MA475 Example Sheet 4

26 February 2019

1. Two ordered bases for a two dimensional real vector space determine the same orientation if the change of basis transformation has a positive determinant. An orientation on a two dimensional real vector space is a choice of an equivalence class of basis. A surface is orientable if we can choose a continuously varying orientation on each tangent space. Show that a Riemann surface is orientable.

2. Let f be a meromorphic function on a Riemann surface R. Let σ be a curve in R from p to q. Show that

$$\int_\sigma df = f(q) - f(p).$$

3. Let f be holomorphic but not constant on a Riemann surface. Show that $\Re(f)$ cannot attain a local maximum.

4. Let $f(z) = \frac{P(z)}{Q(z)}$. Calculate the valence of f at ∞. Explicitly calculate $\sum_{p:f(p)=0} v_f(p)$ and $\sum_{p:f(p)=\infty(} v_f(p)$ in terms of the degrees of P and Q. (This problem asks you to do the explicit calculation rather than to apply the degree theorem.)

5. Let $h : \Delta \to \Delta$ be a holomorphic map from the unit disk to itself. Show that if $|h'(v)|_{hyp} = |v|_{hyp}$ for some non-zero tangent vector v then h is given by a linear fractional transformation. Here $|v|_{hyp} = \delta(z)|v|$ with $\delta(z) = \frac{2}{1-|z|^2}$ for v a tangent vector based at the point $z \in \Delta$.

6. Show that if a compact Riemann surface R has a self-map f of degree greater than one then R is the sphere or torus. Show that if f is a positive degree self-map and R is the torus then f is a covering map.