MA475 Example Sheet 4

12 March, 2018

1. Show that
$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$
is holomorphic and well defined on $\mathbb{C}/\mathbb{Z}$.

2. Let $\wp$ be the Weierstrass $\wp$-function with respect to a lattice $\Lambda \subset \mathbb{C}$. Show that $\wp$ satisfies the differential equation $\wp''(z) = 6\wp(z)^2 + A$ for some constant A. Show that there are at least three points and at most five points in $\mathbb{C}/\Lambda$ at which $\wp'$ is not locally injective.

3. Let a_1, a_2, a_3 be distinct points in $\mathbb{CP}^1$. Show that there is a unique Möbius transformation that takes $0, 1, \infty$ to a_1, a_2, a_3 (respecting the order). Show that there is a Möbius transformation that takes any ordered triple of distinct points to any other. Define the cross ratio of a_1, a_2, a_3, a_4 to be the image of a_4 under the unique map that takes the ordered triple a_1, a_2, a_3 to the ordered triple $0, 1, \infty$. Compute the cross ratio. Show that two ordered quadruples are equivalent if and only if they have the same cross ratio.

4. Let $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be the rational map defined by $f(z) = z^n + 1/z^n$. Compute the critical points, orders of critical points and branch points of π_x. Determine the monodromy of the appropriate cover. (Determine the map from the appropriate fundamental group into the appropriate permutation group.) Compute the monodromy around the point at infinity. Is this a regular cover?

5. Consider the affine curve C defined by $P(x, y) = y^3 - x(x^2 - 1)$. Let $\pi_x = \pi_x|C$ be the restriction of the projection onto the first coordinate to the curve C. Compute the critical points, orders of critical points and branch points of π_x. Explain why C is a curve of finite type. Show that we can add 3 points to create $\bar{C}$ and we can extend π_x to a map from $\bar{C}$ to S^2 which takes the added points to the point at infinity in S^2. Calculate the genus of $\bar{C}$.