MA475 Example Sheet 2

29 January 2019

1. Let γ be a smooth curve in the complex plane. Verify the following calculation:
$$\frac{d}{dt}\left(\gamma(t)\exp\left(-\int_0^t \frac{\gamma'(s)}{\gamma(s)}ds\right)\right) = 0 \tag{1}$$
Use this to derive the fact that:
$$\exp\int_\gamma \frac{dz}{z} = \frac{\gamma(1)}{\gamma(0)} \tag{2}$$
Conclude that when γ is a loop then $\int_\gamma \frac{dz}{z}$ has the form $2\pi i n$ for some integer n.

2. Recall from complex analysis that we define $z \mapsto z^\alpha$ by $z \mapsto \exp(\alpha \log z)$ where log is a branch of the logarithm. Show that when $z = (r, \theta)$ is expressed in polar coordinates then there is a branch of the logarithm that gives $(r, \theta) \mapsto (r^\alpha, \alpha\theta)$ for $z \mapsto z^\alpha$. If α is real show that two different branches of the log produce maps that differ by multiplication by a complex number of norm 1.

3. Let D be a disk. Recall that $v(f, z)$ is the valence of f at the point z. The following formula was used in the proof of the inverse function theorem. Prove it.
$$\frac{1}{2\pi i}\int_{\partial D} \frac{zf'(z)}{f(z) - w}dz = \sum_{f(z_j)=w, z_j \in D} v(f, z_j)z_j \tag{3}$$

4. Write $dz = x + idy$ and $d\bar{z} = x - idy$.

   (a) Show that any complex valued 1-form in $\mathbb{R}^2$ can be written as $fdz + gd\bar{z}$ for some choice of complex valued functions f and g.

   (b) Let J be the linear transformation of $\mathbb{R}^2$ given by $J(x, y) = (y, -x)$. J acts on 1-forms by pulling them back: θ maps to $J^*(\theta)$. Show that dz and $d\bar{z}$ are eigenvectors for this action.

   (c) Say that f is a holomorphic function on $\mathbb{C} = \mathbb{R}^2$. We showed that $f^*(dz) = f'(z)dz$. Compute $f^*(d\bar{z})$.

5. Let D be an open disc divided by a diameter into two half discs D_1 and D_2. Use Morera's theorem to show that if f is holomorphic in each open half-disk and f is continuous on the entire disk then f is holomorphic on the entire disk. (This argument shows that overlap functions are holomorphic in the construction of atlases for polyhedra.)