MA475 Example Sheet 1

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1. Show that the unit disk and the upper half-plane are conformally equivalent. Show that the unit disc and the complex plane are not conformally equivalent.

2. A holomorphic function on the plane has a power series expansion valid for $|z|$ sufficiently large. We say that a homomorphic function on $\mathbb{C}$ is meromorphic at infinity if it extends to a meromorphic function on $S^2 = \mathbb{C} \cup \infty$. What does it mean for the function to be meromorphic at ∞ in terms of the coefficients of this expansion?

3. Let $\Lambda = \langle \tau_1, \tau_2 \rangle$ represent the additive group generated by τ_1 and τ_2. We say Λ is a lattice if the quotient $\mathbb{C}/\Lambda$ is a torus. Let Γ be the group generated by Λ and the transformation $z \mapsto -z$. Find a fundamental domain for the action of Γ on $\mathbb{C}$. Explain why $\mathbb{C}/\Gamma$ is a Riemann surface. What surface is this topologically?

4. Let R be the Riemann surface obtained (as a flat surface with cone points) from the boundary of the regular tetrahedron. Show that R is conformally equivalent to a surface of the form $\mathbb{C}/\Gamma$ as in the previous problem. What are τ_1 and τ_2?

5. Let J be a linear transformation of a real vector space V satisfying $J^2 = -I$. The vector space V is a module over the subring of the matrix ring generated by I and J. Show that this subring is isomorphic to $\mathbb{C}$ and that, as a module over $\mathbb{C}$, V becomes a complex vector space.

6. Let $L : V \to W$ be an $\mathbb{R}$ linear transformation. Say that V and W have complex structures coming from linear transformations J_V and J_W as in the previous problem. Show that L becomes a complex linear map respect to these complex structures if and only if $L \circ J_V = J_W \circ L$.

7. Let $e_1 \dots e_{2n}$ be a basis for $\mathbb{R}^{2n}$. Define a linear transformation J_n by $J(e_{2j-1}) = e_{2j}$ and $J_n(e_{2j}) = -e_{2j-1}$ for $j = 1 \dots n$. This matrix induces the same complex structure on $\mathbb{R}^{2n}$ as one would obtain by identifying it with $\mathbb{C}^n$ in the standard way. Let $L : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ be a linear transformation given by the matrix A. What conditions on the

coefficients does the matrix A need to satisfy in order to be complex linear? Let F be a smooth map from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$. What condition do the partial derivatives of F need to satisfy for F to be holomorphic? (These conditions are the higher dimensional Cauchy-Riemann equations.)

8. Let H be the upper half-plane. Consider the multiplicative group $\{r^n : n \in \mathbb{Z}\}$ acting on H by multiplication. Show that the quotient surface is conformally equivalent to an annulus $\{r_1 < |z| < r_2\}$ for some positive values of r_1 and r_2.