MA475 Example Sheet 4

2 March 2020

We will assume from now on that all Riemann surfaces are Hausdorff.

1. Show that the residue of a pole on a Riemann surface is uniquely defined (as a line integral).

2. Recall our construction of a conformal structure on the boundary of a polyhedron P in $\mathbb{R}^3$. Consider a point in P not lying on a vertex or an edge. What are the values of the coefficients of the first fundamental form E, F and G at such a point?

3. Let f be a holomorphic map from R to S. Explain why f has a lift $\tilde{f}$ to the universal covering surfaces $\tilde{R}$ and $\tilde{S}$ and show $\tilde{f}$ is holomorphic.

4. Suppose that $f : \Delta \to \Delta$ is holomorphic but not in $Aut(\Delta)$. Show that f strictly contracts all hyperbolic distances.

5. Show that $f \mapsto f(1) - f(0)$ is a homomorphism from $Aut(\mathbb{C})$ to the of multiplicative group of non-zero complex numbers. Deduce that the group of translations is a normal subgroup of $Aut(\mathbb{C})$.

6. Using the classification of simply connected Riemann surfaces show that every Riemann surface has a metric of constant curvature.

7. Let R be a Riemann surface. Show that if $\tilde{R}$ and $\tilde{S}$ are $\mathbb{C}$ then an invertible holomorphic map from R to S preserves the constant curvature metric up to a constant scaling factor.

8. Show that a holomorphic map between compact Riemann surfaces with valence 1 at every point is a covering map.

9. We have shown that given a holomorphic map f between compact Riemann surfaces that $\delta(q)$ is constant for call this quantity the degree of f. Let Q, R and S be compact Riemann surfaces. Let $f : Q \to R$ and $g : R \to S$ show that the degree of gf is the product of the degrees of f and g.