MA475 Example Sheet 5

9 March 2020

1. Verify the Riemann-Hurwitz formula for the map $f(z) = z^4 + z^{-4}$ from $\mathbb{C}_\infty$ to itself.

2. Say that we have a non-constant meromorphic function $f : R \to S$ between compact Riemann surfaces. Show that the genus of S cannot be larger than the genus of R. If both surfaces have genus one show that f is a covering map. If both surfaces have the same genus and it is 2 or more show that f is a holomorphic equivalence.

3. Define the divisor of a meromorphic function on a Riemann surface R to be the function from $R \to \mathbb{Z}$ which assigns to each point the order of the zero or pole at that point. Define the degree of a divisor to be the sum of the non-zero values of this function.

   (a) Show that two meromorphic functions with the same divisor differ by multiplication by a non-zero complex number.

   (b) Show that the degree of a divisor on a compact Riemann surface is zero.

4. Let $\wp$ be the Weierstrass $\wp$-function with respect to a lattice $\Lambda \subset \mathbb{C}$. Say that $\wp(z) = z^{-2} + \lambda z^2 + \mu z^4 + O(z^6)$. Compute the value of λ and μ in terms of Λ.

5. Let $\wp$ be the Weierstrass $\wp$-function with respect to a lattice $\Lambda \subset \mathbb{C}$. Show that $\wp$ satisfies the differential equation $\wp''(z) = 6\wp(z)^2 + A$ for some constant A. Show that there are at least three points and at most five points in $\mathbb{C}/\Lambda$ at which $\wp'$ is not locally injective.