MA475 Example Sheet 2

30 January 2020

1. Let $W = \mathbb{C}^2 - \{0\}$ and recall that $\mathbb{CP}^1$ is the quotient space $W/\sim$ where $(u, v) \sim (\lambda u, \lambda v)$. Let $\pi : W \to \mathbb{CP}^1$ be the quotient map. We can identify $\mathbb{CP}^1$ with the space of lines through the origin in $\mathbb{C}^2$. Let A be an invertible matrix 2×2 matrix.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Define $\psi_A : \mathbb{C} \to W$ by $\psi_A(z) = (az + b, cz + d)$.

   (a) Show that the image of ψ does not go throught the origin and intersects every line but one. Denote this line by ℓ_A.
   Let $U_A = \mathbb{CP}^1 - \ell_A$.

   (b) Let $\phi_A = (\pi \circ \psi_A)^{-1}$ and show that $\mathcal{B} = \{\phi_A, U_A\}$ is a holomorphic atlas for $\mathbb{CP}^1$.

   (c) Define $\beta : \mathbb{CP}^1 \to \mathbb{C}_\infty$ by

$$\beta([u : v]) = \begin{cases} u/v & \text{if } v \neq 0 \\ \infty & \text{if } v = 0 \end{cases}$$

   Show that β is a holomorphic equivalence.

   (d) Let A be a non-singular complex matrix. We define an action of A on W by writing an element of W as a column vector.

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}$$

   Thus A acts on W.
   Show that:

$$\beta A \beta^{-1}(z) = \frac{az + b}{cz + d}$$

   (e) Relate the eigenvectors of A to fixed points of the induced transformation.

2. Show that the unit disk and the complex plane are homeomorphic but not conformally equivalent Riemann surfaces.

3. Let λ be a complex number of norm greater than one. Let $f(z) = \lambda z$ define an automorphism of $\mathbb{C} - \{0\}$ and let Γ be the group generated by this automorphism. Show that the quotient of $\mathbb{C} - \{0\}$ by the action of this group Γ is a torus with a natural Riemann surface structure.

4. Let $g(z) = g(x + iy) = 2x + iy/2$. Let Λ be the group generated by this homeomorphism.

   (a) Show that for each point there is a neighbourhood which is disjoint from all its images under Λ. (This shows that the quotient space has the manifold property.)

   (b) Show that the quotient space is not Hausdorff. (Hint: Show that 1 and i do not have disjoint invariant neighbourhoods.)

5. Let $V_P = \{(z, w) : w^2 = P(z)\}$. The point of this exercise is to use the tools described in class to evaluate the topology of V in some examples where the roots of P have higher multiplicity.

   Let X be a topological space. A point $p \in X$ is a *cut point* of X when X is connected but $X - p$ is disconnected. A point p in X is a *local cut point* if p has a neighbourhood U in X so that p is a cut point for U.

   (a) Show that a topological surface has no local cut points.

   (b) If $P(z) = z^2$ show that $(0, 0)$ is a cut point of V_P. Conclude that V_P is not a topological surface.

   (c) If $P(z) = z^2(z + 1)$ show that V_P is connected, $(0, 0)$ is not a cut point for V_P but $(0, 0)$ is a local cut point for V_P. Conclude that V_P is not a topological surface.

   (d) If $P(z) = z^3$ show that V_P is a topologically surface.