MA475 Example Sheet 3

12 March 2018

1. Consider the projective curve corresponding to the affine curve $w^2 = f(z)$ where f is a polynomial of degree d. How many points at ∞ does this curve have (as a function of d)? For which values of d is this curve non-singular at ∞? (Note that the compactification of this affine curve which we constructed in class does not necessarily agree with this compactification.)

2. Let C be the complex plane curve (Riemann surface) given by the equation $x^2+y^2 = 1$. Recall that the map $\phi : z \mapsto (\cos(z), \sin(z))$ gives a map from $\mathbb{C}/2\pi\mathbb{Z}$ to the C which parametrises the curve. We also presented C as a branched cover over the x-plane branched at ± 1. We would like to connect these two presentations. Observe that points of C are not too far from points on the curve $x^2 + y^2 = 0$ which is union of two lines and has solutions $x = iy$ and $x = -iy$. Divide the curve C into three regions. Let R_0 consist of $(x, y) \in C$ x is closer to iy than to $-iy$. Let R_1 consist of $(x, y) \in C$ where x is closer to $-iy$ than to iy, and let R_2 consist of (x, y) where x is equidistant from iy and $-iy$.

   (a) Compute the corresponding decomposition of $\mathbb{C}/2\pi\mathbb{Z}$. That is compute the inverse images under ϕ of the 3 regions.

   (b) Relate this decomposition of C to the projection of the curve to the x-axis and with the line segment drawn between -1 and 1.

3. Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$. Let me write $Ind(f, p)$ for the index of f at p. Show that $Ind(fg, p) = Ind(f, p)+Ind(g, p)$. Say that ϕ is holomorphic with non-zero derivative at p show that $Ind(f \circ \phi, p) = Ind(f, \phi(p))$. Explain why $Ind(f, p)$ is well defined for $f : R \to \mathbb{C} \cup \{\infty\}$ for any Riemann surface R.

4. Let me write $\nu(f, p)$ for the local multiplicity of f at p. Let ϕ and ψ be conformal holomorphic functions. Show that $\nu(\phi \circ f \circ \psi, p) = \nu(f, \psi(p))$. Explain why $\nu(f, p)$ is well defined when $f : R \to S$ is a holomorphic map between Riemann surfaces.

5. Show that the degree of the composition of two proper maps between Riemann surfaces is the product of the degrees.