Equi-homogeneity

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Abstract

In this paper we XYZ

1 Introduction

Let (X, d_X) be a metric space.

We adopt the notation $B_\delta(x)$ for the closed ball of radius δ with centre $x \in X$, and for brevity we refer to closed balls of radius δ as δ-balls.

1.1 Box-counting dimension

Throughout this paper we will use two quantities to describe the geometry of a set $F \subset X$: for each $\delta > 0$ we denote

- the minimum number of δ-balls with centres in F such that F is contained in their union by $N(F, \delta)$, and
- the maximum number of disjoint δ-balls with centres in F by $N'(F, \delta)$.

We say that $F \subset X$ is *totally bounded* if for all $\delta > 0$ the quantity $N(F, \delta) < \infty$, which is to say that F can be covered in finitely many balls of any radius.

These apparently distinct quantities are in fact very closely related:

Lemma 1.1. *Let $F \subset X$ be totally bounded. For all $\delta > 0$*

$$N(F, 2\delta) \leq N'(F, 2\delta) \leq N(F, \delta). \tag{1}$$

Proof. Let $x_1, \ldots, x_{N'(F,\delta)} \in F$ be the centres of disjoint δ-balls. As each δ-ball $B_\delta(y)$ can cover at most one of the x_i, to cover F we need at least as many δ-balls are there are x_i, hence $N(F, \delta) \geq N'(F, \delta)$, which is the first inequality of (1).

Next, with the same points $\{x_i\}$ observe that for each $x \in F$ the distance $\mathrm{d}_X(x, x_i) \leq 2\delta$ for some $i = 1, \ldots, N'(F, \delta)$, otherwise the additional closed ball $B_\delta(x)$ would be disjoint from each of the $B_\delta(x_i)$. Consequently, the balls $B_{2\delta}(x_i)$ cover the set F, hence $N(F, \delta) \leq N'(F, 2\delta)$, which is the second inequality of (1). □

The familiar box-counting dimensions encode the scaling of these quantities as $\delta \to 0$.

Definition 1.2. *For a totally bounded set $F \subset X$ the lower and upper box-counting dimensions are defined by*

$$\dim_{LB} F = \liminf_{\delta \to 0+} \frac{\log N(F,\delta)}{-\log \delta}, \tag{2}$$

$$\text{and} \qquad \dim_{B} F = \limsup_{\delta \to 0+} \frac{\log N(F,\delta)}{-\log \delta} \tag{3}$$

respectively.

In light of the inequalities (1), replacing $N(F,\delta)$ with $N'(F,\delta)$ in the above gives an equivalent definition. The box-counting dimensions essentially capture the exponent $s \in \mathbb{R}^+$ for which $N(F,\delta) \sim \delta^{-s}$. More precisely, it follows from Definition 1.2 that for all $\delta_0 > 0$ and any $\varepsilon > 0$ there exists a constant $C \geq 1$ such that

$$C^{-1}\delta^{-\dim_{LB} F+\varepsilon} \leq N(F,\delta) \leq C\delta^{-\dim_B F-\varepsilon} \qquad \forall\ 0 < \delta \leq \delta_0. \tag{4}$$

For some bounded sets F the bounds (4) also hold for $\varepsilon = 0$ giving precise control of the growth of $N(F,\delta)$. We distinguish this class of sets in the following definition:

Definition 1.3. *We say that a bounded set $F \subset X$* attains *its lower box-counting dimension if for all $\delta_0 > 0$ there exists a positive constant $C \leq 0$ such that*

$$N(F,\delta) \geq C\delta^{-\dim_{LB} F} \qquad \textit{for all} \quad 0 < \delta < \delta_0.$$

Similarly, we say that F attains *its upper box-counting dimension if for all $\delta_0 > 0$ there exists a constant $C \geq 1$ such that*

$$N(F,\delta) \leq C\delta^{-\dim_B F} \qquad \textit{for all} \quad 0 < \delta < \delta_0.$$

We remark that a similar distinction is made with regard to the Hausdorff dimension of sets: recall that the Hausdorff measures are a one-parameter family of measures, denoted $\mathcal{H}^s$ with parameter $s \in \mathbb{R}^+$, and that for each set $F \subset \mathbb{R}^n$ there exists a value $\dim_H F \in \mathbb{R}^+$, called the Hausdorff dimension of F, such that

$$\mathcal{H}^s(F) = \begin{cases} \infty & s < \dim_H F \\ 0 & s > \dim_H F. \end{cases}$$

For a set F to have Hausdorff dimension d it is sufficient, but not necessary, for the Hausdorff measure with parameter d to satisfy $0 < \mathcal{H}^d(F) < \infty$. Sets with this property are sometimes called d-sets (see, for example, [4] pp.32) and are distinguished as they have many convenient properties. For example, the Hausdorff dimension product formula $\dim_H(F \times G) \geq \dim_H F + \dim_H G$ was first proved for sets F and G in this restricted class (see Besicovitch & Moran [2]) before being extended to hold for all sets (see Howroyd [6]).

1.2 Homogeneity and the Assouad dimension

The Assouad dimension is a less familiar notion of dimension, in which we are concerned with 'local' coverings of a set F: for more details see Assouad [1], Bouligand [3], Luukkainen [8] Olson [9], or Robinson [13].

Definition 1.4. *A set $F \subset X$ is s-homogeneous if for all $\delta_0 > 0$ there exists a constant $C > 0$ such that*

$$\sup_{x \in F} N(B_\delta(x) \cap F, \rho) \leq C(\delta/\rho)^s \qquad \forall\, \delta, \rho \quad \text{with} \quad 0 < \rho < \delta \leq \delta_0. \tag{5}$$

Note that we do not require F to be bounded in order to be s-homogeneous, but minimally require each intersection $B_\delta(x) \cap F$ to be totally bounded. This trivially holds if X has *totally bounded balls*, which is to say that every ball $B_\delta(x) \subset X$ is totally bounded (for example, in Euclidean space $X = \mathbb{R}^n$).

The following technical lemma gives a relationship between the minimal size of covers of the set $B_\delta(x) \cap F$ for different length-scales, which will use in many of the subsequent proofs.

Lemma 1.5. *Let $F \subset X$. For all $\delta, \rho, r > 0$ and each $x \in F$*

$$N(B_\delta(x) \cap F, \rho) \leq N(B_\delta(x) \cap F, r) \sup_{x \in F} N(B_r(x) \cap F, \rho) \tag{6}$$

Proof. The only non-trivial case occurs when $\rho < r < \delta$. Further, if $M := N(B_\delta(x) \cap F, r) = \infty$ then there is nothing to prove. Assume that $M < \infty$ and let $x_1, \ldots, x_M \in F$ be the centres of the r-balls $B_r(x_j)$ that cover $B_\delta(x) \cap F$. Clearly

$$B_\delta(x) \cap F \subset \bigcup_{j=1}^{M} B_r(x_j) \cap F$$

so

$$\begin{aligned} N(B_\delta(x) \cap F, \rho) &\leq \sum_{j=1}^{M} N(B_r(x_j) \cap F, \rho) \\ &\leq M \sup_{x \in F} N(B_r(x) \cap F, \rho) \end{aligned}$$

which is precisely (6). □

It will be useful to observe that in some cases s-homogeneity is equivalent to (5) holding only for *some* δ_0, which is easier to check.

Lemma 1.6. *If $F \subset X$ is totally bounded or X has totally bounded balls then $F \subset X$ is s-homogeneous if and only if there exist constants $C, \delta_1 > 0$ such that*

$$\sup_{x \in F} N(B_\delta(x) \cap F, \rho) \leq C(\delta/\rho)^s \qquad \forall\, \delta, \rho \quad \text{with} \quad 0 < \rho < \delta \leq \delta_1. \tag{7}$$

Proof. The 'if' direction is immediate from the definition of s-homogeneity. To prove the converse we let $\delta_0 > 0$ and $x \in F$ be arbitrary. If $\delta_0 \leq \delta_1$ then there is nothing to prove, so we assume that $\delta_0 > \delta_1$. Suppose δ, ρ lie in the range $0 < \rho < \delta_1 < \delta \leq \delta_0$. From Lemma 1.5 with $r = \delta_1$ we obtain

$$\begin{aligned} N(B_\delta(x) \cap F, \rho) &\leq N(B_\delta(x) \cap F, \delta_1) \sup_{x \in F} N(B_{\delta_1}(x) \cap F, \rho) \\ &\leq N(B_\delta(x) \cap F, \delta_1)\, C(\delta_1/\rho)^s \\ &\leq N(B_\delta(x) \cap F, \delta_1)\, C(\delta/\rho)^s \end{aligned} \tag{8}$$

which follows from (7) and the fact that $\delta > \delta_1$.

Now, if X has totally bounded balls then it follows from (8) that for $0 < \rho < \delta_1 < \delta \le \delta_0$

$$N\left(B_\delta\left(x\right)\cap F,\rho\right)\le N\left(B_{\delta_0}\left(0\right),\delta_1\right)C\left(\delta/\rho\right)^s,$$

and trivially for $\delta_1 \le \rho < \delta \le \delta_0$ that

$$N\left(B_\delta\left(x\right)\cap F,\rho\right)\le N\left(B_\delta\left(x\right),\rho\right)\le N\left(B_{\delta_0}\left(x\right),\delta_1\right)\le N\left(B_{\delta_0}\left(0\right),\delta_1\right)\left(\delta/\rho\right)^s$$

as $\delta/\rho > 1$. Consequently, with $C_{\delta_0} = N\left(B_{\delta_0}\left(0\right),\delta_1\right)\max\left(C,1\right)$ we obtain

$$\sup_{x\in F} N\left(B_\delta\left(x\right)\cap F,\rho\right)\le C_{\delta_0}\left(\delta/\rho\right)^s \qquad \forall\ \delta,\rho \quad \text{with} \quad 0<\rho<\delta\le\delta_0,$$

which, as $\delta_0 > 0$ was arbitrary, is precisely that F is s-homogeneous.

Next, if $F \subset X$ is totally bounded then it follows from (8) that for $0 < \rho < \delta_1 < \delta \le \delta_0$

$$N\left(B_\delta\left(x\right)\cap F,\rho\right)\le N\left(F,\delta_1\right)C\left(\delta/\rho\right)^s,$$

and again for $\delta_1 \le \rho < \delta \le \delta_0$ that

$$N\left(B_\delta\left(x\right)\cap F,\rho\right)\le N\left(F,\delta_1\right)\left(\delta/\rho\right)^s.$$

Consequently, the constant $C' = N\left(F,\delta_1\right)\max\left(C,1\right)$ is sufficient to extend (7) to all $0 < \rho < \delta \le \delta_0$, so we conclude that F is s-homogeneous. □

Corollary 1.7. *If $F \subset X$ is totally bounded then F is s-homogeneous if and only if there exists a constant C such that*

$$\sup_{x\in F} N\left(B_\delta\left(x\right)\cap F,\rho\right)\le C\left(\delta/\rho\right)^s \qquad \forall\ \delta,\rho \quad \textit{with} \quad 0<\rho<\delta.$$

Proof. The 'if' direction is immediate from the definition. Conversely, we see in the above proof that the constant C' does not depend upon the upper bound δ_0, so the inequality is valid for all ρ, δ satisfying $0 < \rho < \delta$. □

Definition 1.8. *The Assouad dimension of a set $F \subset X$ is defined by*

$$\dim_A F := \inf\left\{s\in\mathbb{R}^+ :\ F \textit{ is } s\textit{-homogeneous}\right\}$$

It is known that for a bounded set $F \subset \mathbb{R}^n$ the three notions of dimension that we have now introduced satisfy

$$\dim_{LB} F \le \dim_B F \le \dim_A F \tag{9}$$

(see, for example, Lemma 9.6 in Robinson [13]). The inequality (9) also holds for totally bounded subsets in general metric spaces.

Lemma 1.9. *If $F \subset X$ is totally bounded then $\dim_B F \le \dim_A F$.*

Proof. Let $s > \dim_A F$ and let $x_1, \ldots, x_{N(F,1)}$ be the centres of balls of radius 1 that form a cover of F. For all $\rho < 1$

$$\begin{aligned} N(F, \rho) &\leq \sum_{j=1}^{N(F,1)} N(B_1(x_j) \cap F, \rho) \leq N(F, 1) \sup_{x \in F} N(B_1(x) \cap F, \rho) \\ &\leq N(F, 1)\, C\, (1/\rho)^s \end{aligned}$$

for some $C > 0$, hence $\dim_B F \leq s$. As $s > \dim_A F$ was arbitrary we conclude that $\dim_B F \leq \dim_A F$. □

An interesting example is given by the compact countable set $F_\alpha := \{n^{-\alpha}\}_{n \in \mathbb{N}} \cup \{0\} \subset \mathbb{R}$ with $\alpha > 0$ for which

$$\begin{aligned} &\dim_{LB} F_\alpha = \dim_B F_\alpha = (1 + \alpha)^{-1} \\ \text{but} \qquad &\dim_A F_\alpha = 1. \end{aligned}$$

(see Olson [9] and Example 13.4 in Robinson [12]).

2 Equi-homogeneity

From Definition 1.4 we see that homogeneity encodes the *maximum* size of a local optimal cover at a particular length-scale. However, the *minimal* size of a local optimal cover is not captured by homogeneity, and indeed this minimum size can scale very differently, as the following example illustrates:

Example 2.1. *For each $\alpha > 0$ the set $F_\alpha \coloneqq \{n^{-\alpha}\}_{n \in \mathbb{N}} \cup \{0\}$ has Assouad dimension equal to 1, so for all $\varepsilon > 0$*

$$\sup_{x \in F_\alpha} N(B_\delta(x) \cap F_\alpha, \rho)\, (\delta/\rho)^{-(1-\varepsilon)}$$

is unbounded on δ, ρ with $0 < \rho < \delta$.

On the other hand $1 \in F_\alpha$ is an isolated point so

$$\inf_{x \in F_\alpha} N(B_\delta(x) \cap F_\alpha, \rho) = 1$$

for all δ, ρ with $0 < \rho < \delta < 1 - 2^{-\alpha}$ as $B_\delta(1) \cap F_\alpha = \{1\}$ for such δ and this isolated point can be covered by a single ball of any radius.

For a totally bounded set the maximal and minimal sizes of local optimal covers can be estimated using the following relationships.

Lemma 2.2. *For a totally bounded set $F \subset X$ and δ, ρ satisfying $0 < \rho < \delta$*

$$\inf_{x \in F} N(B_\delta(x) \cap F, \rho) \leq \frac{N(F, \rho)}{N(F, 4\delta)} \tag{10}$$

$$\text{and} \qquad \sup_{x \in F} N(B_\delta(x) \cap F, \rho) \geq \frac{N(F, \rho)}{N(F, \delta)}. \tag{11}$$

Proof. Let $x_1, \dots, x_{N(F,\delta)} \in F$ be the centres of δ-balls that form a cover of F. Clearly,

$$N(F,\rho) \le \sum_{j=1}^{N(F,\delta)} N(B_\delta(x_j) \cap F, \rho) \le N(F,\delta) \sup_{x\in F} N(B_\delta(x) \cap F, \rho),$$

which is (11).

Next, let δ, ρ satisfy $0 < \rho < \delta$ and let $x_1, \dots, x_{N'(F,4\delta)} \in F$ be the centres of disjoint 4δ-balls. Observe that an arbitrary ρ-ball $B_\rho(z)$ intersects at most one of the balls $B_\delta(x_i)$: indeed, if there exist $x, y \in B_\rho(z)$ with $x \in B_\delta(x_i)$ and $y \in B_\delta(x_j)$ with $i \ne j$ then

$$\mathrm{d}_X(x_i, x_j) \le \mathrm{d}_X(x_i, x) + \mathrm{d}_X(x, z) + \mathrm{d}_X(z, y) + \mathrm{d}_X(y, x_j) \le 2\delta + 2\rho \le 4\delta$$

and so $x_i \in B_{4\delta}(x_j)$, which is a contradiction. Consequently, as F contains the union $\bigcup_{j=1}^{N'(F,4\delta)} B_\delta(x_j) \cap F$, it follows that

$$\begin{aligned} N(F,\rho) &\ge \sum_{j=1}^{N'(F,4\delta)} N(B_\delta(x_j) \cap F, \rho) \\ &\ge N'(F,4\delta) \inf_{x\in F} N(B_\delta(x) \cap F, \rho), \\ &\ge N(F,4\delta) \inf_{x\in F} N(B_\delta(x) \cap F, \rho) \end{aligned}$$

from (1), which is precisely (10). □

We now define equi-homogeneous sets to be those sets for which the range of the number of sets required in the local covers is uniformly bounded at all length-scales.

Definition 2.3. *We say that a set $F \subset X$ is* equi-homogeneous *if for all $\delta_0 > 0$ there exist constants $M \ge 1$ and $c_1, c_2 > 0$ such that*

$$\sup_{x\in F} N(B_\delta(x) \cap F, \rho) \le M \inf_{x\in F} N(B_{c_1\delta}(x) \cap F, c_2\rho) \tag{12}$$

for all δ, ρ with $0 < \rho < \delta \le \delta_0$.

Note that as $N(B_\delta(x) \cap F, \rho)$ increases with δ and decreases with ρ, by replacing the c_i with 1 if necessary we can assume without loss of generality that $c_2 \le 1 \le c_1$ in (12).

2.1 Equivalent definitions

As with the definition of homogeneity, for a large class of sets it is sufficient that (12) holds only for *some* δ_0.

Lemma 2.4. *If $F \subset X$ is totally bounded or X has totally bounded balls then F is equi-homogeneous if and only if there exist constants $M \ge 1$ and $c_1, c_2, \delta_1 > 0$ such that*

$$\sup_{x\in F} N(B_\delta(x) \cap F, \rho) \le M \inf_{x\in F} N(B_{c_1\delta}(x) \cap F, c_2\rho)$$

for all ρ, δ satisfying $0 < \rho < \delta \le \delta_1$.

Proof. The proof is substantially the same as that of Lemma 1.6. □

Again, if F is totally bounded then we can find $M \geq 1$ such that (12) holds for all ρ, δ with $0 < \rho < \delta$.

In normed spaces with totally bounded balls (such as Euclidean space) there is an even more elementary formulation that does not require the constants c_1, c_2.

Lemma 2.5. *Let X be a normed space with totally bounded balls. A set $F \subset X$ is equi-homogeneous if and only if there exists constants $M \geq 1$, $\delta_1 \geq 1$ such that*

$$\sup_{x\in F} N(B_\delta(x) \cap F, \rho) \leq M \inf_{x\in F} N(B_\delta(x) \cap F, \rho) \tag{13}$$

for all ρ, δ with $0 < \rho < \delta \leq \delta_1$.

Proof. The 'if' direction follows immediately from Lemma 2.4. To prove the converse fix $\delta_0 > 0$ and let $M \geq 1$ and $c_1, c_2 > 0$ with $c_2 \leq 1 \leq c_1$ be such that

$$\sup_{x\in F} N(B_\delta(x) \cap F, \rho) \leq M \inf_{x\in F} N(B_{c_1\delta}(x) \cap F, c_2\rho)$$

for all $0 < \rho < \delta \leq \delta_0$.

First, observe that replacing δ by δ/c_1 we can assume that

$$\sup_{x\in F} N(B_{\delta/c_1}(x) \cap F, \rho) \leq M \inf_{x\in F} N(B_\delta(x) \cap F, c_2\rho) \tag{14}$$

for all δ, ρ with $0 < \rho < \delta/c_1$, $\delta \leq c_1\delta_0$. Note that if $\rho \geq \delta/c_1$ then the above inequality holds trivially, since the left-hand side is 1 and the right-hand side is at least $M \geq 1$; so in fact (14) holds for all $0 < \rho < \delta \leq \delta_1 := c_1\delta_0$.

Now, it follows from (6) with $r = \delta/c_1$ that

$$N(B_\delta(x) \cap F, \rho) \leq N(B_\delta(x), \delta/c_1) \sup_{x\in F} N\left(B_{\delta/c_1}(x) \cap F, \rho\right)$$

for all $x \in F$, so setting $N_1 := N(B_\delta(x), \delta/c_1) = N(B_1(0), 1/c_1)$, which follows as X is a normed space, we obtain

$$\sup_{x\in F} N(B_\delta(x) \cap F, \rho) \leq N_1 \sup_{x\in F} N\left(B_{\delta/c_1}(x) \cap F, \rho\right). \tag{15}$$

It also follows from (6) that for any $r > 0$

$$\begin{aligned} N(B_\delta(x) \cap F, c_2\rho) &\leq N(B_\delta(x) \cap F, r) \sup_{x\in F} N(B_r(x) \cap F, c_2\rho) \\ &\leq N(B_\delta(x) \cap F, r) \sup_{x\in F} N(B_r(x), c_2\rho) \\ &= N(B_\delta(x) \cap F, r)\, N(B_r(0), c_2\rho) \end{aligned}$$

so taking $r = \rho$, setting $N_2 = N(B_\rho(0), c_2\rho) = N(B_1(0), c_2)$, which again follows as X is a normed space, and taking the infimum over $x \in F$ we obtain

$$\inf_{x\in F} N(B_\delta(x) \cap F, c_2\rho) \leq \inf_{x\in F} N(B_\delta(x) \cap F, \rho)\, N_2. \tag{16}$$

It follows from (14), (15) and (16) that for all ρ, δ with $0 < \rho < \delta \leq \delta_1$

$$\sup_{x \in F} N\left(B_\delta\left(x\right) \cap F, \rho\right) \leq M \frac{N_1}{N_2} \inf_{x \in F} N\left(B_\delta\left(x\right) \cap F, \rho\right)$$

so we conclude from Lemma 2.5 that F is equi-homogeneous. $\square$

In [11] the authors demonstrate that for reasonable choices of product metric, the product of two equi-homogeneous sets is also equi-homogeneous.

We will demonstrate that this notion of equi-homogeneity is not overly restrictive: it is enjoyed, at least, by all self-similar sets that satisfy the Moran open set condition.

Self-similar sets are a much studied and canonical class of fractal sets. A (contracting) similarity is a map $f_i : \mathbb{R}^d \to \mathbb{R}^d$ of the form

$$f_i(x) = \sigma_i O_i x + \beta_i,$$

where $\sigma_i \in (0,1)$, $\beta_i \in \mathbb{R}^d$, and $O_i \in O(d)$, the set of all $d \times d$ orthogonal matrices. Given a family $\{f_i\}_{i=1}^n$ of similarities, there exists a unique set F, known as the attractor of this family, such that

$$F = \bigcup_{i=1}^{n} f_i(F). \tag{17}$$

(See Falconer [4], for example.)

These sets are easier to analyse when we impose some separation properties, i.e. we insist that (in some sense) the images of the f_i do not overlap. The simplest such property is the Moran open-set condition: there exists an open set U such that $F \subset \overline{U}$, $f_i(U) \subseteq U$, and

$$f_i(U) \cap f_j(U) = \varnothing \qquad \text{when} \qquad i \neq j.$$

Lemma 2.6. *Self-similar sets that satisfy the Moran open-set condition are equi-homogeneous.*

Proof. Let $\mathcal{I} = \{1, \ldots, n\}$, define $\mathcal{I}^* = \bigcup_{n=1}^{\infty} \mathcal{I}^n$, and for $\alpha = (i_1, \ldots, i_n) \in \mathcal{I}$ let

$$f_\alpha = f_{i_1} \circ \cdots \circ f_{i_n}, \quad O_\alpha = O_{i_1} \cdots O_{i_n} \quad \text{and} \quad \sigma_\alpha = \sigma_{i_1} \cdots \sigma_{i_n}.$$

For $n \geq 2$ we denote $(i_1, \ldots, i_{n-1})$ by α'. Let $\sigma_{\min} = \min\{\, \sigma_i : i \in \mathcal{I} \,\}$ and $\eta = \operatorname{diam}(\overline{U})$. For $\delta \leq \sigma_{\min}\eta$ define

$$S_\delta = \{\, \alpha \in \mathcal{I} : \sigma_\alpha \eta < \delta \leq \sigma_{\alpha'} \eta \,\}.$$

Note that $n \geq 2$ for any $\alpha \in S_\delta$. Moreover, for $\alpha, \beta \in S_\delta$ we have

$$f_\alpha(U) \cap f_\beta(U) = \varnothing \qquad \text{when} \qquad \alpha \neq \beta. \tag{18}$$

This follows from the open set condition as we now show.

Write

$$\alpha = (i_1, \ldots, i_n) \qquad \text{and} \qquad \beta = (j_1, \ldots, j_m)$$

and assume without loss of generality that $m \leq n$. Let k be the smallest integer such that $i_k \neq j_k$. Such a k exists because if not, then $\alpha \neq \beta$ would imply $m < n$

and consequently that $\sigma_{\alpha'} \le \sigma_\beta$. This would imply that $\delta \le \sigma_{\alpha'}\eta < \sigma_\beta\eta < \delta$, which is a contradiction. If $k = 1$ then $i_1 \ne j_1$ and the open set condition implies that

$$f_\alpha(U) \cap f_\beta(U) \subseteq f_{i_1}(U) \cap f_{j_1}(U) = \varnothing.$$

If $k > 1$ define $\gamma = (i_1, \ldots, i_{k-1})$ and again

$$f_\alpha(U) \cap f_\beta(U) \subseteq f_\gamma \circ f_{i_k}(U) \cap f_\gamma \circ f_{j_k}(U) = f_\gamma(\varnothing) = \varnothing.$$

Thus we have shown that (18) holds.

We next claim that

$$F = \bigcup_{\alpha\in S_\delta} f_\alpha(F) \qquad \text{where} \qquad f_\alpha(x) = \sigma_\alpha O_\alpha(x) + f_\alpha(0). \tag{19}$$

This follows from (17) and induction. Given $x \in F$ choose $i_1 \in \mathcal{I}$ such that $x \in f_{i_1}(F)$. Assume that $x \in f_{i_1} \circ \cdots \circ f_{i_k}(F)$; then $f_{i_k}^{-1} \circ \cdots \circ f_{i_1}^{-1}(x) \in F$ implies that we can choose $i_{k+1} \in \mathcal{I}$ such that $f_{i_k}^{-1} \circ \cdots \circ f_{i_1}^{-1}(x) \in f_{i_{k+1}}(F)$. It follows that $x \in f_{i_1} \circ \cdots \circ f_{i_{k+1}}(F)$. Given the sequence i_k chosen above, there is exactly one choice of n such that $\alpha = (i_1, \cdots, i_n)$ satisfies $\sigma_\alpha\eta < \delta \le \sigma_{\alpha'}\eta$. We conclude that $x \in f_\alpha(F)$ for some $\alpha \in S_\delta$, which completes the proof of the claim.

We now use (18) and (19) to show that F is equi-homogeneous. Let $x \in F$ be arbitrary. Then $x \in f_\alpha(F)$ for some $\alpha \in S_\delta$ and consequently

$$\operatorname{diam}(f_\alpha(F)) = \sigma_\alpha \operatorname{diam}(F) \le \sigma_\alpha\eta < \delta,$$

which implies that $f_\alpha(F) \subseteq B_\delta(x)$. It follows that

$$B_\delta(x) \cap F = B_\delta(x) \cap \bigcup_{\beta\in S_\delta} f_\beta(F) \supseteq B_\delta(x) \cap f_\alpha(F) = f_\alpha(F).$$

Therefore

$$N(B_\delta(x) \cap F, \rho) \ge N(f_\alpha(F), \rho) = N(F, \rho/\sigma_\alpha) \ge N(F, c_1\rho/\delta)$$

where $c_1 = \eta/\sigma_{\min}$ implies that

$$\inf_{x\in F} N(B_\delta(x) \cap F, \rho) \ge N(F, c_1\rho/\delta).$$

Now let $A_\delta = \{\, \alpha \in S_\delta : B_\delta(x) \cap f_\alpha(\overline{U}) \ne \varnothing \,\}$. Then $\alpha \in A_\delta$ implies that

$$f_\alpha(\overline{U}) \subseteq B_{\delta+\operatorname{diam} f_\alpha(\overline{U})}(x) \subseteq B_{2\delta}(x).$$

Therefore by (18) we obtain

$$\begin{aligned}\lambda(B_{2\delta}(x)) \ge \lambda\Big(\bigcup_{\alpha\in A_\delta} f_\alpha(U)\Big) &= \sum_{\alpha\in A_\delta} \lambda\big(f_\alpha(U)\big) \\ &= \lambda(U) \sum_{\alpha\in A_\delta} (\sigma_a)^d \ge \lambda(U)(\delta/c_1)^d \operatorname{card}(A_\delta)\end{aligned}$$

where λ is the d-dimensional Lebesgue measure. Consequently

$$\begin{aligned}N(B_\delta(x) \cap F, \rho) \le \sum_{\alpha\in A_\delta} N\big(f_\alpha(F), \rho\big) &= \sum_{\alpha\in A_\delta} N(F, \rho/\sigma_\alpha) \\ &\le \operatorname{card}(A_\delta) N(F, \eta\rho/\delta) \le M N(F, \eta\rho/\delta)\end{aligned}$$

where $M = (2c_1\eta)^d \lambda(B_1(x))/\lambda(U)$. It follows that

$$\sup_{x\in F} N(B_\delta(x)\cap F,\rho) \le MN(F,\eta\rho/\delta),$$

which completes the proof of the theorem. □

2.2 Equi-homogeneity and the Assouad dimension

For equi-homogeneous sets F we obtain from (10) an upper bound for the maximal size of the local coverings $\sup_{x\in F} N(B_\delta(x)\cap F,\rho)$ in terms of the minimum number of sets required to cover F. In fact, with this bound we can precisely find the Assouad dimension of equi-homogeneous sets provided that their box-counting dimensions are suitably 'well behaved', which is the content of the following theorem.

Theorem 2.7. *If a totally bounded set $F \subset X$ is equi-homogeneous, F attains both its upper and lower box-counting dimensions, and $\dim_{LB} F = \dim_B F$, then $\dim_A F = \dim_B F = \dim_{LB} F$.*

Proof. As F attains both its upper and lower box-counting dimensions and these dimensions are equal it is clear from Definition 1.3 that there exists a constant $C \ge 1$ and a $\delta_0 > 0$ such that

$$\frac{1}{C}\delta^{-\dim_B F} \le N(F,\delta) \le C\delta^{-\dim_B F} \qquad \forall 0<\delta\le\delta_0. \tag{20}$$

Next, as F is equi-homogeneous there exist $M \ge 1$ and $c_1, c_2 > 0$ with $c_2 \le 1 \le c_1$ such that for all δ,ρ with $0<\rho<\delta\le\delta_0$

$$\sup_{x\in F} N(B_\delta(x)\cap F,\rho) \le M \inf_{x\in F} N(B_{c_1\delta}(x)\cap F, c_2\rho).$$

As $0 < c_2\rho < c_1\delta$ we can apply Lemma 2.2 to obtain

$$\begin{aligned}\sup_{x\in F} N(B_\delta(x)\cap F,\rho) \le M\frac{N(F,c_2\rho)}{N(F,4c_1\delta)} &\le MC^2\frac{(c_2\rho)^{-\dim_B F}}{(4c_1\delta)^{-\dim_B F}}\\ &= MC^2\,(4c_1/c_2)^{\dim_B F}\,(\delta/\rho)^{\dim_B F}\end{aligned}$$

from (20), so the set F is $(\dim_B F)$-homogeneous. Consequently, $\dim_A F \le \dim_B F$, but from (9) the Assouad dimension dominates the upper box-counting dimension so we obtain the equality $\dim_A F = \dim_B F$. □

Note that our notion of equi-homogeneous is related to, but distinctly different from, the coincidence of the Assouad dimension with the minimal dimension number $\dim_{\mathrm{LA}}(F)$ defined by Larman in [7] as as the supremum over all s for which there exists constants c and $\delta_0 > 0$ such that

$$\inf_{x\in f} N(B_\delta(x)\cap F,\rho) \ge c(\delta/\rho)^s \qquad \text{for all} \qquad 0<\rho<\delta\le\delta_0.$$

Corollary 2.11 of Fraser [5] shows that self-similar sets F that satisfy the open-set condition also satisfy $\dim_{\mathrm{A}}(F) = \dim_{\mathrm{LA}}(F)$. Given that we have just shown that such sets are equi-homogeneous, this raises the possibility that equi-homogeneity is equivalent to the condition $\dim_{\mathrm{A}}(F) = \dim_{\mathrm{LA}}(F)$.

However, the generalised Cantor sets that we will construct in Section 3 are equi-homogeneous but have minimal dimension numbers that differ from their Assouad dimensions, and we now give a simple example of a set that satisfies $\dim_{\mathrm{A}}(F) = \dim_{\mathrm{LA}}(F)$ but that is not equi-homogeneous. Taken together these two examples demonstrate that the notion of equi-homogeneity is entirely distinct from the coincidence of these two dimensions.

Proposition 2.8. *Let* $F = \{0,1\} \cup \{\, 2^{-n} : n \in \mathbb{N}\,\}$*. Then*

$$\dim_{\mathrm{A}}(F) = \dim_{\mathrm{LA}}(F) = 0$$

but F *is not equi-homogeneous.*

Proof. Let $\delta = 1/2$. Then

$$B_\delta(1) \cap F = \{1\} \quad \text{implies that} \quad \inf_{x \in F} N(B_\delta(x) \cap F, \rho) = 1.$$

for every $\rho > 0$. On the other hand, for $0 < \rho < 1/4$, let K be chosen so that

$$2^{-K-1} \leq \rho < 2^{-K}.$$

Then $K \geq 2$ and

$$B_\delta(0) \cap F \supseteq \{\, 2^{-n} : n = 2, \dots, K\,\}.$$

Moreover $2^{-n+1} - 2^{-n} = 2^{-n} \geq 2^{-K} > \rho$ for $n \leq K$ implies that at least one set of diameter ρ is required to cover each of the $K-1$ points above. Therefore

$$\sup_{x \in F} N(B_\delta(x) \cap F, \rho) \geq K - 1 \geq \frac{\log(1/\rho)}{\log 2} - 2.$$

This shows there is no value for M independent of ρ that could appear in Definition 2.3 for this set, and so F is not equi-homogeneous.

Clearly $\dim_{\mathrm{LA}}(F) = 0$. The equality $\dim_{\mathrm{A}}(F) = 0$ is stated as Fact 4.3 in Olson [9] without proof. We include the proof here and remark that the logarithmic terms that occur in the course of the argument can also be used to show that F does not 'attain' its box-counting dimension (in the sense of Definition 1.3).

Let $x \in [0,1]$ and $0 < \rho < \delta < 1/4$. Define

$$G = \{\, 2^{-n} : \max(0, x-\delta) < 2^{-n} \leq \rho \,\}$$

and

$$H = \{\, 2^{-n} : \max(\rho, x-\delta) < 2^{-n} < \min(x+\delta, 1)\,\}.$$

Then $B_\delta(x) \cap F \subseteq \{0,1\} \cup G \cup H$. Now depending on ρ, x, and δ it may happen that either or both of the sets H and G are empty. As covering an empty set is trivial, we need only consider the cases when these sets are non-empty.

If $G \neq \varnothing$ then $x - \delta < \rho$, and it follows that

$$N(G, \rho) \leq \frac{\rho - \max(0, x-\delta)}{\rho} + 1 \leq 2. \tag{21}$$

Similarly if $H \neq \varnothing$ then

$$N(H, \rho) \leq \frac{1}{\log 2} \log \left\{ \frac{\min(x+\delta, 1)}{\max(\rho, x-\delta)} \right\} + 1.$$

If $x+\delta \geq 1$ then $x-\delta \geq 1-2\delta \geq 1/2$. Thus $N(H,\rho) \leq 2$. If $x-\delta \leq \rho$ then $x+\delta \leq \rho+2\delta < 3\delta < 1$. Thus $N(H,\rho) \leq (\log 2)^{-1}\log(3\delta/\rho)+1$. Otherwise, $\rho+\delta < x < 1-\delta$. On this interval $x \mapsto \log\{(x+\delta)/(x-\delta)\}$ is a decreasing function. Therefore, in general,

$$N(H,\rho) \leq 2\log(\delta/\rho)+3. \tag{22}$$

Combining (21) with (22) we obtain

$$N(B_\delta(x)\cap F,\rho) \leq 2\log(\delta/\rho)+7$$

Since for every $s>0$ there exists $C>0$ such that

$$2\log(\delta/\rho)+7 \leq C(\delta/\rho)^s \qquad \text{for every} \qquad 0<\rho<\delta<1/4,$$

taking $\delta_0 = 1/4$ in Lemma 1.6 shows that $\dim_A(F)=0$. □

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