

\[
ν|Au| \leq |u_t| + |B(u, u)| + |f|,
\]

or with (12.25)

\[
ν|Au| \leq |u_t| + k|Au|^{1/2} ||u||^{3/2} + |f|,
\]

and so after using Young’s inequality and rearranging we have

\[
|Au| \leq C(|u_t| + ||u||^3 + |f|).
\]

Together with (S12.4) we obtain

\[
|Au(t)| \leq ρ_D
\]

for all \( t \geq 1 + t_0(||u_0||) \). So we have an absorbing set in \( D(A) \) and hence a global attractor for the 3D equations.

Chapter 13

13.1 Let \( G(X, ε) \) be the number of boxes in a fixed cubic lattice, with sides \( ε \), that are necessary to cover \( X \). Since each cube with side \( ε \) sits inside a ball of radius \( ε \), \( N(X, ε) \leq G(X, ε) \), and so

\[
d_f(X) \leq d_{\text{box}}(X).
\]

Also, since any ball with side \( ε \) is contained within at most \( 2^m \) different boxes in the grid, we have \( G(X, ε) \leq 2^m N(X, ε) \). Therefore

\[
d_{\text{box}}(X) = \limsup_{ε \to 0} \frac{\log G(X, ε)}{− \log ε}
\leq \limsup_{ε \to 0} \frac{m \log 2 + \log N(X, ε)}{− \log ε}
\leq \limsup_{ε \to 0} \frac{\log N(X, ε)}{− \log ε}
= d_f(X),
\]
Solutions to Exercises

giving equality between box-counting dimension and fractal dimension in $\mathbb{R}^m$.

13.2 If $\epsilon_{n+1} \leq \epsilon < \epsilon_n$ then we have

$$\frac{\log N(X, \epsilon)}{-\log \epsilon} \leq \frac{\log N(X, \epsilon_{n+1})}{-\log \epsilon_n} \leq \frac{\log N(X, \epsilon_{n+1})}{-\log \epsilon_{n+1} + \log(\epsilon_{n+1}/\epsilon_n)} \leq \frac{\log N(X, \epsilon_{n+1})}{-\log \epsilon_{n+1} + \log \alpha},$$

and so

$$\limsup_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{-\log \epsilon} \leq \limsup_{n \to \infty} \frac{\log N(X, \epsilon_n)}{-\log \epsilon_n}.$$

That this inequality holds in the opposite sense is straightforward, and hence we obtain the desired equality.

13.3 The sequence $\epsilon_m = (\sqrt{2} \log m)^{-1}$, $m \geq 2$, satisfies

$$\frac{\epsilon_{m+1}}{\epsilon_m} = \frac{\log m}{\log (m+1)} \geq \frac{\log 2}{\log 3},$$

and so we can use the result of the previous exercise. Note that we have

$$\left| \frac{e_n}{\log n} - \frac{e_k}{\log k} \right|^2 = \frac{1}{(\log n)^2} + \frac{1}{(\log k)^2} \leq \frac{2}{(\log n)^2}$$

for $n > k$, and so the first $m - 1$ elements from $H_{\log}$ will belong to distinct balls of radius $\epsilon_m$. It follows that

$$N(H_{\log}) \geq m - 1,$$

and so

$$d_f(H_{\log}) \geq \limsup_{m \to \infty} \frac{\log N(H_{\log}, \epsilon_m)}{\log \epsilon_m} \geq \limsup_{m \to \infty} \frac{\log(m-1)}{\log(\sqrt{2} \log m)} = \infty,$$

which implies that $d_f(H_{\log}) = \infty$, as claimed.

13.4 At the $j$th stage of construction the middle-$\alpha$ set $C_\alpha$ consists of $2^j$ intervals of length $\beta^j$, where $\beta = (1 - \alpha)/2$. It follows that

$$N(C_\alpha, \beta^j) = 2^j.$$
Chapter 13

Therefore, using the result of Exercise 13.2 we can calculate
\[ df(C_{\alpha}) = \lim_{j \to \infty} \frac{\log 2^j}{\log \beta^j} = \frac{\log 2}{\log \beta}. \]

13.5 Clearly
\[ \mu\left( \bigcup_{k=1}^{\infty} X_k, d, \epsilon \right) \leq \sum_{k=1}^{\infty} \mu(X_k, d, \epsilon). \]

Since \( \mu(X_k, d, \epsilon) \) is nondecreasing in \( \epsilon \) we have
\[ \mu(X_k, d, \epsilon) \leq \mathcal{H}^d(X_k) \]
for each \( k \), and so for every \( \epsilon > 0 \) we have
\[ \mu\left( \bigcup_{k=1}^{\infty} X_k, d, \epsilon \right) \leq \sum_{k=1}^{\infty} \mathcal{H}^d(X_k). \]

We can now take the limit as \( \epsilon \to 0 \) on the left-hand side to obtain
\[ \mathcal{H}^d\left( \bigcup_{k=1}^{\infty} X_k \right) \leq \sum_{k=1}^{\infty} \mathcal{H}^d(X_k) \]
as claimed.

13.6 The map \( L \) taking \( e^{(i)} \) into \( v^{(i)} \) (\( 1 \leq i \leq n \)) is given by
\[ L = \sum_{k=1}^{n} v^{(k)}(e^{(k)})^T, \]
and since \( e^{(k)}_i = \delta_{ik} \), the components of \( L \) are \( L_{ij} = v^{(j)}_i \):
\[ (L^T L)_{ij} = v^{(i)}_k v^{(j)}_k = v^{(i)} \cdot v^{(j)} = M_{ij}. \]

13.7 \( M \) is real and symmetric since
\[ M_{ij} = \delta X^{(i)} \cdot \delta X^{(j)}. \]

It follows that its eigenvalues \( \lambda_j \) are real, and one can find an orthonormal set of eigenvectors \( e^{(k)} \) with
\[ Me^{(k)} = \lambda_k e^{(k)}. \]
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To show that \( \lambda_k > 0 \), consider

\[
\lambda_k = e^{(k)T} M e^{(k)} = e_i^{(k)} \delta x_s^{(i)} e_j^{(k)} = |v^{(k)}|^2 \geq 0,
\]

where \( v^{(k)} \) is the vector given by its components

\[
v_s^{(k)} = e_i^{(k)} \delta x_s^{(i)}.
\]

If \( v^{(k)} = 0 \) then the two different initial conditions

\[
\delta x(0) = 0 \quad \text{and} \quad \delta x(0) = n \sum_{i=1}^n e_i^{(k)} \delta x^{(i)}
\]

have the same solution at time \( t \), contradicting uniqueness. So all the eigenvalues are strictly positive.

13.8 Writing \( M \) as

\[
M = \sum_{j=1}^n \lambda_j e_j e_j^T,
\]

we have

\[
\log M = \sum_{j=1}^n \log \lambda_j e_j e_j^T.
\]

Clearly

\[
\text{Tr}[\log M] = \sum_{j=1}^n \log \lambda_j, \quad (S13.1)
\]

and since

\[
\det M = \prod_{j=1}^n \lambda_j
\]

the required result follows immediately.

Since \( \text{Tr}[\log M] \) is given by (S13.1), we have

\[
\frac{d}{dt} \text{Tr}[\log M] = \sum_{i=1}^n \frac{\dot{\lambda}_i}{\lambda_i}.
\]
Chapter 13

The right-hand side of (13.34) is

\[
\sum_{i=1}^{n} \left( e_i, M^{-1} \frac{dM}{dt} e_i \right) = \sum_{i=1}^{n} \left( e_i, \sum_{j=1}^{n} \lambda_j^{-1} \dot{e}_j e_j^T + \sum_{k=1}^{n} \lambda_k \dot{e}_k e_k^T + \lambda_k \dot{e}_k e_k^T e_i \right) = \sum_{i=1}^{n} \left( \lambda_i^{-1} \dot{e}_i + \sum_{k=1}^{n} \lambda_k \dot{e}_k e_k^T \right) e_i = \sum_{i=1}^{n} \left[ \lambda_i^{-1} \dot{e}_i + \dot{e}_i e_i \right] = \sum_{i=1}^{n} \lambda_i^{-1} \dot{\lambda}_i,
\]

since \( \frac{d}{dt}(e_i, e_i) = 0 \).

13.9 Since the eigenvalues are proportional to the sums of squares of \( m \) integers, we will have reached the eigenvalue \( mk^2 \) once we have taken \( km \) combinations of integers. Thus

\[ \lambda_{km} = Cmk^2, \]

and so if \( k^m < n < (k + 1)^m \) we obtain

\[ Cmk^2 \leq \lambda_n \leq Cm(k + 1)^m. \]

We now have

\[ k < n^{1/m} < (k + 1) \]

and so

\[ \frac{1}{2} n^{1/m} < k < k + 1 < 2n^{1/m}. \]

This gives

\[ cn^{2/m} \leq \lambda_n \leq Cn^{2/m}, \]

as required.
Solutions to Exercises

13.10 Taking the inner product of (13.27) with \( U \) we obtain

\[
\frac{1}{2} \frac{d}{dt} |U|^2 + v \|U\|^2 = -b(U, u, U),
\]

and so

\[
\frac{1}{2} \frac{d}{dt} |U|^2 + v \|U\|^2 \leq k |U||\|U\||u|.
\]

Using Young's inequality and rearranging we get

\[
\frac{d}{dt} |U|^2 + v \|U\|^2 \leq C |U|^2. \tag{S13.2}
\]

That bounded sets in \( L^2 \) are mapped into bounded sets in \( L^2 \) follows by neglecting the term in \( \|U\|^2 \) and applying Gronwall's inequality (Lemma 2.8),

\[
|U(t)|^2 \leq e^{Ct} |U(0)|^2 = e^{Ct}|\xi|^2. \tag{S13.3}
\]

To show that we in fact obtain a bounded set in \( H^1 \), we first return to (S13.2) and integrate between \( t/2 \) and \( t \) to obtain

\[
v \int_{t/2}^t \|U(s)\|^2 \, ds \leq C \int_{t/2}^t |U(s)|^2 \, ds + |U(t/2)|^2 \leq C(t)|U(t/2)|^2, \tag{S13.4}
\]

using (S13.3). Now we take the inner product of (13.27) with \( AU \), which gives

\[
\frac{1}{2} \frac{d}{dt} \|U\|^2 + v \|AU\|^2 = -b(u, U, AU) - b(U, u, AU).
\]

Using (9.27) we obtain

\[
\frac{1}{2} \frac{d}{dt} \|U\|^2 + v \|AU\|^2 \leq k \left( |u|^{1/2} \|u\|^{1/2} \|U\|^{1/2} \|AU\|^{3/2} + |U|^{1/2} \|U\|^{1/2} \|u\|^{1/2} \|AU\|^{1/2} \right),
\]

and after using Young's inequality and rearranging we have

\[
\frac{d}{dt} \|U\|^2 + v \|AU\|^2 \leq C \|U\|^2.
\]

Expression (S13.4) allows us to use the “uniform Gronwall” trick and find a bound on \( \|U\| \) valid for all \( t > 0 \). Thus \( \Lambda(t; u_0) \) is compact for all \( t > 0 \).
Chapter 14

13.11 If we integrate (12.6) between 0 and \( T \) we obtain
\[
\nu \int_0^T |Au(s)|^2 \, ds \leq \frac{T|f|^2}{\nu} + \|u(0)\|^2.
\]
Dividing by \( T \) and taking the limit as \( T \to \infty \) yields
\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T |Au(s)|^2 \, ds \leq \frac{|f|^2}{\nu^2},
\]
since there is an absorbing set in \( V \). Therefore
\[
\chi \leq \frac{|f|^2}{L^2\nu} = \nu^3 L^{-6} G^2.
\]
The only length that can be formed from \( \chi \) and \( \nu \) is
\[
L_{\chi} = \left( \frac{\nu^3}{\chi} \right)^{1/6},
\]
and this implies (13.35).

Chapter 14

14.1 Since \( \mathcal{A} \) is compact, it is bounded and certainly contained in \( B(0, r) \) for some \( r > 0 \). So \( N_r(\mathcal{A}) = 1 \). We consider
\[
S(B(0, r) \cap \mathcal{A}),
\]
which by our assumption can be covered by \( K_0 \) balls, centred in \( \mathcal{A} \), and of radius \( r/2 \). So
\[
N(\mathcal{A}, r/2) = K_0.
\]
Now consider each one of the balls in this covering, and apply our assumption again to show that
\[
S(B(a_i, r/2) \cap \mathcal{A})
\]
can be covered by \( K_0 \) balls of radius \( r/4 \), so that
\[
N(\mathcal{A}, r/4) = K_0^2.
\]
Iterating this argument, we can see that
\[
N(\mathcal{A}, 2^{-k} r) = K_0^k.
\]
Solutions to Exercises

So therefore, using the result of Exercise 13.2, we have

\[ d_f(A) = \lim_{k \to \infty} \frac{\log(N(A, 2^{-kr}))}{\log(2^k)} \]

\[ \leq \frac{k \log K_0}{k \log 2} \]

\[ \leq n_0 \frac{\log \alpha}{\log 2}, \]

precisely (14.33).

14.2 We have, for any \( u \in D(A^{1/2}) \),

\[ \|u\|^2 = a(u, u) = (A^{1/2}u, A^{1/2}u) = |A^{1/2}u|. \]

Expanding \( p \) in terms of the eigenfunctions of \( A \) gives

\[ p = \sum_{j=1}^{n} (p, w_j)w_j, \]

and so

\[ \|p\|^2 = \sum_{j=1}^{n} \lambda_j |(p, w_j)|^2 \leq \lambda_n |p|^2. \]

Similarly,

\[ \|q\|^2 = \sum_{j=1}^{n} \lambda_j |(q, w_j)|^2 \geq \lambda_{n+1} |q|^2. \]

The other two inequalities in the exercise follow easily from these.

14.3 Differentiating \( \Phi \) gives

\[ \frac{d\Phi}{dt} = \exp(\lambda a / C(a+b)) \left[ \frac{da}{dt} \left( 1 - \frac{\lambda b}{C(a+b)} \right) + \frac{db}{dt} \left( 1 + \frac{\lambda a}{C(a+b)} \right) \right]. \]

Since we have (14.34), the coefficient of \( da/dt \) is negative, whereas the coefficient of \( db/dt \) is positive. It follows that we can substitute in the inequalities for \( da/dt \) and \( db/dt \), which gives \( d\Phi/dt \leq 0 \).

14.4 Write

\[ |u(x) - u(y)| \leq \sum_{k \in \mathbb{Z}^2} |e^{2\pi i k x / L} - e^{2\pi i k y / L}||c_k|, \]
and use (14.35) to deduce that
\[ |u(x) - u(y)| \leq C|x - y|^{1/2} \sum_{k \in \mathbb{Z}^2} |c_k||k|^{1/2} \]
\[ \leq C|x - y|^{1/2} \left( \sum_{k \in \mathbb{Z}^2} (1 + |k|^4)|c_k|^2 \right)^{1/2} \left( \sum_{k \in \mathbb{Z}^2} \frac{|k|}{(1 + |k|^4)} \right)^{1/2} \]
\[ \leq C\|u\|_{H^2}|x - y|^{1/2}. \]
\[ \sum_{k \in \mathbb{Z}^2} \frac{|k|}{(1 + |k|^4)} \text{ is finite.} \]

14.5 Since (6.14) shows that \( \|u\|_{H^2} = C|Au| \) for \( u \in D(A) \), we can use the result of the previous exercise to deduce that
\[ |u(x) - u(y)| \leq c|Au||x - y|^{1/2}. \]
Expression (14.36) follows immediately from this and the definitions of \( d(N) \) and \( \eta(u) \).

14.6 Choose \( \epsilon > 0 \). Then there exists a \( T \) such that \( b(t) \leq \epsilon/2 \) for all \( t \geq T \). Hence for \( t \geq T \),
\[ \frac{dX}{dt} + aX \leq \epsilon/2. \]
By Gronwall’s inequality (Lemma 2.8),
\[ X(T + t) \leq X(T)e^{-at} + \epsilon/2, \]
and so choosing \( \tau \) large enough that
\[ ke^{-at} < \epsilon/2, \]
we have
\[ X(t) \leq \epsilon \quad \text{for all} \quad t \geq T + \tau, \]
so that \( X(t) \to 0 \).

14.7 Using the bound on \( b \) given in (9.25), we can write
\[ \frac{1}{2} \frac{d}{dt} \|w\|^2 + v|Aw|^2 \leq \|w\|_{\infty}\|w\||Au| \]
\[ \leq [\eta(w) + cd(N)^{1/2}|Aw|]\|w\||Au| \]
\[ \leq \eta(w)\|w\||Au| + cd(N)^{1/2}1^{-1/2}|Aw|^2|Au|, \]
Solutions to Exercises

using (14.36), and therefore

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2 + \left[ v - c \lambda_1^{-1/2} d(N)^{1/2} |Au| \right] \lambda_1 \|w\|^2 \leq \eta(w) \|w\| |Au|.
\]

Now, we know that \( A \) is bounded in \( V \) and \( D(A) \), so that

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2 + [v - c \lambda_1^{-1/2} \rho_A d(N)^{1/2}] \lambda_1 \|w\|^2 \leq 2 \rho_V \rho_A \eta(w).
\]

Now, choose \( \delta \) such that

\[
\mu = v - c \lambda_1^{1/2} \rho_A \delta^{1/2} > 0.
\]

Then we have, for \( d(N) < \delta \),

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2 + \mu \|w\|^2 \leq 2 \rho_V \rho_A \eta(w).
\]

By assumption, we know that \( \eta(w) \to 0 \), and since the attractor is bounded in \( V \) we have \( \|w(t)\|^2 \leq 4 \rho_V^2 \). The result of the previous exercise applied to (S14.1) now shows (14.37).

14.8 (i) Take the inner product of (14.38) with \( q_n = Q_n u \) to obtain

\[
\frac{1}{2} \frac{d}{dt} |q_n|^2 + (Au, q_n) = (F(u), q_n).
\]

Now, notice that

\[
|(Au, q_n)| = |(Aq_n, q_n)| \geq \lambda_{n+1} |q_n|^2,
\]

and so

\[
\frac{1}{2} \frac{d}{dt} |q_n|^2 + \lambda_{n+1} |q_n|^2 \leq C_0 |q_n|,
\]

from which, using the result of Exercise 2.5, we see that

\[
\frac{d}{dt} |q_n| \leq -\lambda_{n+1} |q_n| + C_0,
\]

which gives

\[
|Q_n u(t)| \leq \frac{C_0}{\lambda_{n+1}} + |Q_n u(0)|,
\]

using the Gronwall lemma (Lemma 2.8).
Chapter 15

(ii) Writing \( p(t) = P_n u(t) \) and \( q(t) = Q_n u(t) \), \( p \) solves the equation

\[
\frac{dp}{dt} + Ap = P_n F(p + q).
\]

Thus the equation for \( w = p - p_n \) is

\[
\frac{dw}{dt} + Aw = P_n F(p_n) - P_n F(p + q).
\]

Taking the inner product with \( w \) and using the Lipschitz property of \( F \) gives

\[
\frac{1}{2} \frac{d}{dt} |w|^2 + \|w\|^2 \leq C_1 |w|^2 + C_1 |q||w|.
\]

Hence

\[
\frac{d}{dt} |w| \leq C_1 |w| + C_1 |q|.
\]

and so, using the bound in (S14.2) and the Gronwall lemma as above we obtain

\[
|P_n u(t) - p_n(t)| \leq C_1^{-1} \left[ \frac{C_0}{\lambda_{n+1}} + |Q_n u(0)| \right] e^{C_1 t}.
\]

Combining this with (S14.2) yields

\[
|u(t) - p_n(t)| \leq C_1^{-1} \left[ \frac{C_0}{\lambda_{n+1}} + |Q_n u(0)| \right] (C_1 + e^{C_1 t}),
\]

and since we know that \( \lambda_{n+1} \to \infty \) and \( |Q_n u(0)| \to 0 \) as \( n \to \infty \), it follows that \( p_n(t) \) converges to \( u(t) \) as claimed.

Chapter 15

15.1 For any point \( v \in H \),

\[
\text{dist}(v, \mathcal{M})^2 = \inf_{p \in PH} (|P v - p|^2 + |Q v - \phi(p)|^2)
\]

and

\[
|Q v - \phi(P v)|^2 = |Q v - \phi(p) + \phi(p) - \phi(P v)|^2
\leq 2|Q v - \phi(p)|^2 + 2|\phi(p) - \phi(P v)|^2
\leq 2|Q v - \phi(p)|^2 + 2l^2|P v - p|^2
\leq c^2 (|Q v - \phi(p)|^2 + |P v - p|^2)
\]
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for all \( p \in PH \), where \( c^2 = 2 \max(l_2, 1) \). Therefore

\[ |Qv - \phi(Pv)| \leq c \text{dist}(v, M). \]

The other implication is obvious.

15.2 Using Proposition 15.3 we see that the attractor lies in the graph of some Lipschitz function \( \Phi : P_nH \to Q_nH \). We can therefore project the dynamics on \( A \) onto \( P_nH \) by writing

\[ dp/dt + Ap = P_n F(p + \Phi(p)). \quad (S15.1) \]

It is easy to show that (S15.1) is a Lipschitz ODE on \( P_nH \), since

\[
|P_n F(p + \Phi(p)) - P_n F(\overline{p} + \Phi(\overline{p}))| \leq |F(p + \Phi(p)) - F(\overline{p} + \Phi(\overline{p}))| \\
\leq C |p + \Phi(p) - \overline{p} - \Phi(\overline{p})| \\
\leq C (|p - \overline{p}| + |\Phi(p) - \Phi(\overline{p})|) \\
\leq 2C |p - \overline{p}|.
\]

We know that if \( u(t) \) is a solution in \( A \) then \( p(t) = P_n u(t) \) is a solution of (S15.1) lying in \( P_nA \). Since (S15.1) is Lipschitz its solutions are unique, and so in particular \( P_nA \) is an invariant set. Thus (S15.1) is a finite-dimensional system that reproduces the dynamics on \( A \). [The advantage of the inertial form over (S15.1) is that \( P_nA \) is the attractor of the finite-dimensional system, not just an invariant set.]

15.3 Since \( F = 0 \) outside \( B(0, \rho) \),

\[ \Sigma_{t_0} \subset S(t_0)\{u : u \in P_nH : \rho \leq |u| \leq \rho e^{\lambda_n t_0}\}. \]

The cone invariance part of the strong squeezing property then shows that for any two points \( u_1 \) and \( u_2 \) in \( \Sigma_{t_0} \), we must have

\[ |Q_n(u_1 - u_2)| \leq |P_n(u_1 - u_2)|. \]

If we write

\[ \Sigma = \bigcup_{0 \leq t < \infty} S(t) \Gamma \]

then the function \( \Phi \) defined by

\[ \Phi(P_n u) = Q_n u \quad \text{for all} \quad u \in \Sigma \]
is Lipschitz on its domain of definition, \( P_n \Sigma = P_n B(0, \rho) \). Clearly \( \Sigma \) is positively invariant, and so \( M \) is invariant.

To show that \( A \subset M \), suppose that \( u \in A \) and \( v \in M \) with \( P_n u = P_n v \) but \( Q_n u \neq Q_n v \). Then, using the invariance of \( \Sigma \) and \( A \), we have \( u = S(t)u_t \) with \( u_t \in M \), and \( v = S(t)v_t \) with \( v_t \in A \). Thus

\[
|Q_n(u - v)| \leq |Q_n(u_t - v_t)| e^{-kt} \\
\leq 2\rho e^{-kt},
\]

(S15.2)

since both \( A \) and \( \Sigma \) are subsets of \( B(0, \rho) \). Since (S15.2) holds for all \( t \geq 0 \), we must have \( Q_n u = Q_n v \). Thus \( u = v \) and \( A \subset M \) as claimed.

15.4 We have

\[
135 = 1^2 + 2^2 + 3^2 + 11^2, \\
136 = 6^2 + 10^2, \\
137 = 4^2 + 11^2, \\
138 = 1^2 + 3^2 + 8^2 + 8^2,
\]

all as sums of (at most) four squares.

15.5 (i) If \( u(t) \) is a solution of (15.24), then \( p(t) = P_n u(t) \) is the solution of the equation

\[
dp/dt + Ap = P_n F(p(t) + q(t)).
\]

Since \( F \) is Lipschitz, it follows that

\[
|P_n F(p(t) + q(t)) - P_n F(p(t) + \Phi(p(t)))| \leq C_1|q(t) - \Phi(p(t))| \\
\leq C_1Ce^{-kt},
\]

where the result of Exercise 15.1 has been used.

(ii) Let \( \overline{u}(t) = \overline{p}(t) + \Phi(\overline{p}(t)) \). Then \( \overline{u}(t) \in M \), so we just have to show the exponential convergence in (15.26). To do this, we write

\[
|u(t) - \overline{u}(t)| \leq |p(t) + q(t) - p(t) - \Phi(p(t))| \\
+ |p(t) + \Phi(p(t)) - \overline{p}(t) - \Phi(\overline{p}(t))| \\
\leq |q(t) - \Phi(p(t))| + 2|p(t) - \overline{p}(t)| \\
\leq Ce^{-kt} + 2De^{-kt} = Me^{-kt},
\]

where we have used the result of Exercise 15.1 again and the Lipschitz property of \( \Phi \).
Chapter 16

16.1 \( \omega(r) \) is clearly well defined, since the set

\[ \{(x, y) \in X \times X : |x - y| \leq r\} \]

is a compact subset of \( X \times X \). The convexity property follows easily, since

\[
\omega(r + s) = \sup_{|x - z| \leq r + s} |f(x) - f(z)| \\
\leq \sup_{|x - y| \leq r, |y - z| \leq s} |f(x) - f(y)| + |f(y) - f(z)| \\
\leq \sup_{|x - y| \leq r} |f(x) - f(y)| + \sup_{|y - z| \leq s} |f(y) - f(z)| \\
= \omega(r) + \omega(s),
\]

where to prevent too clumsy notation we have assumed throughout that \( x, y, z \in X \).

16.2 (i) \( X \) can be covered by \( N(X, \epsilon) \) balls of radius \( \epsilon \) and, in particular, lies within \( \epsilon \) of the space spanned by the centres of these balls. Therefore \( d(X, \epsilon) \leq N(X, \epsilon) \), and the inequality follows.

(ii) Simply choose any open subset \( O \) in \( \mathbb{R}^n \). Then \( d_f(O) = n \) but since \( O \subset \mathbb{R}^n \) we must have \( \tau(O) = 0 \).

16.3 Consider the projection \( P_n \) onto the space spanned by the first \( n \) eigenfunctions of \( A \),

\[
P_n u = \sum_{j=1}^{n} (u, w_j) w_j,
\]

and its orthogonal complement \( Q_n = I - P_n \). Then

\[
|u - P_n u| = |Q_n u| \\
= |Q_n A^{s/2} A^{-s/2} u| \\
\leq \|Q_n A^{s/2} \|_{\text{op}} |A^{-s/2} u| \\
\leq \lambda_{n+1}^{-s/2} \|u\|_{H^s} \\
\leq C n^{-2s/m}
\]

for some constant \( C \). Clearly,

\[
\log d(X, \epsilon) \leq \frac{\log \epsilon}{2s/m} + \frac{\log C}{2s/m},
\]
and so one obtain (16.23). If $X$ is bounded in $D(A')$ for any $r$ then it follows from (16.23) that $r(X) = 0$, and so one can obtain any $\theta$ in the range

$$0 < \theta < 1 - \frac{2d_f(X)}{k}.$$  

We can now obtain any $\theta < 1$ by choosing $k$ large enough.

16.4 Write $w = u - v$ for $u, v \in \mathcal{A}$. If $A$ is Lipschitz continuous from $\mathcal{A}$ into $H$ then

$$|Aw| \leq L|w|$$

for some $L$. Now split $w = P_n w + Q_n w$, and observe that we have both

$$|Aw|^2 = |A(P_n w + Q_n w)|^2 = |A(P_n w)|^2 + |A(Q_n w)|^2 \geq \lambda_{n+1}^2 |Q_n w|^2$$

and

$$|Aw|^2 \leq L^2 |w|^2 \leq L^2 |P_n w|^2 + L^2 |Q_n w|^2.$$  

Since $\lambda_n \to \infty$ as $n \to \infty$, we can choose $n$ large enough that $\lambda_{n+1} > L$, and then write

$$\lambda_{n+1}^2 - L^2 |Q_n w|^2 \leq L^2 |P_n w|^2,$$

that is,

$$|Q_n w| \leq \left( \frac{L^2}{\lambda_{n+1}^2 - L^2} \right)^{1/2} |P_n w|.$$  

It follows that we can define $\Phi(P_n u) = Q_n u$ uniquely for each $u \in \mathcal{A}$, and then

$$|\Phi(p_1) - \Phi(p_2)| \leq \left( \frac{L^2}{\lambda_{n+1}^2 - L^2} \right)^{1/2} |p_1 - p_2|,$$

so that (cf. Proposition 15.3) the attractor is a subset of a Lipschitz graph over $P_n H$.

16.5 Since $X$ is the attractor for $\dot{x} = g(x)$, given $\epsilon > 0$, there exists a $\delta > 0$ such that if $x(0) \in N(X, \delta)$ then the solution $x(t)$ of $\dot{x} = g(x)$ remains within $N(X, \epsilon)$ for all $t \geq 0$. 

Solutions to Exercises

Define $\tilde{f}(x)$ on a closed subset of $\mathbb{R}^n$ by

$$f(x) = \begin{cases} 
  f(x), & \text{dist}(x, X) \leq \delta/4, \\
  0, & \text{dist}(x, X) \geq \delta/2.
\end{cases}$$

Since $\tilde{f}$ is Lipschitz on its domain of definition, it can be extended using Theorem 16.4 to a function $F(x)$ that is Lipschitz on $\mathbb{R}^n$. Now consider

$$\dot{x} = F(x) + g(x). \quad (S16.1)$$

Clearly $X$ is an invariant subset for (S16.1), since $F(x) + g(x) = F(x)$ on $X$. To show that the attractor of (S16.1) lies within an $N(X, \epsilon)_{\mathbb{R}^n}$ it suffices to show that $N(X, \epsilon)$ is absorbing. This follows from the choice of $\delta$ and the fact that $F(x) + g(x) = g(x)$ outside $N(X, \delta/2)$.

Chapter 17

17.1 Integrating (17.3) between 0 and $L$ and using the periodic boundary conditions gives

$$\int_0^L |Du|^2 = - \int_0^L uD^2u \, dx,$$

which implies (17.4) after an application of the Cauchy–Schwarz inequality. For $u \in \dot{H}_p^2$ the result follows by finding a sequence $\{u_n\} \in \dot{C}_p^2$ that converges to $u$ in the norm of $H_p^2$.

17.2 Multiplying (17.5) by a function $\phi$ in $\dot{C}_p^2$ and integrating by parts twice gives

$$\int_0^L (D^2u)(D^2\phi) \, dx = \int_0^L f(x)\phi(x) \, dx. \quad (S17.1)$$

Define a bilinear form $a(u, v) : \dot{H}_p^2 \times \dot{H}_p^2 \to \mathbb{R}$ by

$$a(u, v) = \int_0^L (D^2u)(D^2v) \, dx,$$

and then, using the density of $\dot{C}_p^2$ in $\dot{H}_p^2$, we see that (S17.1) becomes (17.6).
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17.3 Since \( a(u, v) \) is equivalent to the inner product on \( H^2_p \) (by the general Poincaré inequality from Exercise 5.4), we can use the Riesz representation theorem to deduce the existence of a unique solution \( u \in H^2_p \) of (17.6) for any \( f \in H^{-2} \).

In particular if \( f \in L^2 \) then \( u \in H^2_p \), which is a compact subset of \( L^2 \), using the Rellich–Kondrachov compactness theorem (Theorem 5.32). It follows that the inverse of \( A \) is compact, and \( A \) itself is clearly symmetric. We can therefore apply Corollary 3.26 to deduce that \( A \) has an orthonormal set of eigenfunctions \( \{ w_j \} \) that form a basis for \( L^2 \).

17.4 The orthogonality property (17.8) follows easily, since for \( u \in C^2_p \),

\[
b(u, u, u) = \int_0^L u(x)^3 \frac{du}{dx} dx = \frac{1}{4} \int_0^L \frac{d}{dx} u(x)^3 \, dx = 0,
\]

using the periodic boundary conditions. The result follows for all \( u \in H^2_p \) by taking limits. Similarly for the cyclic equality, after an integration by parts, we have

\[
\int_0^L uv_x w \, dx = - \int_0^L (uv)_x w \, dx = - \int_0^L vw_x u + w u_x v \, dx.
\]

The inequalities in (17.10) follow from the estimate

\[
\int uvw \, dx \leq \|u\|_\infty \|v\| \|w\| \leq \|Du\| \|v\| \|w\|,
\]

since \( H^1 \subset C^0 \) on a one-dimensional domain (Theorem 5.31).

17.5 Taking the inner product of (17.12) with \( u_n \) gives

\[
\frac{1}{2} \frac{d}{dt} |u_n|^2 + a(u_n, u_n) + (D^2 u_n, u_n) + (P_n B(u_n, u_n), u_n) = 0.
\]

Since

\[
(P_n B(u_n, u_n), u_n) = (B(u_n, u_n), P_n u_n) = (B(u_n, u_n), u_n) = 0
\]

by (17.8), we obtain

\[
\frac{1}{2} \frac{d}{dt} |u_n|^2 + |D^2 u_n|^2 = |Du_n|^2.
\]

Using (17.4) we have

\[
\frac{1}{2} \frac{d}{dt} |u_n|^2 + |D^2 u_n|^2 \leq |u_n| |D^2 u_n| \leq \frac{1}{2} |u_n|^2 + \frac{1}{2} |D^2 u_n|^2.
\]
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and so
\[ \frac{d}{dt} |u_n|^2 + |D^2 u_n|^2 \leq |u_n|^2. \] (S17.2)

Dropping the term in $|D^2 u_n|^2$ and integrating we get
\[ |u_n(t)|^2 \leq e^t |u_n(0)|^2, \]
so clearly
\[ u_n \] is uniformly bounded in $L^\infty(0, T; \dot{L}^2)$.

Integrating (S17.2) as it stands then gives
\[ |u_n(t)|^2 + \int_0^t |D^2 u_n(s)|^2 \, ds \leq \int_0^t |u_n(s)|^2 \, ds + |u_n(0)| \]
and in particular shows that
\[ u_n \] is uniformly bounded in $L^2(0, T; \dot{H}_p^2)$.

It follows from these estimates, the equality
\[ du_n/dt = -Au - D^2 u - B(u, u), \]
and Poincaré’s inequality (17.2) that
\[ du_n/dt \] is uniformly bounded in $L^2(0, T; H^{-2})$,
and we have obtained the bounds in (17.13).

Extracting subsequences from the $\{u_n\}$ and relabelling as necessary we find a $u$ such that
\[ u \in L^2(0, T; \dot{H}_p^2) \cap L^\infty(0, T; \dot{L}^2) \quad \text{with} \quad du/dt \in L^2(0, T; H^{-2}), \]
and
\[ u_n \rightharpoonup u \quad \text{in} \quad L^2(0, T; \dot{H}_p^2), \]
\[ u_n \rightharpoonup u \quad \text{in} \quad L^\infty(0, T; \dot{L}^2), \]
\[ du_n/dt \rightharpoonup du/dt \quad \text{in} \quad L^2(0, T; H^{-2}). \]

We can also use the compactness theorem (Theorem 8.1) to find a subsequence with the additional strong convergence
\[ u_n \rightarrow u \quad \text{in} \quad L^2(0, T; \dot{H}_p^1). \]
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since $H^2_p \subset \dot{H}^1_p \subset H^{-2}$. It is simple to show the weak-* convergence in $L^2(0, T; H^{-2})$ of all the terms in the equation, except for the non-linear term. For this we need the strong convergence in $L^2(0, T; \dot{H}^1_p)$ and the uniform bound on $u_n$ in $L^\infty(0, T; \dot{L}^2)$. We need to show that

$$\int_0^T b(u_n, u_n, v) \, dt \to \int_0^T b(u, u, v) \, dt \quad \text{for all} \quad v \in L^2(0, T; H^2_p).$$

Using (17.9) we write

$$b(u_n, u_n, v) - b(u, u, v) = b(u_n - u, u_n, v) + b(u, u_n - u, v)$$

$$= -b(u_n, v, u_n - u) - b(v, u_n - u, u_n) + b(u, u_n - u, v),$$

and then for the first term

$$\int_0^T |b(u_n, v, u_n - u)| \, dt \leq k \int_0^T |u_n||D^2v||u_n - u| \, dt$$

$$\leq k \|u_n\|_{L^\infty(0, T; L^2)} \|v\|_{L^2(0, T; H^2_p)} \|u_n - u\|_{L^2(0, T; L^2)}$$

$$\to 0,$$

and for the second and third terms

$$\int_0^T |b(v, u_n - u, u_n)| \, dt \leq k \int_0^T |v||D(u_n - u)||u_n| \, dt$$

$$\leq k \|u_n\|_{L^\infty(0, T; L^2)} \|v\|_{L^2(0, T; H^2_p)} \|u_n - u\|_{L^2(0, T; H^2_p)}$$

$$\to 0,$$

giving the required convergence. That

$${P_n B(u_n, u_n)} \xrightarrow{\star} B(u, u)$$

follows as in Exercise 9.5.

Finally, the continuity of $u$ into $\dot{L}^2$ follows from the generalisation of Theorem 7.2 discussed after its formal statement in Chapter 7.

17.7 The equation for the difference $w$ of two solutions, $w = u - v$, is

$$w_t + w_{xxxx} + w_{xx} + uw_x + vw_x = 0.$$

Taking the inner product with $w$ we obtain

$$\frac{1}{2} \frac{d}{dt} |w|^2 + |D^2w|^2 - |Dw|^2 = -b(w, u, w) - b(v, w, w).$$
Estimating the terms on the right-hand side by using (17.10) we have
\[
\frac{1}{2} \frac{d}{dt} |w|^2 + |D^2 w|^2 \leq |Dw|^2 + |D^2 u||w|^2 + |v||Dw|^2.
\]
Using (17.4) and Young’s inequality gives
\[
\frac{1}{2} \frac{d}{dt} |w|^2 + |D^2 w|^2 \leq (1 + |v|)|w||D^2 w| + |D^2 u||w|^2 \leq \frac{1}{2}|D^2 w|^2 + C(1 + |D^2 u| + |v|^2)|w|^2,
\]
and so
\[
\frac{d}{dt} |w|^2 + |D^2 w|^2 \leq C(1 + |D^2 u| + |v|^2)|w|^2. \quad \text{(S17.3)}
\]
Neglecting the term in $|D^2 w|^2$ and integrating from 0 to $t$ shows (17.15). Since $u, v \in L^2(0, T; H^2_p)$, it follows that $w(t) = 0$ for all $t$ if $w(0) = 0$, which gives uniqueness.

### 17.8 Choosing $\alpha = 6$

We have
\[
\frac{d}{dt} |v|^2 + \frac{1}{2}|D^2 v|^2 + 2|v|^2 \leq \frac{1}{2}|g|^2, \quad \text{(S17.4)}
\]
and so in particular
\[
\frac{d}{dt} |v|^2 \leq -2|v|^2 + \frac{1}{2}|g|^2.
\]

The Gronwall inequality (Lemma 2.8) now shows that
\[
|v(t)|^2 \leq |v(0)|^2 e^{-2t} + \frac{1}{4}|g|^2 (1 - e^{-2t}). \quad \text{(S17.5)}
\]
Since $u = \phi + v$ and $\phi \in \hat{C}_p^\infty$ is constant, it follows that there is an absorbing set for $u(t)$ in $L^2$.

We can also obtain from (S17.4) a bound on the integral of $|D^2 v|^2$,
\[
\frac{1}{2} \int_t^{t+1} |D^2 v(s)|^2 \, ds \leq \frac{1}{2}|g|^2 + |v(t)|^2,
\]
or for $|D^2 u|^2$ the bound
\[
\int_t^{t+1} |D^2 u(s)|^2 \, ds \leq |g|^2 + |D^2 \phi|^2 + |v(t)|^2.
\]

It follows from (S17.5) that if $t$ is large enough then
\[
\int_t^{t+1} |D^2 u(s)|^2 \, ds \leq M, \quad \text{(S17.6)}
\]
and we have both bounds in (17.18).
Taking the inner product of (17.11) with $-D^2u$ we obtain
\[ \frac{1}{2} \frac{d}{dt} |Du|^2 + |D^3u|^2 = |D^2u|^2 + b(u, u, D^2u). \]

We now estimate the right-hand side by using (17.10),
\[ \frac{1}{2} \frac{d}{dt} |Du|^2 + |D^3u|^2 \leq |D^2u|^2 + |D^2u||Du|^2. \]

Neglecting the term in $|D^3u|^2$ we have
\[ \frac{d}{dt} |Du|^2 \leq |D^2u|^2 + |D^2u||Du|^2. \]

Note that this is in the form in which the uniform Gronwall lemma of Exercise 11.2 is applicable, since we have a uniform estimate on the integral of $|D^2u|$ provided in (S17.6) above. It follows that there is an absorbing set in $H^1_0$.

We have therefore obtained a compact absorbing set in $L^2$ and proved the existence of a global attractor.

As in the proof of Theorem 13.20, we consider the equation for $\theta = u - v - U$,
\[ \theta_t + \theta_{xxxx} + \theta_{xx} + \theta u_x + w w_x = 0, \]
where $w = u - v$. Taking the inner product with $\theta$ yields
\[ \frac{1}{2} \frac{d}{dt} |\theta|^2 + |D^2\theta|^2 = |D\theta|^2 - b(\theta, u, \theta) - b(w, w, \theta). \]

Using (17.4) and (17.10) on the right-hand side we obtain
\[ \frac{1}{2} \frac{d}{dt} |\theta|^2 + |D^2\theta|^2 \leq |\theta| |D^2\theta| + |\theta|^2 |D^2u| + |Dw|^2 |\theta| \]
\[ \leq \frac{1}{2} |\theta|^2 + \frac{1}{2} |D^2\theta|^2 + |D^2u||\theta|^2 + \frac{1}{2} |Dw|^4 + \frac{1}{2} |\theta|^2, \]
and so
\[ \frac{d}{dt} |\theta|^2 + |D^2\theta|^2 \leq 2(1 + |D^2u||\theta|^2 + |Dw|^4). \]

It follows from Gronwall’s inequality (Lemma 2.8), since $\theta(0) = 0$, that
\[ |\theta(t)|^2 \leq k(t) \int_0^t |Dw(s)|^4 ds, \]
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and so, using (17.4), we get

$$|\theta(t)|^2 \leq k \int_0^t |w(s)|^2 |D^2 w(s)|^2 \, ds.$$  

Returning to (S17.3),

$$\frac{d}{dt} |w|^2 + |D^2 w|^2 \leq C(1 + |D^2 u| + |v|^2)|w|^2,$$

multiplying both sides by $|w|^2$, and integrating we obtain

$$\int_0^t |w(s)|^2 |D^2 w(s)|^2 \, ds \leq C \int_0^t |w(s)|^4 \, ds + \frac{1}{4} |w(0)|^4.$$  

Using (17.15) we have

$$\int_0^t |w(s)|^2 |D^2 w(s)|^2 \, ds \leq C(t)|w(0)|^4,$$

and hence

$$|\theta(t)|^2 \leq K(t)|w(0)|^4.$$  

The uniform differentiability property now follows.

17.11 To show that $\Lambda(t; u_0)$ is compact take the inner product of (17.19) with $U$ to obtain

$$\frac{1}{2} \frac{d}{dt} |U|^2 + |D^2 U|^2 - |DU|^2 + b(U, u, U) + b(u, U, U) = 0.$$  

Using the cyclic property (17.9) and the bound in (17.10) we have

$$\frac{1}{2} \frac{d}{dt} |U|^2 + |D^2 U|^2 \leq C |DU|^2.$$  

Using (17.4) and Young’s inequality we end up with

$$\frac{d}{dt} |U|^2 + |D^2 U|^2 \leq C |U|^2. \quad (S17.7)$$  

Dropping the term in $|D^2 U|^2$ shows that

$$|U(t)|^2 \leq e^{C_1} |\xi|^2, \quad (S17.8)$$

and integrating between $t/2$ and $t$ shows that (cf. Exercise 13.10)

$$\int_{t/2}^t |D^2 U(s)|^2 \, ds \leq C(t)|U(t/2)|^2. \quad (S17.9)$$
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Now take the inner product of (17.19) with $-D^2U$ and obtain

$$\frac{1}{2} \frac{d}{dt} |DU|^2 + |D^3U|^2 = |D^2U|^2 + b(U, u, D^2U) + b(u, U, D^2U)$$

$$\leq |D^2U|^2 + |DU||Du||D^2U| + |u||D^2U|^2,$$

by using (17.10). We can use the Poincaré inequality (17.2) and drop the term in $|D^3U|^2$ to give

$$\frac{d}{dt} |DU|^2 \leq C|D^2U|^2.$$

Using (S17.9) and the uniform Gronwall “trick” shows that a bounded set in $L^2$ becomes a bounded set in $H^1$, and so $\Lambda(t; u_0)$ is compact for all $t > 0$ as claimed.

17.12 We use (17.4) to estimate the second term on the right-hand side by

$$\sum_{j=1}^{n} |D\phi_j|^2 \leq \sum_{j=1}^{n} |\phi_j||D^2\phi_j| \leq \left( \sum_{j=1}^{n} |\phi_j|^2 \right)^{1/2} \left( \sum_{j=1}^{n} |D^2\phi_j|^2 \right)^{1/2}.$$

Since the $\{\phi_j\}$ are orthonormal, $|\phi_j|^2 = 1$, giving

$$\sum_{j=1}^{n} |D\phi_j|^2 \leq n^{1/2} \left( \sum_{j=1}^{n} |D^2\phi_j|^2 \right)^{1/2} \leq n + \frac{1}{2} \sum_{j=1}^{n} |D^2\phi_j|^2.$$

To estimate the final term, we use the Cauchy–Schwarz inequality.

$$\int_0^L \phi_j^2 \, Du \, dx \leq |\phi_j|^2 |Du| = ||\phi_j||_{L^2}^2 |Du| \leq C|D\phi_j|^2,$$

since $|Du|$ is bounded on $A$ and $H^1 \subset L^4$. Now, using (17.4), we have

$$\int_0^L \phi_j^2 \, Du \, dx \leq C|\phi_j||D^2\phi_j|$$

$$\leq C|\phi_j|^2 + \frac{1}{2} |D^2\phi_j|^2.$$

Combining these estimates we have

$$\sum_{j=1}^{n} (L\phi_j, \phi_j) \leq -\frac{1}{2} \sum_{j=1}^{n} |D^2\phi_j|^2 + Mn.$$
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Since the eigenvalues $\lambda_j$ of $A = D^4$ are proportional to $j^4$, it follows (cf. final part of the argument in the proof of Lemma 13.17) that

$$\sum_{j=1}^{n} |D^2 \phi_j|^2 \geq Cn^5.$$ 

Therefore we need

$$-Cn^5 + Mn < 0,$$

which occurs provided that $n > (M/C)^{1/4}$. The KSE therefore has a finite-dimensional attractor.

17.13 For $v \in D(A^{1/2})$ we have

$$(N(u), v) = \int_0^L u(Du)v + (D^2 u)v \, dx$$

$$= -\int_0^L \frac{1}{2}u^2 Dv - uD^2 v \, dx,$$

and so

$$|(N(u), v)| \leq \frac{1}{2}|u|^2 \|Dv\|_{L^\infty} + |u||D^2 v|.$$

Since $H^1 \subset L^\infty$ and $D(A^{1/2}) \subset H^2$ then

$$|(N(u), v)| \leq C(|u| + 1)|u||A^{1/2} v|,$$

as required.

17.14 For $w \in D(A^{1/2}),$

$$(N(u) - N(v), w) = \int_0^L (uDu - vDv)w + D^2(u - v)w \, dx$$

$$= \int_0^L \frac{1}{2}(u^2 - v^2)Dw + (u - v)(D^2 w) \, dx,$$

and so

$$|(N(u) - N(v), w)| \leq \frac{1}{2}(|u + v| + 1)|u|\|Dw\|_{L^\infty}$$

$$\leq c(|u + v| + 1)|u||A^{1/2} w|,$$

where the same embedding results as those given above were used.