MA475 Example Sheet 3

15 February 2020

1. Let R be a connected Riemann surface and let f and g be holomorphic functions from R to a Riemann surface S. Show that if f and g coincide on some sequence that contains a limit point then $f = g$.

2. Show that a rational function $P(z)/Q(z)$ on $\mathbb{C}$ has a meromorphic extension to $\mathbb{C}_\infty$. Compute the order of the zero or pole at ∞ in terms of P and Q.

3. Let $D = \{z : 0 < |z| < 1\}$. Let G be the group generated by $z \mapsto z\exp(2\pi i/n)$ for some fixed n. Identify D/G up to conformal equivalence.

4. Recall that $\nu_f(z)$ is the valence of f at the point p. Let f and g be $\mathbb{C}$ valued holomorphic functions defined on domains in $\mathbb{C}$. Show that $\nu_{fg}(z) = \nu_f(g(z)) \cdot \nu_g(z)$. Use this result to show that $\nu_f(p)$ is well defined when $f : R \to S$ is a holomorphic map between Riemann surfaces and $p \in R$.

5. Draw the set of real points that satisfy the equation $w^2 = z^2(z+1)$. Let V be the set of complex points that satisfy the equation. Let $\bar{V}$ be the closure of V in $\mathbb{C}_\infty \times \mathbb{C}_\infty$. Describe $\bar{V}$ as a Riemann surface with points identified. What is the genus of this Riemann surface and how many points are identified?

6. Draw the real points that satisfy the equation $w^2 = z^n$ for $n = 1, 2, 3, 4$. Find an explicit parametrisation for the complex curve $w^2 = z^n$. Let S^3 denote the sphere of radius $\sqrt{2}$ centred at the origin. Prove that the intersection of V with S^3 is contained in a torus in S^3.