MA475 Example Sheet 2

29 February 2018

1. Let $F : \mathbb{C}^n \to \mathbb{C}^m$ be a function which is differential as a real valued function. In particular we know that its partial derivatives exist. Let $p \in \mathbb{C}^n$ and let $q = F(p)$. Let $\iota_j : \mathbb{C} \to \mathbb{C}^n$ send z to $p + ze_j$. Let $\pi_k : \mathbb{C}^m \to \mathbb{C}$ be the coordinate function which sends the vector v to the coefficient v_j. Assume that F has continuous partial derivatives (in the ordinary sense). Show that $F : \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic if and only if for all j and k the functions $\pi_k \circ F \circ \iota_j$ are holomorphic as maps from $\mathbb{C}$ to $\mathbb{C}$.

2. Show that the linear action of $GL(n, \mathbb{C})$ on $\mathbb{C}^n$ induces an action by holomorphic automorphisms of $\mathbb{CP}^{n-1}$. Show that when $n = 2$ these automorphisms are just linear fractional transformations (Möbius transformations) when expressed in terms of the variable determined by the coordinate chart U_1.

   (a) What is the subgroup of $GL(2, \mathbb{C})$ that fixes $\mathbb{CP}^1$ pointwise? Show that every such automorphism is induced by a matrix with determinant one. How many matrices with determinant one induce the same automorphism?

   (b) Identify the group of linear fractional transformations with the group $PSL(2, \mathbb{C})$.

3. Find the singular points for each of the curves in $\mathbb{C}^2$

   (a) $y^3 - y^2 + x^3 - x^2 + 3y^2x + 2xy = 0$

   (b) $x^4 + y^4 - x^2y^2 = 0$

   Find the points at infinity for the curve $x^3 - xy^2 + 10x^2 + 20y + 30$

4. Find the singular points for each of the curves in $\mathbb{CP}^2$

   (a) $xy^4 + yz^4 + xz^4 = 0$

   (b) $x^2y^3 + x^2z^3 + y^2x^3 = 0$

   (c) $x^n + y^n + z^n = 0$

5. Two bases for a two dimensional real vector space determine the same orientation if the change of basis transformation has a positive determinant. An orientation on a two dimensional real vector space is a choice of an equivalence class of basis. A surface is orientable if we can choose a continuously varying orientation on each tangent space. Show that $\mathbb{RP}^2$ is not an orientable surface. Show that a Riemann surface is an orientable surface.