Loosely, the question is whether the dynamics on a finite-dimensional attractor of an infinite-dimensional flow can be any more complicated than the dynamics on the attractor of a finite-dimensional flow (for a general discussion see Eden et al., 1994; Robinson, 1999; Romanov, 2000). In the context of homeomorphisms, we have the following result (Pinto de Moura et al., 2010, after Robinson, 1999).

Theorem 1. *Let H be a Hilbert space and $F : H \to H$ a continuous map that has a global attractor $\mathscr{A}$. Suppose in addition that $F|_{\mathscr{A}}$ is a homeomorphism, and that $\mathscr{A}$ has finite topological dimension d. Let $n = 2d + 4$. Given any $\varepsilon > 0$, there exists a homeomorphism h of $\mathscr{A}$ onto a subset X of $\mathbb{R}^n$, and a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$f(x) = h \circ F \circ h^{-1}(x) \qquad \text{for all} \quad x \in X,$$

and the global attractor of f lies within the ε-neighbourhood of X.

Proof. $\mathscr{A}$ has the shape of a point (e.g. Kapitanski & Rodnianski, 2000).
If the topological dimension of $\mathscr{A}$ is finite then it can be embedded into $\mathbb{R}^{2d+1}$ via some homeomorphism h (e.g. Hurewicz & Wallman, 1948).
Shape is topologically invariant, so $h(\mathscr{A}) \subset \mathbb{R}^{2d+1}$ has the shape of a point. Consequently $h(\mathscr{A}) \times \{0\} \subset \mathbb{R}^{2d+2}$ is cellular (McMillan, 1964); so is $X = h(\mathscr{A}) \times \{0\}^3 \subset \mathbb{R}^{2d+4}$, which is also a Z_1-set. Set $n = 2d + 4$.
Since X is a cellular subset of $\mathbb{R}^n$, there exists a flow on $\mathbb{R}^n$ for which X is the global attractor, and for which all points on X are fixed (Garay, 1991); in particular there is a homeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ that is the identity on X and for which X is the attractor.
Since X is a Z_1-set, any homeomorphism from X into itself can be extended to a homeomorphism from $\mathbb{R}^n$ to itself (Geoghegan & Summerhill, 1973); apply this to $h \circ F \circ H^{-1} : X \to X$.
These last two facts can be combined to produce a map f as in the statement of the theorem. □

In this setting, the open problem is whether one can construct a map f for which X itself is the attractor. This boils down to whether one can extend a homeomorphism $f : X \to X$ (recall that X is cellular) to a homeomorphism from $\mathbb{R}^n$ to itself that respects a given neighbourhood basis of X (here this would be related to the dynamics of φ).

There is a harder problem, which asks the same when $\mathscr{A}$ is the attractor of a semiflow on H whose restriction to $\mathscr{A}$ is a flow; one would then ask for a flow on $\mathbb{R}^n$ for which $X = h(\mathscr{A})$ is the global attractor, on which the dynamics is conjugate to those of the original semiflow on $\mathscr{A}$. In this version even the 'attractor within ε' version of the result is unknown. It would thus be interesting to have a 'flow extension' theorem, i.e. what properties of $X \subset \mathbb{R}^n$ ensure that any flow on X can be extended to a flow on $\mathbb{R}^n$?

The motivating problem behind this appears to be even harder. Global attractors with finite box-counting dimension $d_B(\mathscr{A}) = \limsup_{\epsilon\to 0} \frac{\log N(A,\epsilon)}{-\log\epsilon}$, with $N(A,\epsilon)$ is the minimum number of ϵ balls required to cover $\mathscr{A}$, arise for the semigroups generated by various dissipative parabolic PDEs, most notably the 2D incompressible Navier–Stokes equations (see Temam, 1988, for example). One can show (following an argument of Kukavica, 2007) that on these attractors the underlying PDE can be written as an infinite-dimensional ODE $\dot{u} = \mathscr{F}(u)$, where $\mathscr{F}$ is 1-log-Lipschitz on $\mathscr{A}$, i.e.

$$|\mathscr{F}(u) - \mathscr{F}(v)| \leq C|u - v| \log(R/|u - v|) \qquad \text{for all} \quad u, v \in \mathscr{A}. \tag{2}$$

Ideally in this case one would construct an ODE on $\mathbb{R}^n$ for which X (the embedded version of $\mathscr{A}$) is an invariant set, or better still the global attractor. This involves extending the embedded vector field $h \circ \mathscr{F} \circ h^{-1}$ from X to a vector field f on $\mathbb{R}^n$; since uniqueness of $\dot{x} = f(x)$ when $|f(x) - f(y)| \leq \omega(|x-y|)$ requires $\int_0^1 \omega(r)^{-1}\,\mathrm{d}r = \infty$, one would like f to be at least 1-log-Lipschitz. But given (2) this would require h (the homeomorphism from $\mathscr{A}$ onto X) to be bi-Lipschitz, which is extremely strong and unlikely to be true in general; conditions guaranteeing that an arbitrary compact subset of a Hilbert space can be embedded in a bi-Lipschitz way into some $\mathbb{R}^n$ are not known.

When $d_B(\mathscr{A}) = d$ and $\mathscr{A}$ is the attractor of an equation like (1), it is known that if $k > 2d$ then 'most' linear maps from H into $\mathbb{R}^k$ have a Hölder continuous inverse (Hunt & Kaloshin, 1999). But this is not sufficient for uniqueness of the embedded ODE. If $d_A(\mathscr{A} - \mathscr{A}) = d$, where d_A is the Assouad dimension,

$$d_A(X) = \inf\{s : N(\mathscr{A}\cap B(x,\rho), r) \leq M(r/\rho)^s \text{ for some } M > 0, \text{ for all } \rho < r\},$$

then it is known that 'most' linear maps from H into $\mathbb{R}^k$ are α-log-Lipschitz with $\alpha > (k+2)/2(k-d)$ (Robinson, 2011; after Olson & Robinson, 2010). But (i) there is no known method for showing that the attractors of PDEs satisfy $d_A(\mathscr{A} - \mathscr{A}) < \infty$, and (ii) even coupled with (2) such smoothness of L^{-1} is not sufficient to guarantee uniqueness of the embedded vector field (which could be made almost 3/2-log-Lipschitz).

How to carry forward the programme in this continuous case is a mystery. How to improve (2) is unclear; and it is known that the exponent 1/2 in (3) is sharp in general (Pinto de Moura & Robinson, 2010).

References

A Eden, C Foias, B Nicolaenko, & R Temam (1994) *Exponential attractors for dissipative evolution equations* (Wiley, New York).

B M Garay (1991) Strong cellularity and global asymptotic stability. *Fund. Math.* **138**, 147–154.

R Geoghegan & R R Summerhil (1973) Concerning the shapes of finite-dimensional compacta. *Trans. Amer. Math. Soc.* **179**, 281–292.

B R Hunt & V Y Kaloshin (1999) Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces. *Nonlinearity* **12**, 1263–1275.

W Hurewicz & H Wallman (1948) *Dimension Theory.* Princeton University Press.

L Kapitanski & I Rodnianski (2000) Shape and Morse Theory of Attractors. *Comm. Pure Appl. Math.* **53**, 218–242.

I Kukavica (2007) Log-log convexity and backward uniqueness. *Proc. Amer. Math. Soc.* **135**, 2415–2421.

D R McMillan (1964) A criterion for cellularity in a manifold. *Ann. of Math.* **79**, 327–337.

E J Olson & J C Robinson (2010) Almost bi-Lipschitz embeddings and almost homogeneous sets. *Trans. Amer. Math. Soc.* **362**, 145–168.

E Pinto de Moura & J C Robinson (2010) Orthogonal sequences and regularity of embeddings into finite-dimensional spaces. *J. Math. Anal. Appl.* **368**, 254–262.

E Pinto de Moura, J C Robinson, & J J Sánchez-Gabites (2010) Embedding of global attractors and their dynamics, *Proc. Amer. Math. Soc.*, to appear.

J C Robinson (1999) Global Attractors: Topology and finite-dimensional dynamics. *J. Dynam. Differential Equations* **11**, 557–581.

J C Robinson (2011) *Dimensions, Embeddings, and Attractors.* Cambridge University Press.

A V Romanov (2000) Finite-dimensional limiting dynamics for dissipative parabolic equations. *Math. Sbornik* **191**, 415–429.

R Temam (1988) *Infinite-dimensional dynamical systems in mechanics and physics.* Springer.